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ON THE DEGREE OF CONVERGENCE OF LAPLACE'S SERIES*

BY

T. H. GRONWALL

§ 1. Introduction.

Let S denote a sphere of unit radius, and refer its points to a geographical system of coördinates, θ being the polar distance and φ the longitude; the north pole and the zero meridian may be fixed arbitrarily. Further let $f(\theta, \varphi)$ be a uniform function of the position on the sphere, and suppose this function to be absolutely integrable in the Riemann sense, so that

$$\int_S |f(\theta, \varphi)|\, d\sigma$$

exists, where $d\sigma = \sin\theta\, d\theta\, d\varphi$ is the surface element of S. We finally denote by θ', φ' any point on S, the corresponding surface element being $d\sigma'$, and by γ the angle between the two vectors from the center of the sphere to the points θ, φ and θ', φ' respectively (or, in other words, the distance between the two points in question, measured on the great circle joining them). This angle γ is completely determined by the condition $0 \leqq \gamma \leqq \pi$, and we have

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi' - \varphi).$$

Then *the Laplace series corresponding to* $f(\theta, \varphi)$ is

$$(1) \qquad f(\theta, \varphi) \backsim \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_S f(\theta', \varphi') P_n(\cos\gamma)\, d\sigma',$$

P_n being the nth Legendre polynomial; the equivalence sign $\backsim$ sign fies that the series is considered in a purely formal way, and implies no statement in regard to its convergence.

An important particular case is obtained by supposing $f(\theta, \varphi)$ independent of φ; writing $\cos\theta = x$ and $f(\theta, \varphi) = f(\cos\theta) = f(x)$, the Laplace series (1) becomes *the Legendre series corresponding to* $f(x)$:

$$(2) \qquad f(x) \backsim \sum_{n=0}^{\infty} c_n P_n(x), \qquad c_n = \frac{2n+1}{2} \int_{-1}^{+1} f(y) P_n(y)\, dy.$$

* Presented to the Society February 22, 1913.

The present paper is concerned with the degree of approximation obtained by replacing $f(\theta, \varphi)$ by the sum of the first $n+1$ terms in its Laplace series, or in other words, with the investigation of the function

$$(3) \qquad r_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - \sum_{\nu=0}^{n} \frac{2\nu+1}{4\pi} \int_S f(\theta', \varphi') P_\nu(\cos\gamma)\, d\sigma'$$

under the assumption that for any two points θ, φ and θ', φ' the function $f(\theta, \varphi)$ satisfies a generalized Lipschitz condition of the form

$$(4) \qquad |f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma),$$

where, for γ increasing, $\omega(\gamma)$ never decreases, and $\omega(\gamma)/\gamma$ never increases, while $\lim_{\gamma=0} \omega(\gamma) = 0$. On account of the last condition, $f(\theta, \varphi)$ is obviously continuous all over the sphere S.

In a recent paper, Jackson* has treated the corresponding problem for the Legendre series (2), assuming that $f(x)$ satisfies the condition

$$(5) \quad |f(x_2) - f(x_1)| \leqq \omega(|x_2 - x_1|), \qquad -1 \leqq x_1 \leqq 1, \qquad -1 \leqq x_2 \leqq 1,$$

the function ω being subject to the same restrictions as above. He confines his investigation, however, to the points of the interval

$$(6) \qquad -1 + \epsilon \leqq x \leqq 1 - \epsilon,$$

where ϵ is any *positive* quantity. The methods of the present paper apply, on convenient specialization, to Legendre's series for $x = \pm 1$; whenever the occasion arises, we shall compare our results with those of Jackson and thus bring out the fundamental difference between the behavior of the Legendre series in the interior points of the interval $(-1, 1)$ and in the end points.

Section 2 of the present paper gives a lemma of fundamental importance: supposing $f(\theta, \varphi)$ in (1) limited by the condition that $|f(\theta, \varphi)| \leqq 1$ at every point of S, then there exists a function $f(\theta, \varphi)$ of this kind which makes the absolute value of the sum of the first $n+1$ terms in Laplace's series a maximum ρ_n at a given point θ, φ, and it is shown that these constants ρ_n, which we shall call the Lebesgue constants for Laplace's series,† tend towards infinity with n as $\sqrt{n}$,‡ or more accurately, that §

* Dunham Jackson, I. *On the Degree of Convergence of the Development of a Continuous Function According to Legendre's Polynomials*, these Transactions, vol. 13 (1912), pp. 305–318.

† Lebesgue first showed the fundamental importance of the corresponding constants for Fourier's series; see *Leçons sur les séries trigonométriques*, Paris, 1906, § 45.

‡ Jackson shows (l. c., pp. 306–309) that the corresponding maximum for Legendre's series in the interval (6) is inferior to a constant multiple of $\log n$.

§ The first proof of this relation may be found in § 2 of my paper *Über die Laplace'sche Reihe* (Mathematische Annalen, vol. 74 (1913), pp. 213–270); the present proof contains several simplifications and also some new developments which will be used in § 4.

$$\lim_{n=\infty} \frac{\rho_n}{\sqrt{n}} = 2\sqrt{\frac{2}{\pi}}.$$

In § 3 it will be shown that $f(\theta, \varphi)$ may be approximated by a sum $T_n(\theta, \varphi)$ of spherical harmonics of degree not exceeding n in such a way that

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| \overline{\overline{<}} K\omega\left(\frac{1}{n}\right) \qquad (n = 1, 2, 3, \cdots)$$

at every point of the sphere S, K being a constant independent of n. The method of proof is closely related to that of Jackson* for the corresponding problem regarding $f(x)$.

In § 4, the preceding results are used to show that for all points of S

$$|r_n\{f(\theta, \varphi)\}| \overline{\overline{<}} K'\omega\left(\frac{1}{n}\right)\sqrt{n} \qquad (n = 1, 2, 3, \cdots),$$

K' being a constant independent of n, and that, conversely, if $\Omega(\gamma)$ is a function subject to the same restrictions as $\omega(\gamma)$ and furthermore

$$\lim_{\gamma=0} \frac{\Omega(\gamma)}{\omega(\gamma)} = 0,$$

then there exists a function $f(\theta, \varphi)$ satisfying (4) and such that at a given point θ_0, φ_0 the inequality

$$|r_n\{f(\theta_0, \varphi_0)\}| \overline{\overline{>}} K''\Omega\left(\frac{1}{n}\right)\sqrt{n}$$

is satisfied for an infinite number of values of n, the constant K'' being independent of n.†

§ 2. The Lebesgue constants for Laplace's series.

The sum of the first $n + 1$ terms in (1) is equal to

$$s_n\{f(\theta, \varphi)\} = \frac{1}{4\pi}\int_S f(\theta', \varphi')\, s_n(\cos\gamma)\, d\omega'$$

where

$$s_n(x) = \sum_{\nu=0}^{n} (2\nu + 1) P_\nu(x). \tag{7}$$

Limiting $f(\theta, \varphi)$ to the class of absolutely integrable functions which are not

* Dunham Jackson, II. *Uber die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Inaugural-Dissertation, Göttingen, 1911. III. *On Approximation by Trigonometric Sums and Polynomials*, these Transactions, vol. 13 (1912), pp. 491–515.

† For $f(x)$ and x limited by (6), $\sqrt{n}$ may be replaced by $\log n$ and $\Omega(1/n)$ by $\omega(1/n)$ in the two statements regarding r_n. (Jackson, I, theorems 1 and 3.)

greater than unity in absolute value at every point of S, we see at once that

$$|s_n\{f(\theta, \varphi)\}| \leqq \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\, d\sigma',$$

and the absolute maximum

$$\rho_n(\theta, \varphi) = \max. |s_n\{f(\theta, \varphi)\}| = \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\, d\sigma'$$

is reached at the point θ, φ by defining f so that

$$f(\theta', \varphi') = \text{sgn. } s_n(\cos\gamma),$$

where, in the notation of Kronecker,

$$\text{sgn. } a = \begin{cases} +1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0. \end{cases}$$

Moving the north pole of our coördinate system to the point θ, φ, we obtain $\gamma = \theta'$ and

$$\rho_n(\theta, \varphi) = \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\, d\sigma' = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} |s_n(\cos\theta')| \sin\theta'\, d\theta'\, d\varphi'$$
$$= \tfrac{1}{2}\int_0^\pi |s_n(\cos\theta')| \sin\theta'\, d\theta',$$

so that $\rho_n(\theta, \varphi) = \rho_n$ is independent of θ, φ; writing $\cos\theta' = x$, we find

$$\rho_n = \tfrac{1}{2}\int_{-1}^{+1} |s_n(x)|\, dx, \qquad (n = 0, 1, 2, \cdots). \tag{8}$$

In order to investigate the order of magnitude of ρ_n with respect to n, we begin by developing an asymptotic expression for $s_n(\cos\theta)$, which will furnish approximations to the roots of the equation $s_n(\cos\theta) = 0$.

From the well-known equation of definition of Legendre's polynomials

$$\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n, \qquad |z| < 1, \tag{9}$$

where the square root equals unity for $z = 0$, the equally well-known formula

$$\frac{1 - z^2}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) P_n(x) z^n$$

is easily obtained, and on multiplying both sides by

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$$

and using (7), it is seen at once that

$$\frac{1+z}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} s_n(x) z^n, \qquad |z| < 1. \tag{10}$$

If we write $1/z$ instead of z and introduce $x = \cos\theta$, this becomes

$$\frac{z^2(z+1)}{(z-e^{\theta i})^{3/2}(z-e^{-\theta i})^{3/2}} = \sum_{n=0}^{\infty} s_n(\cos\theta) z^{-n}, \qquad |z| > 1, \tag{11}$$

the radicals being determined so that their real parts tend toward $+\infty$ when z tends toward infinity by real and positive values. Then the application of Cauchy's theorem gives

$$s_n(\cos\theta) = \frac{1}{2\pi i}\int_C \frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}}dz, \tag{11}$$

the contour of integration C enclosing both of the singular points $z = e^{\theta i}$ and $z = e^{-\theta i}$. Now let ϵ be a positive quantity which will later be made to tend toward zero, and let C consist of the circuit C_1 around $z = e^{\theta i}$, composed of the three parts: (1) the straight line from $z = 0$ to $z = (1-\epsilon)e^{\theta i}$, (2) the circle $z = e^{\theta i} + \epsilon e^{\phi i}$, $\theta - \pi \leqq \phi \leqq \theta + \pi$, (3) the straight line from $z = (1-\epsilon)e^{\theta i}$ to $z = 0$, followed by the circuit C_2 around $z = e^{-\theta i}$, composed of the three parts: (4) the straight line from $z = 0$ to $z = (1-\epsilon)e^{-\theta i}$, (5) the circle $z = e^{-\theta i} + \epsilon e^{\phi i}$, $-\theta - \pi \leqq \phi \leqq \pi - \theta$, (6) the straight line from $z = (1-\epsilon)e^{-\theta i}$ to $z = 0$.

To make the passage to the limit $\epsilon = 0$ possible, (11) must be transformed so that the integrand in (11) becomes infinite of an order less than one at the singular points, and to this purpose we use the identity

$$\frac{z+1}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}} = -\frac{1}{z-1}\left[\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}} + 2z\frac{d}{dz}\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\right]$$

which gives, upon integration by parts,

$$\begin{aligned}
\int \frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}}dz &= -\int \frac{z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}dz \\
&\quad - 2\int \frac{z^{n+2}}{(z-1)}\frac{d}{dz}\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}dz \\
&= -\frac{2z^{n+2}}{z-1}\cdot\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}} \\
&\quad + \int \frac{(2n+3)z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}dz \\
&\quad - \int \frac{2z^{n+2}}{(z-1)^2(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}dz.
\end{aligned}$$

Integrate over the circuit C_1; the expression outside of the integral signs vanishes at the beginning and the end of C_1 $(z = 0)$; making $\epsilon = 0$, the integrals over (2) vanish, and those over (3) become equal to the corresponding ones over (1), owing to the fact that $(z - e^{\theta i})^{\frac{1}{2}}$ changes its sign when z describes the circle (2). We therefore obtain

$$\frac{1}{2\pi i}\int_{C_1}\frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}}dz = \frac{1}{\pi i}\int_0^{e^{\theta i}}\frac{(2n+3)z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}dz$$

$$-\frac{1}{\pi i}\int_0^{e^{\theta i}}\frac{2z^{n+2}}{(z-1)^2(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}dz,$$

the path of integration being the straight line from $z = 0$ to $z = e^{\theta i}$, and the initial values of the radicals for $z = 0$ having their real parts positive (being positive for $z = +\infty$ and never passing through zero as z moves along the real axis). We now introduce a new variable of integration, making

$$z = e^{\theta i}(1-u), \qquad 0 \leqq u \leqq 1;$$

then, on account of the manner of fixing the values of the radicals for $z = 0$,

$$(z-e^{\theta i})^{\frac{1}{2}} = e^{\frac{\theta-\pi}{2}i}\cdot u^{\frac{1}{2}}, \qquad u^{\frac{1}{2}} \geqq 0,$$

$$(z-e^{-\theta i})^{\frac{1}{2}} = \sqrt{2\sin\theta}\cdot e^{\frac{\pi}{4}i}\left(1-\frac{ue^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}\right)^{\frac{1}{2}},$$

where

$$\left(1-\frac{ue^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}\right)^{\frac{1}{2}} = +1 \quad \text{for} \quad u = 0.$$

Introducing all this in the integrals above, and writing

$$\alpha = \frac{e^{\frac{\theta-\pi}{2}}}{2\sin\frac{\theta}{2}}, \qquad \beta = \frac{e^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}, \tag{12}$$

we obtain

$$\frac{1}{2\pi i}\int_{C_1}\frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}}dz = \frac{2n+3}{\pi}\cdot\frac{e^{(n+1)\theta i-\frac{3}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\int_0^1\frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}}du$$

$$-\frac{2}{\pi}\cdot\frac{e^{(n+\frac{3}{2})\theta i-\frac{1}{4}\pi i}}{\left(2\sin\frac{\theta}{2}\right)^2(2\sin\theta)^{\frac{1}{2}}}\int_0^1\frac{u^{-\frac{1}{2}}(1-u)^{n+2}}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}}du.$$

In exactly the same way it is seen that the corresponding integral over the

circuit C_2 has a value which is conjugate to the one just obtained for C_1, so that, by (11),

$$
\begin{aligned}
s_n(\cos\theta) = {} & \frac{2(2n+3)}{\pi} \boldsymbol{R} \frac{e^{(n+1)\theta i - \frac{3}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} du \\
& - \frac{4}{\pi} \boldsymbol{R} \frac{e^{(n+\frac{3}{2})\theta i - \frac{1}{4}\pi i}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+2}}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}} du,
\end{aligned} \tag{13}
$$

where $\boldsymbol{R}$ denotes the real part of the expression following it.

This formula will now easily give the desired asymptotic expression for $s_n(\cos\theta)$. On account of (12), we have for $0 \leqq u \leqq 1$

$$|1-\alpha u|^2 = \left(1-\frac{u}{2}\right)^2 + \frac{u^2}{4}\cot^2\frac{\theta}{2} \geqq \left(1-\frac{u}{2}\right)^2 \geqq \frac{1}{4},$$

$$|1-\beta u|^2 = \left(1-\frac{u}{2}\right)^2 + \frac{u^2}{4}\cot^2\theta \geqq \left(1-\frac{u}{2}\right)^2 \geqq \frac{1}{4},$$

so that

$$\frac{1}{|1-\alpha u|} \leqq 2, \qquad \frac{1}{|1-\beta u|} \leqq 2; \tag{14}$$

furthermore the identity

$$\frac{1}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} - 1 = \int_0 \left(\frac{\alpha}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}} + \frac{1}{2}\cdot\frac{\beta}{(1-\alpha u)(1-\beta u)^{\frac{3}{2}}} \right) du$$

shows, by the aid of (14), that

$$\left|\frac{1}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} - 1\right| < \int_0^u (2^{\frac{5}{2}}|\alpha| + 2^{\frac{3}{2}}|\beta|)du = \left(\frac{2^{\frac{5}{2}}}{2\sin\frac{\theta}{2}} + \frac{2^{\frac{3}{2}}}{2\sin\theta}\right)u < \frac{8u}{\sin\theta}.$$

From the first integral in (13) we now obtain

$$
\begin{aligned}
\Bigg| & \frac{2(2n+3)}{\pi} \boldsymbol{R} \frac{e^{(n+1)\theta i - \frac{3}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} du \\
& - \frac{2(2n+3)}{\pi} \boldsymbol{R} \frac{e^{(n+1)\theta i - \frac{3}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 u^{-\frac{1}{2}}(1-u)^{n+1} du \Bigg| \\
& < \frac{2(2n+3)}{\pi} \cdot \frac{1}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \cdot \frac{8}{\sin\theta} \int_0^1 u^{\frac{1}{2}}(1-u)^{n+1} du,
\end{aligned}
$$

or evaluating the Euler integrals of the first kind occurring in this formula and

reducing the result by means of $\Gamma(x+1) = x\Gamma(x)$,

$$\left| \frac{2(2n+3)}{\pi} \boldsymbol{R} \frac{e^{(n+1)\theta i - \frac{3}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} du \right.$$

$$\left. - \frac{4}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)} \cdot \frac{\sin\left((n+1)\theta - \frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \right|$$

$$< \frac{4}{\pi}\left(n+\frac{3}{2}\right) \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{7}{2}\right)} \cdot \frac{16}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{3}{2}}}$$

$$< \frac{4}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)} \cdot \frac{8}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{3}{2}}}.$$

The second integral in (13) may be estimated directly by (14), which gives

$$\left| \frac{4}{\pi} \boldsymbol{R} \frac{e^{(n+\frac{3}{2})\theta i - \pi i}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-1}(1-u)^{n+2}}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}} du \right|$$

$$< \frac{4}{\pi} \cdot \frac{2^2 \cdot 2^{\frac{1}{2}}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \int_0^1 u^{-\frac{1}{2}}(i-u)^{n+2} du$$

$$= \frac{4}{\pi} \cdot \frac{2^{\frac{5}{2}}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \cdot (n+2) \cdot \Gamma(n+2)}{\left(n+\frac{5}{2}\right)\Gamma\left(n+\frac{5}{2}\right)}$$

$$< \frac{4}{\pi} \cdot \frac{2 \cdot 2}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{3}{2}}} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)}.$$

Introducing these approximations in (13), we obtain the asymptotic expression

$$s_n(\cos\theta) = \frac{4}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)} \frac{\sin\left((n+1)\theta - \frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}$$

(15)

$$+ \frac{4}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)} \cdot \frac{24\,\eta(n,\theta)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{3}{2}}},$$

where

$$|\eta(n,\theta)| < 1 \qquad \text{for} \qquad 0 \leqq \theta \leqq \pi \qquad \text{and} \qquad n = 1, 2, 3, \cdots.$$

To obtain some information regarding the zeros of $s_n(\cos\theta)$, we write (15) in the form

$$\frac{\pi}{8}\,\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}\cdot 2\sin\frac{\theta}{2}\,(2\sin\theta)^{\frac{3}{2}}\cdot s_n(\cos\theta) \tag{16}$$

$$= \sin\theta\sin\left((n+1)\theta-\frac{\pi}{4}\right)+\frac{24\eta(n,\theta)}{2n+3}.$$

Subjecting the integers n and ν to the inequalities

$$n \geqq 4^5, \tag{17}$$

$$(n+1)^{\frac{2}{5}} \leqq \nu \leqq n-(n+1)^{\frac{2}{5}},$$

we introduce the special value of θ

$$\theta_0 = \frac{4\nu+1}{n+1}\,\frac{\pi}{4}+\frac{\kappa\pi}{(n+1)^{\frac{6}{5}}}, \qquad \kappa = \pm 1, \tag{18}$$

in the expression (16). From (18) it follows that

$$(n+1)\theta_0-\frac{\pi}{4} = \nu\pi+\frac{\kappa\pi}{(n+1)^{\frac{1}{5}}},$$

and

$$\sin\left((n+1)\theta_0-\frac{\pi}{4}\right) = (-1)^{\nu}\kappa\sin\frac{\pi}{(n+1)^{\frac{1}{5}}}.$$

Furthermore, we have, by (17) and (18),

$$0 < \frac{\nu\pi}{n+1} < \theta_0 < \frac{\nu+1}{n+1}\pi < \pi,$$

and, on account of $\sin x > 2x/\pi$ for $0 < x < \pi$, we find, in case $0 < \theta_0 \leqq \pi/2$,

$$\sin\theta_0 > \sin\frac{\nu\pi}{n+1} > \frac{2}{\pi}\cdot\frac{\nu\pi}{n+1} \geqq \frac{2}{(n+1)^{\frac{3}{5}}},$$

and in case $\pi/2 < \theta_0 < \pi$,

$$\sin\theta_0 > \sin\frac{\nu+1}{n+1}\pi = \sin\frac{n-\nu}{n+1}\pi > \frac{2}{\pi}\cdot\frac{n-\nu}{n+1}\pi \geqq \frac{2}{(n+1)^{\frac{3}{5}}},$$

and finally

$$\left|\sin\left((n+1)\theta_0-\frac{\pi}{4}\right)\right| = \sin\frac{\pi}{(n+1)^{\frac{1}{5}}} > \frac{2}{\pi}\cdot\frac{\pi}{(n+1)^{\frac{1}{5}}} = \frac{2}{(n+1)^{\frac{1}{5}}}.$$

From these inequalities it follows that

$$\left|\sin\theta_0 \sin\left((n+1)\theta_0 - \frac{\pi}{4}\right)\right| > \frac{2}{(n+1)^{\frac{3}{5}}}\cdot\frac{2}{(n+1)^{\frac{1}{5}}}$$

$$= \frac{4}{(n+1)^{\frac{4}{5}}} > \frac{12}{n+1} > \frac{24}{2n+3} > \left|\frac{24\eta(n,\theta_0)}{2n+3}\right|,$$

and consequently, for $\theta = \theta_0$, the expression (16) has the same sign as the first term on its right side, so that

$$\operatorname{sgn.} s_n(\theta_0) = (-1)^\nu \cdot \kappa,$$

whence we conclude that $s_n(\theta)$ changes sign an odd number of times in the interval

$$\frac{4\nu+1}{n+1}\frac{\pi}{4} - \frac{\pi}{(n+1)^{\frac{6}{5}}} < \theta < \frac{4\nu+1}{n+1}\frac{\pi}{4} + \frac{\pi}{(n+1)^{\frac{6}{5}}}, \tag{19}$$

whenever n and ν satisfy the inequalities (17). In case the interval (19) should contain several zeros of odd order of $s_n(\theta)$, let θ_ν be the smallest among them; we then obviously have

$$\theta_\nu = \frac{4\nu+1}{n+1}\frac{\pi}{4} + \frac{h_\nu \pi}{(n+1)^{\frac{6}{5}}}, \qquad |h_\nu| < 1. \tag{20}$$

Returning to the expression (8), let $x_1, x_2, \cdots, x_m$, where $1 > x_1 > x_2 > \cdots > x_m > -1$, be the points at which $s_n(x)$ changes sign in the interval $-1 \overline{\overline{<}} x \overline{\overline{<}} 1$. Such points exist, on account of (20), and on the other hand, $s_n(x)$ being, by definition, a polynomial of degree n, we have $m \overline{\overline{<}} n$.* For reasons of symmetry, we also write $x_0 = 1$ and $x_{m+1} = -1$; as $P_n(1) = 1$, $s_n(1)$ is positive, and consequently

$$|s_n(x)| = (-1)^\lambda s_n(x), \qquad x_\lambda \leqq x \leqq x_{\lambda+1}, \qquad \lambda = 0, 1, \cdots, m,$$

so that

$$\rho_n = \frac{1}{2}\int_{-1}^{1} |s_n(x)|\,dx = \left(\frac{1}{2}\int_{x_1}^{x_0} + \frac{1}{2}\int_{x_2}^{x_1} + \cdots + \frac{1}{2}\int_{x_{m+1}}^{x_m}\right)|s_n(x)|\,dx$$

$$= \frac{1}{2}\sum_{\lambda=0}^{m}(-1)^\lambda \int_{x_{\lambda+1}}^{x_\lambda} s_n(x)\,dx.$$

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* We do not need to use the fact that $m = n$, which is easily proved by Christoffel's formula

$$s_n(x) = (n+1)\frac{P_n(x) - P_{n+1}(x)}{1-x}$$

and the fact that the zeros of $P_n(x)$ lie between -1 and $+1$ and separate those of $P_{n+1}(x)$. Substituting for x two consecutive roots of $P_{n+1}(x) = 0$, we obtain values of $s_n(x)$ of opposite signs, and consequently the zeros of $s_n(x)$ separate those of $P_{n+1}(x)$, or $m = n$.

Using the formula

$$s_n(x) = \frac{dP_n(x)}{dx} + \frac{dP_{n+1}(x)}{dx}, \tag{21}$$

we obtain

$$\rho_n = \frac{1}{2}\sum_{\lambda=0}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda) - P_n(x_{\lambda+1}) - P_{n+1}(x_{\lambda+1})],$$

or rearranging the terms,

$$\rho_n = \frac{1}{2}[P_n(x_0) + P_{n+1}(x_0)] + \sum_{\lambda=1}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)] + \frac{(-1)^{m+1}}{2}[P_n(x_{m+1}) + P_{n+1}(x_{m+1})],$$

or finally, observing that $P_n(x_0) = 1$, $P_n(x_{m+1}) = (-1)^n$,

$$\rho_n = 1 + \sum_{\lambda=1}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)]. \tag{22}$$

Let ν_1 be the smallest, and ν_2 the greatest, of the integers ν satisfying (17); then to each ν, where $\nu_1 \overline{\overline{<}} \nu \overline{\overline{<}} \nu_2$, there corresponds, according to (20), an index λ_ν $(1 \leqq \lambda_\nu \leqq m)$ such that

$$\cos\theta_\nu = x_{\lambda_\nu}, \qquad \nu_1 \leqq \nu \leqq \nu_2.$$

We obviously have $\lambda_{\nu+1} > \lambda_\nu$, and by the definition of θ_ν, there is an even number of roots of $s_n(\cos\theta) = 0$ (or none) between θ_ν and $\theta_{\nu+1}$, so that

$$\lambda_{\nu+1} - \lambda_\nu \equiv 1 \pmod{2},$$

or

$$\lambda_{\nu+1} - (\nu+1) \equiv \lambda_\nu - \nu \pmod{2}.$$

We may therefore write (22) in the form

$$\rho_n = (-1)^{\lambda_{\nu_1}-\nu_1}\sum_{\nu=\nu_1}^{\nu_2}(-1)^\nu[P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu)] + 1 + \sum_\lambda{}'(-1)^\lambda[(P_n(x_\lambda) + P_{n+1}(x_\lambda)],$$

the second sum extending over such values of λ as are not of the form λ_ν, $(\nu_1 \leqq \nu \leqq \nu_2)$. As $m \leqq n$, the number of terms in Σ' does not exceed $n - (\nu_2 - \nu_1 + 1)$, which is less than or equal to $2(n+1)^{\frac{2}{3}}$, by (17). Each term in Σ' not exceeding $|P_n(x_\lambda)| + |P_{n+1}(x_\lambda)| < 1 + 1 = 2$ in absolute value, we have *

$$\left|1 + \sum_\lambda{}'(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)]\right| < 1 + 2(n+1)^{\frac{2}{3}} \cdot 2 = O(n^{\frac{2}{3}}),$$

* The notation $f(n) = O(g(n))$ signifies that for n sufficiently large, $|f(n)| < Ag(n)$, where A is a constant independent of n. Similarly, $f(n) = o(g(n))$ stands for $\lim_{n=\infty}\frac{f(n)}{g(n)} = 0$.

and consequently

$$\rho_n = (-1)^{\lambda_{\nu_1}-\nu_1} \sum_{\nu=\nu_1}^{\nu_2} (-1)^\nu [P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu)] + O(n^{\frac{3}{2}}). \tag{23}$$

Our next step will obviously be the asymptotic determination of the general term in the series above, and to this purpose we use the following formula, due to Stieltjes* and constituting a generalization of the well-known asymptotic formula of Laplace:

$$\begin{aligned} P_n(\cos\theta) = \frac{2}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg[& \frac{\cos\left(\frac{2n+1}{2}\theta - \frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}} \\ & + \frac{1^2}{2(2n+3)} \cdot \frac{\cos\left(\frac{2n+3}{2} - \frac{3\pi}{4}\right)}{(2\sin\theta)^{\frac{3}{2}}} + \cdots \\ & + \frac{1^2\cdot 3^2\cdot 5^2\cdots(2p-3)^2}{2\cdot 4\cdots(2p-2)(2n+3)\cdots(2n+2p-1)} \frac{\cos\left(\frac{2n+2p-1}{2}\theta - \frac{2p-1}{4}\pi\right)}{(2\sin\theta)^{\frac{2p-1}{2}}} \\ & + \frac{1^2\cdot 3^2\cdot 5^2\cdots(2p-1)^2}{2\cdot 4\cdots 2p(2n+3)(2n+5)\cdots(2n+2p+1)} \frac{M(p,n,\theta)}{(2\sin\theta)^{\frac{2p+1}{2}}} \Bigg], \end{aligned} \tag{24}$$

where

$$|M(p,n,\theta)| < 2 \quad \text{for} \quad 0 \leqq \theta \leqq \pi.$$

For our purpose, it is sufficiently accurate to make $p = 1$ in (24), and we obtain

$$\begin{aligned} P_n(\cos\theta) + P_{n+1}(\cos\theta) = \frac{2}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg\{ & \frac{2\cos\frac{\theta}{2}\cos\left((n+1)\theta - \frac{\pi}{4}\right)}{(2\sin\theta)^1} \\ & + \frac{1}{(2n+3)(2\sin\theta)^{\frac{3}{2}}}\Bigg[-2\sin\theta\cos\left(\frac{2n+3}{2}\theta - \frac{\pi}{4}\right) \\ & + \frac{M(1,n,\theta)}{2} + \frac{2n+2}{2n+5}\frac{M(1,n+1,\theta)}{2}\Bigg]\Bigg\} \\ = \frac{2}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg\{ & \cos\left((n+1)\theta - \frac{\pi}{4}\right)\sqrt{\cot\frac{\theta}{2}} \\ & + \frac{M'(n,\theta)}{(2n+3)(2\sin\theta)^{\frac{3}{2}}}\Bigg\}, \end{aligned} \tag{25}$$

* T. J. Stieltjes, *Sur les polynômes de Legendre*, Annales de la Faculté des Sciences de Toulouse, ser. I, vol. 4 (1890), pp. $G1 - G17$. The method used above for arriving at the asymptotic expression for $s_n(\cos\theta)$ was suggested by Stieltjes' proof of (24).

where

$$|M'(n,\theta)| < 4 \quad \text{for} \quad 0 \leqq \theta \leqq \pi.$$

By (20) and the various inequalities following (18), we have

$$\cos\left((n+1)\theta_\nu - \frac{\pi}{4}\right) = (-1)^\nu \cos\frac{h_\nu \pi}{(n+1)^{\frac{1}{6}}}$$

$$= (-1)^\nu\left[1 + O\left(\frac{h_\nu^2 \pi^2}{(n+1)^{\frac{2}{3}}}\right)\right] = (-1)^\nu\left[1 + O\left(\frac{1}{n^{\frac{2}{3}}}\right)\right],$$

$$\cot\frac{\theta_\nu}{2} < \frac{1}{\sin\frac{\theta_\nu}{2}} < \frac{1}{\frac{2}{\pi}\cdot\frac{\nu\pi}{2(n+1)}} \leqq (n+1)^{\frac{3}{2}};$$

consequently

$$\cos\left((n+1)\theta_\nu - \frac{\pi}{4}\right)\sqrt{\cot\frac{\theta_\nu}{2}} = (-1)^\nu\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{1}{10}}}\right),$$

and furthermore

$$\frac{M'(n,\theta_\nu)}{(2n+3)(2\sin\theta_\nu)^{\frac{3}{2}}} = O\left(\frac{1}{2n+3}\cdot(n+1)^{\frac{9}{10}}\right) = O\left(\frac{1}{n^{\frac{1}{10}}}\right);$$

finally Stirling's formula,

$$\log\Gamma(x+1) = (x+\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + \sum_{\nu=1}^{p}\frac{(-1)^{\nu-1}B_\nu}{(2\nu-1)\cdot 2\nu}\cdot\frac{1}{x^{2\nu-1}}$$

$$+ \frac{(-1)^p B_{p+1}}{(2p+1)(2p+2)}\frac{\vartheta}{x^{2p+1}}, \qquad 0 < \vartheta < 1,$$

gives, for $p = 1$, and since $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$\frac{2}{\pi}\frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} = \frac{2}{\sqrt{\pi n}}\left[1 - \frac{1}{4n} + O\left(\frac{1}{n^2}\right)\right]. \tag{26}$$

Introducing all this in (25), we obtain

$$(-1)^\nu[P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu)] = \frac{2}{\sqrt{\pi n}}\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right),$$

whence, by (23),

$$(-1)^{\lambda_{\nu_1}-\nu_1}\rho_n = \frac{2}{\sqrt{\pi n}}\sum_{\nu=\nu_1}^{\nu_2}\sqrt{\cot\frac{\theta_\nu}{2}} + \sum_{\nu=\nu_1}^{\nu_2} O\left(\frac{1}{n^{\frac{3}{2}}}\right) + O(n^{\frac{2}{3}}).$$

In the second sum, each term is less than a constant multiple of $n^{-\frac{3}{6}}$, and the number of terms is less than n, so that

$$\sum_{\nu=\nu_1}^{\nu_2} O\left(\frac{1}{n^{\frac{3}{2}}}\right) = O\left(n\cdot\frac{1}{n^{\frac{3}{2}}}\right) = O(n^{\frac{2}{3}}),$$

and consequently

$$(-1)^{\lambda_{\nu_1}-\nu_1}\rho_n = \frac{2}{\sqrt{\pi n}}\sum_{\nu=\nu_1}^{\nu_2}\sqrt{\cot\frac{\theta_\nu}{2}} + O(n^{\frac{2}{5}}),$$

which may be written

$$(-1)^{\lambda_{\nu_1}-\nu_1}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\cdot\frac{n+1}{n}\sum_{\nu=\nu_1}^{\nu_2}\left(\frac{4\nu+3}{n+1}\frac{\pi}{4} - \frac{4\nu-1}{n+1}\frac{\pi}{4}\right)\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{1}{10}}}\right),$$

By (17) and (20), we have

$$\frac{4\nu+3}{n+1}\frac{\pi}{4} > \theta_\nu > \frac{4\nu-1}{n+1}\frac{\pi}{4},$$

and the preceding equation gives, when we increase n indefinitely,*

$$\lim_{n=\infty}(-1)^{\lambda_{\nu_1}-\nu_1}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta.$$

From the equation of definition (8), it follows that ρ_n is positive, and consequently

$$(-1)^{\lambda_{\nu_1}-\nu_1} = 1$$

for all sufficiently large values of n, so that

$$\lim_{n=\infty}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta.$$

Making the substitution $\cot\theta/2 = u^2$, we obtain

$$\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta = \int_0^\infty\frac{4u^2\,du}{1+u^4} = \int_{-\infty}^{+\infty}\frac{2u^2\,du}{1+u^4},$$

and by a classical application of Cauchy's theorem, the latter integral equals $2\pi i$ times the sum of the residues of the integrand in the upper half plane, or

$$2\pi i\left\{\left(\frac{2u^2}{4u^3}\right)_{u=\frac{1+i}{\sqrt{2}}} + \left(\frac{2u^2}{4u^3}\right)_{u=\frac{-1+i}{\sqrt{2}}}\right\} = \pi i\sqrt{2}\left(\frac{1}{1+i} + \frac{1}{-1+i}\right) = \pi\sqrt{2},$$

so that finally,

$$\lim_{n=\infty}\frac{\rho_n}{\sqrt{n}} = 2\sqrt{\frac{2}{\pi}}. \tag{27}$$

§3. Approximation to functions satisfying a generalized Lipschitz condition by finite sums of spherical harmonics.

THEOREM I. *Let $f(\theta, \varphi)$ be a uniform function of the position on the sphere S, satisfying the generalized Lipschitz condition for any two points θ, φ and θ', φ':*

$$|f(\theta', \varphi') - f(\theta, \varphi)| \overline{\overline{<}} \omega(\gamma), \tag{28}$$

* The integrand decreasing monotonously as θ increases, this passage to the limit is seen at once to be legitimate, although the integral obtained is an improper one.

*the function $\omega(\gamma)$ being subject to the conditions**

$$(29)\qquad \begin{gathered} \omega(\gamma) \leqq \omega(\gamma'), \qquad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \quad \text{for} \quad 0 < \gamma \leqq \gamma' < \pi; \\ \lim_{\gamma=0} \omega(\gamma) = 0. \end{gathered}$$

Then there exists, for every positive integer n, a sum $T_n(\theta, \varphi)$, of a finite number of spherical harmonics of order $\overline{\overline{<}}\, n$, approximating $f(\theta, \varphi)$ in such a way that

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| \overline{\overline{<}}\, K\omega(1/n)$$

at every point of the sphere S, K being a constant independent of n.

In order to prove this theorem, we shall use a special type of functions $T_n(\theta, \varphi)$, defined by †

$$(30)\qquad T_n(\theta, \varphi) = \frac{\displaystyle\int_S f(\theta', \varphi') \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\displaystyle\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'},$$

where the integer m is related to n by

$$(31)\qquad 2(m-1) \overline{\overline{<}}\, n < 2m.$$

The function $T_n(\theta, \varphi)$ is actually a sum of a finite number of spherical harmonics of order $\leqq n$; to prove this assertion, we start from the relation

$$(32)\qquad \frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}} = \frac{e^{\frac{m\gamma i}{2}} - e^{-\frac{m\gamma i}{2}}}{e^{\frac{\gamma i}{2}} - e^{-\frac{\gamma i}{2}}} = e^{-(m-1)\frac{\gamma i}{2}} \cdot \frac{1 - e^{m\gamma i}}{1 - e^{\gamma i}} = e^{-(m-1)\frac{\gamma i}{2}} \sum_{\nu=0}^{m-1} e^{\nu\gamma i},$$

whence

$$\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 = e^{-2(m-1)\gamma i} \sum_{\nu_1=0}^{m-1}\sum_{\nu_2=0}^{m-1}\sum_{\nu_3=0}^{m-1}\sum_{\nu_4=0}^{m-1} e^{(\nu_1+\nu_2+\nu_3+\nu_4)\gamma i} = \sum_{\lambda=-2(m-1)}^{+2(m-1)} c_\lambda e^{\lambda\gamma i}.$$

Changing γ and λ into $-\gamma$ and $-\lambda$, we see at once that $c_\lambda = c_{-\lambda}$ and the

* From the last condition, it follows that $f(\theta, \varphi)$ is continuous in every point of S.

† The corresponding finite trigonometric sums for a function $f(x)$ satisfying the ordinary Lipschitz condition $|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$, $\lambda = \text{const.}$, were first introduced by Jackson (see papers II and III, quoted in § 1).

last equation becomes

$$(33)\qquad \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 = c_0 + \sum_{\lambda=1}^{2(m-1)} c_\lambda\,(e^{\lambda\gamma i} + e^{-\lambda\gamma i}) = c_0 + 2\sum_{\lambda=1}^{2(m-1)} c_\lambda \cos\lambda\gamma\,.$$

Expressing $\cos\lambda\gamma$ as a polynomial of degree λ in $\cos\gamma$, and this polynomial being expressed by the Legendre polynomials in $\cos\gamma$ of degree $\leqq \lambda$, we reduce (30) to the form

$$T_n(\theta,\varphi) = \sum_{\nu=0}^{2(m-1)} a_\nu \int_S f(\theta',\varphi')\,P_\nu(\cos\gamma)\,d\sigma',$$

the general term on the right side being obviously a spherical harmonic of order $\nu \overline{\overline{<}}\, 2(m-1) \overline{\overline{<}}\, n$.

Having established this point, we may write (30) in the form

$$f(\theta,\varphi) - T_n(\theta,\varphi) = \frac{\displaystyle\int_S [f(\theta,\varphi) - f(\theta',\varphi')]\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\displaystyle\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'},$$

and by (28) we obtain

$$(34)\qquad |f(\theta,\varphi) - T_n(\theta,\varphi)| \overline{\overline{<}} \frac{\displaystyle\int_S \omega(\gamma)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\displaystyle\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}.$$

In the integrals, we now move the north pole to the point θ, φ; then $\gamma = \theta'$ and $d\sigma' = \sin\gamma\, d\gamma d\varphi$, so that

$$\int_S \omega(\gamma)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma' = \int_0^\pi\int_0^{2\pi} \omega(\gamma)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma\, d\varphi$$

$$= 2\pi\int_0^\pi \omega(\gamma)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma\,,$$

$$\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma' = 2\pi\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma.$$

By the aid of (29), we obtain

$$\int_0^\pi \omega(\gamma)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma = \int_0^{1/n} + \int_{1/n}^\pi \leqq \int_0^{1/n} \omega\left(\frac{1}{n}\right)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$+ \int_{1/n}^\pi \frac{\omega\left(\frac{1}{n}\right)}{\frac{1}{n}}\cdot\gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$< \omega\left(\frac{1}{n}\right)\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma + n\omega\left(\frac{1}{n}\right)\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$< \omega\left(\frac{1}{n}\right)\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma + 2m\omega\left(\frac{1}{n}\right)\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma.$$

nce $n < 2m$, by (31), it follows from (34) that

$$(35)\qquad |f(\theta,\varphi) - T_n(\theta,\varphi)| < \omega\left(\frac{1}{n}\right)\left[1 + \frac{2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma}{\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma}\right].$$

Observing that $\sin x > 2x/\pi$ for $0 < x < \pi/2$ and $\sin x < x$ for $0 < x < \pi$, we find

$$2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma < 2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\frac{\gamma}{\pi}}\right]^4 \gamma\, d\gamma = 2\pi^4 m\int_0^\pi \frac{\sin^4\frac{m\gamma}{2}}{\gamma^2}\, d\gamma$$

$$= \pi^4 m^2\int_0^{\frac{m\pi}{2}} \frac{\sin^4\gamma}{\gamma^2}\, d\gamma < \pi^4 m^2\int_0^\infty \frac{\sin^4\gamma}{\gamma^2}\, d\gamma,$$

$$\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma > \int_0^{\frac{\pi}{2}} \left[\frac{\sin\frac{u\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma > \int_0^{\frac{\pi}{2}} \left[\frac{\sin\frac{m\gamma}{2}}{\frac{\gamma}{2}}\right]^4 \cdot \frac{2}{\pi}\gamma\, d\gamma$$

$$= \frac{32}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^4\frac{m\gamma}{2}}{\gamma^3}\, d\gamma = \frac{8}{\pi} m^2 \int_0^{\frac{m\pi}{4}} \frac{\sin^4\gamma}{\gamma^3}\, d\gamma \geqq \frac{8}{\pi} m^2 \int_0^{\frac{\pi}{4}} \frac{\sin^4\gamma}{\gamma^3}\, d\gamma,$$

and from (34) we obtain

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| < \omega\left(\frac{1}{n}\right)\left\{1 + \frac{\pi^5 \int_0^\infty \frac{\sin^4\gamma}{\gamma^2}\, d\gamma}{8\int_0^{\frac{\pi}{4}} \frac{\sin^4\gamma}{\gamma^3}\, d\gamma}\right\} \quad (n = 1, 2, 3, \cdots),$$

which proves our theorem.

Although not necessary for the application in § 4, it is interesting to derive from (35) a fairly small numerical upper limit for the constant K in theorem I. With the notations

$$I_m = \frac{1}{4m^2}\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma, \qquad I'_m = \frac{1}{2m}\int_0^\pi \gamma \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

equation (35) may be written

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| < \left(1 + \frac{I'_m}{I_m}\right)\omega\left(\frac{1}{n}\right), \tag{36}$$

and making $\gamma = 2u$ and using the inequality $u\cos u < \sin u \left(0 < u < \frac{\pi}{2}\right)$, we have

$$\begin{aligned} I_m &= \frac{1}{m^2}\int_0^{\frac{\pi}{2}} \frac{\sin^4 mu \cos u}{\sin^3 u}\, du, \\ I'_m &= \frac{4}{m}\int_0^{\frac{\pi}{2}} \frac{u \sin^4 mu \cos u}{\sin^3 u}\, du < \frac{4}{m}\int_0^{\frac{\pi}{2}} \frac{\sin^4 mu}{\sin^2 u}\, du. \end{aligned} \tag{37}$$

Considering first the integral I_m, we integrate by parts and obtain

$$m^2 I_m = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^4 mu \cdot d\left(\frac{1}{\sin^2 u}\right) = -\frac{1}{2}\sin^4\frac{m\pi}{2} + 2m\int_0^{\frac{\pi}{2}} \frac{\sin^3 mu \cos mu}{\sin^2 u}\, du$$

$$= \frac{(-1)^m - 1}{4} - 2m\int_0^{\frac{\pi}{2}} \sin^3 mu \cos mu \cdot d\cot u$$

$$= \frac{(-1)^m - 1}{4} + 2m^2\int_0^{\frac{\pi}{2}} (3\sin^2 mu \cos^2 mu - \sin^4 mu)\cot u\, du;$$

as $3 \sin^2 mu \cos^2 mu - \sin^4 mu = \frac{1}{2}(\cos 2\,mu - \cos 4mu)$, this gives

$$I_m = \frac{(-1)^m - 1}{4m^2} + j_m,$$

where

$$j_m = \int_0^{\frac{\pi}{2}} \cot u\,(\cos 2\,mu - \cos 4mu)\,du.$$

Consider the difference $j_m - j_{m-1}$; the identities

$$\begin{aligned}
&\cot u\,[\cos 2mu - \cos 4mu - (\cos 2\,(m-1)\,u - \cos 4\,(m-1)\,u)] \\
&\quad = \cot u\,[-2 \sin u \sin (2m-1)\,u + 2 \sin 2u \sin (4m-2)\,u] \\
&\quad = -2 \cos u \sin (2m-1)\,u + 4 \cos^2 u \sin (4m-2)u \\
&\quad = -2 \cos u \sin (2m-1)\,u + 2\,(1 + \cos 2u) \sin (4m-2)\,u \\
&\quad = -\sin 2mu - \sin (2m-2)\,u + 2 \sin (4m-2)\,u + \sin 4mu \\
&\qquad\qquad + \sin (4m-4)\,u
\end{aligned}$$

give, upon integration between the limits 0 and $\pi/2$,

$$j_m - j_{m-1} = \frac{1}{2m}[(-1)^m - 1] + \frac{1}{2m-2}[(-1)^{m-1} - 1] + \frac{2}{2m-1},$$

where, for $m = 1$, the second term on the right side should be replaced by zero. On account of $j_0 = 0$, we conclude that

$$\begin{aligned}
j_m &= \sum_{\nu=1}^{m}\left[\frac{1}{2\nu}[(-1)^\nu - 1] + \frac{1}{2\nu-2}[(-1)^{\nu-1} - 1] + \frac{2}{2\nu-1}\right] \\
&= \frac{(-1)^m - 1}{2m} + \frac{2}{2m-1} + \sum_{\nu=1}^{m-1}\left[\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu-1}\right],
\end{aligned}$$

the result being also true for $m = 1$, if the sum on the right side is replaced by zero. It is now seen that

$$(38)\quad I_m = \sum_{\nu=1}^{m-1}\left(\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu-1}\right) + \frac{(-1)^m - 1}{2m} + \frac{2}{2m-1} + \frac{(-1)^m - 1}{4m^2},$$

and

$$I_{m+1} - I_m = \frac{(-1)^m - 1}{2m} - \frac{(-1)^m - 1}{4m^2} - \frac{(-1)^m + 1}{2\,(m+1)} + \frac{2}{2m+1} - \frac{(-1)^m + 1}{4(m+1)^2}$$

or, when the cases m odd and m even are separated,

$$I_{m+1} - I_m = \begin{cases} \dfrac{1}{2m^2\,(2m+1)}, & m \text{ odd}, \\[2ex] \dfrac{1}{2(m+1)^2\,(2m+1)}, & m \text{ even}, \end{cases}$$

so that in both cases

$$I_{m+1} > I_m. \tag{39}$$

For $m = 2$, (38) gives the value $2/3$, whence by (39)

$$I_m \overline{\overline{>}} \tfrac{2}{3} = 0.6667 - \quad \text{for} \quad m \overline{\overline{>}} 2. \tag{40}$$

To form an idea of the approximation furnished by (40), we observe that by equations (39) and (38),

$$\begin{aligned} I_m < I_{2m} < \lim_{m=\infty} I_{2m} &= \lim_{m=\infty} \sum_{\nu=1}^{2m-1} \left(\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu - 1} \right) \\ &= \lim_{m=\infty} \left(\sum_{\nu=1}^{2m-1} \frac{2}{2\nu - 1} - \sum_{\nu=1}^{m} \frac{2}{2\nu - 1} \right) = \lim_{m=\infty} \sum_{\nu=m+1}^{2m-1} \frac{2}{2\nu - 1} \\ &= \lim_{m=8} \frac{1}{2m-1} \sum_{\nu=1}^{m-1} \frac{1}{\frac{1}{2} + \frac{\nu}{2m-1}} = \int_{\frac{1}{2}}^{1} \frac{dx}{x} \\ &= \log 2 = 0.6931 +. \end{aligned}$$

Passing to the last integral in (37), we have

$$\frac{4}{m} \int_0^{\frac{\pi}{2}} \frac{\sin^4 mu}{\sin^2 u} du = \frac{2}{m} \int_0^{\frac{\pi}{2}} (1 - \cos 2mu) \left(\frac{\sin mu}{\sin u} \right)^2 du.$$

Writing $\gamma = 2u$ in (32) and squaring, we obtain

$$\left(\frac{\sin mu}{\sin u} \right)^2 = e^{-2(m-1)ui} \sum_{\nu_1=0}^{m-1} \sum_{\nu_2=0}^{m-1} e^{2(\nu_1+\nu_2)ui} = \sum_{\lambda=-(m-1)}^{+(m-1)} c'_\lambda e^{2\lambda ui};$$

changing u and λ into $-u$ and $-\lambda$, we find $c'_\lambda = c'_{-\lambda}$, so that

$$\left(\frac{\sin mu}{\sin u} \right)^2 = c'_0 + 2 \sum_{\lambda=1}^{m-1} c'_\lambda \cos 2\lambda u,$$

whence

$$\begin{aligned} \frac{4}{m} \int_0^{\frac{\pi}{2}} \frac{\sin^4 mu}{\sin^2 u} du &= \frac{2}{m} \int_0^{\frac{\pi}{2}} (1 - \cos 2mu) \left(c'_0 + 2 \sum_{\lambda=1}^{m-1} c'_\lambda \cos 2\lambda u \right) du \\ &= \frac{2}{m} \cdot \frac{\pi}{2} c'_0. \end{aligned}$$

From (32) it is seen immediately that c'_0 is the constant term in the expansion of

$$x^{-(m-1)} \left(\frac{1 - x^m}{1 - x} \right)^2 = x^{-(m-1)} (1 - 2x^m + x^{2m}) \sum_{\nu=0}^{\infty} (\nu + 1) x^\nu,$$

or $c'_0 = m$, so that we obtain from (37)

$$I'_m < \pi. \tag{41}$$

Equations (40) and (41) now give

$$1 + \frac{I'_m}{I_m} < 1 + \frac{3\pi}{2} < 6 \quad \text{for} \quad m \overset{=}{>} 2;$$

that is, on account of (31), for $n \overset{=}{>} 2$, and from (36) we derive the desired result

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| < 6\omega\left(\frac{1}{n}\right) \quad (n = 2, 3, 4, \cdots). \tag{42}$$

Before leaving this subject, we shall evaluate the integral

$$J_m = \frac{1}{m^3}\int_0^{\frac{\pi}{2}} \left(\frac{\sin mu}{\sin u}\right)^4 du \tag{43}$$

occurring in Jackson's investigation of the approximation to a function $f(x)$ of period 2π and satisfying a Lipschitz condition

$$f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$$

by finite trigonometric sums. By an asymptotic process, Jackson* shows that

$$J_m > J_{m+1}, \qquad \lim_{m=\infty} J_m = \frac{\pi}{3}. \tag{44}$$

Writing $\gamma = 2u$, we obtain at once from (33)

$$J_m = \frac{c_0}{m^3} \cdot \frac{\pi}{2}.$$

By (32) and (33), c_0 is obviously the constant term in the expansion of

$$x^{-2(m-1)}\left(\frac{1 - x^m}{1 - x}\right)^4 = x^{-2(m-1)}(1 - 4x^m + 6x^{2m} - 4x^{3m} + x^{4m}) \times \sum_{\nu=0}^{\infty} \frac{(\nu+1)(\nu+2)(\nu+3)}{1 \cdot 2 \cdot 3} x^\nu,$$

so that, the terms in the infinite series with $\nu = 2(m-1)$ and $\nu = m - 2$ being the only ones contributing to c_0,

$$c_0 = \frac{(2m-1) \cdot 2m(2m+1)}{1 \cdot 2 \cdot 3} - 4 \cdot \frac{(m-1) \cdot m(m+1)}{1 \cdot 2 \cdot 3} = \frac{m}{3}(2m^2 + 1),$$

and consequently

$$J_m = \frac{\pi}{3}\left(1 + \frac{1}{2m^2}\right). \tag{45}$$

From this expression, the two properties (44) follow at once.

* In the paper previously quoted as III, pp. 503–507.

§ 4. The degree of convergence of Laplace's series for functions satisfying a generalized Lipschitz condition.

THEOREM II. *When $f(\theta, \varphi)$ satisfies the generalized Lipschitz condition*

$$|f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma) \tag{28}$$

for any two points θ, φ and θ', φ' on the sphere S, the function $\omega(\gamma)$ being subject to the conditions

$$\omega(\gamma) \leqq \omega(\gamma'), \quad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \text{ for } 0 < \gamma \leqq \gamma' < \pi, \lim_{\gamma=0} \omega(\gamma) = 0, \tag{29}$$

then $f(\theta, \varphi)$ is approximated by the first $n+1$ terms of its Laplace series in such a way that the remainder

$$r_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - s_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - \frac{1}{4\pi}\int_S f(\theta', \varphi')\, s_n(\cos\gamma)\, d\sigma'$$

satisfies the inequality

$$|r_n\{f(\theta, \varphi)\}| \leqq K'\omega\left(\frac{1}{n}\right)\sqrt{n} \qquad (n = 1, 2, 3, \cdots),$$

K' being a constant independent of n.

The principle of the following proof is due to Lebesgue, in the case of Fourier's series.* From the identity

$$f(\theta, \varphi) = T_n(\theta, \varphi) + [f(\theta, \varphi) - T_n(\theta, \varphi)]$$

and the definition of the remainder r_n, it follows that

$$r_n\{f(\theta, \varphi)\} = r_n\{T_n(\theta, \varphi)\} + r_n\{f(\theta, \varphi) - T_n(\theta, \varphi)\};$$

the Laplace series of $T_n(\theta, \varphi)$ being obviously this function itself, it is seen that

$$r_n\{T_n(\theta, \varphi)\} = 0.$$

The definition of r_n then gives

$$|r_n\{f(\theta, \varphi)| = |r_n\{f(\theta, \varphi) - T_n(\theta, \varphi)\}| \leqq |f(\theta, \varphi) - T_n(\theta, \varphi)|$$
$$+ \frac{1}{4\pi}\int_S |f(\theta', \varphi') - T_n(\theta', \varphi')|\,|s_n(\cos\gamma)|\, d\sigma',$$

and if we apply theorem I,

$$|r_n\{f(\theta, \varphi)\}| \leqq K\omega\left(\frac{1}{n}\right) + K\omega\left(\frac{1}{n}\right)\cdot\frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\, d\sigma'$$
$$= K\omega\left(\frac{1}{n}\right)(1 + \rho_n),$$

* H. Lebesgue, *Sur la représentation trigonométrique des fonctions, etc.*, Bulletin de la Société Mathématique de France, vol. 38 (1910), pp. 184–210; pp. 196–197; *Sur les intégrales singulières*, Annales de la Faculté des Sciences de Toulouse, ser. 3, vol. 1 (1910), pp. 25–117; pp. 116–117. Compare also Jackson, I, II, III.

by the definition of ρ_n at the beginning of § 2. On account of (27), there exists a constant c independent of n and such that

$$\rho_n \leqq c\sqrt{n} \qquad (n = 1, 2, 3, \cdots).$$

The preceding inequality now shows that

$$|r_n\{f(\theta, \varphi)\}| \leqq K(1+c\sqrt{n})\omega\left(\frac{1}{n}\right) \leqq K(1+c)\,\omega\left(\frac{1}{n}\right)\sqrt{n} = K'\,\omega\left(\frac{1}{n}\right)\sqrt{n},$$

which proves our theorem.

It follows, in particular, from theorem II that *whenever*

$$\lim_{n=\infty} \omega\left(\frac{1}{n}\right)\sqrt{n} = 0,$$

the Laplace series corresponding to $f(\theta, \varphi)$ *is convergent and equal to this function at every point of the sphere* S.

In the particular case of the Legendre series corresponding to a function $f(x) = f(\cos\theta)$ satisfying the condition (5), which is analogous to (28), Jackson* has shown that for any positive ϵ there exists a constant $K'(\epsilon)$ independent of n (but depending, of course, on ϵ) such that

$$|r_n\{f(x)\}| < K'(\epsilon)\,\omega\left(\frac{1}{n}\right)\log n \qquad (n = 2, 3, 4, \cdots),$$

uniformly in the interval

$$-1+\epsilon \leqq x \leqq 1-\epsilon.$$

This result cannot be extended to the limiting case $\epsilon = 0$; on the contrary, we have the following theorem, which may be considered as the reciprocal of theorem II:

THEOREM III. *Let* $\omega(\gamma)$ *be a function subject to the conditions*

$$(29) \qquad \omega(\gamma) \leqq \omega(\gamma'), \quad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \quad \textit{for } 0 < \gamma \leqq \gamma' < \pi; \lim_{\gamma=0} \omega(\gamma) = 0,$$

and $\Omega(\gamma)$ *any other function such that*

$$(46) \qquad \lim_{\gamma=0} \frac{\Omega(\gamma)}{\omega(\gamma)} = 0;$$

there exists a function $f(\theta, \varphi)$ *satisfying the generalized Lipschitz condition*

$$(28) \qquad |f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma)$$

for any two points θ, φ *and* θ', φ' *on the sphere* S, *and such that, at a given point*

* Paper referred to as I, theorem (1).

θ_0, φ_0, the remainder in its Laplace series satisfies the inequality

$$| r_n \{ f(\theta_0, \varphi_0) \} | \geqq K'' \Omega \left(\frac{1}{n}\right) \sqrt{n}$$

for an infinity of values of n, the constant K'' being independent of n.

To prove this theorem, we shall construct a special function $f(\theta, \varphi)$ having the required properties,* choosing the given point at $\theta_0 = 0$, $\varphi_0 = 0$, and making $f(\theta, \varphi) = F(\theta)$ independent of φ, so that our Laplace series will reduce to a Legendre series. Our function $F(\theta)$ will be built up from elements of the form $F_n(\theta)$, this latter function being defined by

$$(47) \quad F_n(\theta) = \sin\left((n+1)\theta - \frac{\pi}{4}\right) \quad \text{for} \quad \frac{1}{n+1}\frac{\pi}{4} \leqq \theta \leqq \frac{4\mu+1}{n+1}\frac{\pi}{4},$$

$$= 0 \text{ outside of this interval,}$$

μ and n being positive integers (or zero) to be determined later. $F_n(\theta)$ is continuous, as it vanishes at the end points of the interval above, which we may denote by (μ, n).

By (47), we have $F_n(\theta) = 0$ for $\theta = 0$, so that, developing $F_n(\theta)$ in a Laplace series, we find

$$r_n \{ F_n(0) \} = - s_n \{ F_n(0) \} = -\frac{1}{4\pi}\int_S F_n(\theta') s_n(\cos\gamma)\, d\sigma',$$

or, introducing $d\sigma' = \sin\theta' \, d\theta' \, d\varphi'$ and using the well-known formula

$$\int_0^{2\pi} P_n(\cos\gamma)\, d\varphi' = 2\pi P_n(\cos\theta) P_n(\cos\theta'),$$

making $\theta = 0$, so that $P_n(\cos\theta) = 1$, and finally writing θ instead of θ',

$$r_n \{ F_n(0) \} = -\frac{1}{2}\int_0^{\pi} F_n(\theta) s_n(\cos\theta) \sin\theta \, d\theta.$$

Since $F_n(\theta)$ is identically equal to zero in part of the interval of integration, we may write

$$r_n \{ F_n(0) \} = -\frac{1}{2}\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} F_n(\theta) s_n(\cos\theta) \sin\theta \, d\theta;$$

or introducing the value $F_n(\theta) = \sin\left((n+1)\theta - \frac{\pi}{4}\right)$ in the interval in question, and using (15),

* In principle, the method of proof is due to Lebesgue, who applied it to Fourier's series; see papers by Lebesgue and Jackson previously quoted.

$$r_n\{F_n(0)\} = -\frac{2}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)}\times$$

(48)
$$\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\left[\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}+\frac{24\eta(n,\theta)}{2n+3}\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\right]d\theta.$$

Now assume $\mu \leqq \frac{n}{2}$; we then have, in the interval of integration in (48)

(49) $$\cos\frac{\theta}{2} > \frac{1}{\sqrt{2}}, \qquad \theta > \sin\theta > \frac{2}{\pi}\theta, \qquad 2\sin\frac{\theta}{2} > \frac{2}{\pi}\theta,$$

so that

$$\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}\,d\theta > \frac{1}{2}\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\theta}}\,d\theta$$

$$= \frac{1}{2}\sum_{\nu=1}^{\mu}\int_{\frac{4\nu-3}{n+1}\frac{\pi}{4}}^{\frac{4\nu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\theta}}\,d\theta$$

(50) $$> \frac{1}{2}\sum_{\nu=1}^{\mu}\int_{\frac{4\nu-3}{n+1}\frac{\pi}{4}}^{\frac{4\nu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\frac{4\nu+1}{n+1}\frac{\pi}{4}}}\,d\theta$$

$$= \frac{\sqrt{\pi}}{8}\cdot\frac{1}{\sqrt{n+1}}\sum_{\nu=1}^{\mu}\frac{1}{\sqrt{4\nu+1}}$$

$$> \frac{\sqrt{\pi}}{8}\cdot\frac{1}{\sqrt{n+1}}\int_2^{\mu+1}\frac{du}{\sqrt{4u+1}}$$

$$= \frac{\sqrt{\pi}}{16}\cdot\frac{1}{\sqrt{n+1}}\left(\sqrt{4(\mu+1)+1}-3\right).$$

For the second part of the integral in (48) we obtain, using (49) and observing that $|\eta(n,\theta)| < 1$,

$$\left|\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{24\eta(n,\theta)}{2n+3}\,\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\,d\theta\right| < \int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{24}{2n+2}\cdot\frac{1}{\frac{2}{\pi}\theta\cdot\left(\frac{4}{\pi}\theta\right)^{\frac{1}{2}}}\,d\theta$$

51)
$$= \frac{3\pi^{\frac{3}{2}}}{n+1}\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{d\theta}{\theta^{\frac{3}{2}}}$$

$$= \frac{12\pi}{\sqrt{n+1}}\left(1-\frac{1}{\sqrt{4\mu+1}}\right);$$

and on combining (50) with (51), it is seen that

$$\left|\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \left[\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}+\frac{24\eta(n,\theta)}{2n+3}\,\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\right]d\theta\right|$$

$$> \frac{\sqrt{\pi}}{16}\cdot\frac{1}{\sqrt{n+1}}\left(\sqrt{4(\mu+1)+1}-3\right)-\frac{12\pi}{\sqrt{n+1}}\left(1-\frac{1}{\sqrt{4\mu+1}}\right)$$

$$> c_1\sqrt{\frac{\mu}{n+1}}$$

for n and μ sufficiently large, c_1, as well as $c_2, c_3, \cdots$ which will be introduced below, signifying a certain positive constant, independent of n and μ. By (26) we have, for n sufficiently large,

$$\frac{2}{\pi}\,\frac{\Gamma(\frac{1}{2})\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} > c_2\sqrt{n+1},$$

so that we finally obtain from (48)

(52) $\quad |r_n\{F_n(0)\}| > c_3\sqrt{4\mu+5} \quad$ for $\quad n \geqq n', \quad \mu \geqq \mu', \quad 2\mu \leqq n.$

On account of (27) there exists a c_4 such that

(53) $$\rho_n < c_4\sqrt{n+1} \qquad (n=1,2,\cdots).$$

We now select an infinite sequence of positive integers $n_1, n_2, n_3, \cdots$ subject to the conditions

$$n_1 \geqq n',$$

(54)
$$\omega\left(\frac{\pi}{n_i+1}\right) < \frac{c_3}{4c_4}\cdot\frac{1}{\sqrt{n_{i-2}+1}}\cdot\omega\left(\frac{\pi}{n_{i-1}+1}\right),$$

$$\Omega\left(\frac{1}{n_i}\right) < \frac{1}{\sqrt{n_{i-1}+1}}\,\omega\left(\frac{1}{n}\right) \qquad (i=3,4,5,\cdots).$$

The determination of such a sequence is possible, because after determining $n_1, n_2, \cdots, n_{i-1}$, we may satisfy the last two inequalities (54) by making n_i large enough, on account of $\lim_{\gamma=0} \omega(\gamma) = 0$ and $\lim_{\gamma=0} \Omega(\gamma)/\omega(\gamma) = 0$.

To each n_i we now associate a μ_i by the condition

$$\frac{4\mu_i + 1}{n_i + 1} < \frac{1}{n_{i-1} + 1} \leqq \frac{4\mu_i + 5}{n_i + 1}; \tag{55}$$

then any two intervals (μ_i, n_i) and (μ_j, n_j), in which, by (47), $F_{n_i}(\theta)$ and $F_{n_j}(\theta)$ respectively are not identically zero, have no points in common. Finally, we make v_i equal to zero or unity, according as

$$\left| r_{n_i} \left\{ \sum_{j=1}^{i-1} \frac{1}{2\pi} v_j \, \omega\left(\frac{\pi}{n_j + 1}\right) F_{n_j}(0) \right\} \right|$$

exceeds

$$\frac{c_3}{4\pi} \sqrt{\frac{n_i + 1}{n_{i-1} + 1}} \cdot \omega\left(\frac{\pi}{n_i + 1}\right)$$

or not. Then the expression

$$F(\theta) = \sum_{i=1}^{\infty} \frac{1}{2\pi} v_i \, \omega\left(\frac{\pi}{n_i + 1}\right) F_{n_i}(\theta) \tag{56}$$

has the properties required by theorem III.

Since for any given value of θ not more than one of the terms in the series (56) is different from zero (the intervals in which any two different functions $F_{n_i}(\theta)$ are not identically zero, having no point in common), the series is convergent. Furthermore, $F(\theta)$ satisfies the condition (28). To prove this, we first assume that θ and θ' belong to the same interval (μ_i, n_i); in this interval we have

$$F(\theta) = \frac{v_i}{2\pi} \omega\left(\frac{\pi}{n_i + 1}\right) \sin\left((n_i + 1)\theta - \frac{\pi}{4}\right).$$

We now determine an integer λ by the condition

$$\theta + \frac{2\lambda\pi}{n_i + 1} \leqq \theta' < \theta + \frac{2\lambda + 2}{n_i + 1}\pi$$

and denote by θ'' that one of the two quantities

$$\theta + \frac{2\lambda\pi}{n_i + 1} \qquad \text{and} \qquad \theta + \frac{2\lambda + 2}{n_i + 1}\pi$$

which is nearer to θ'; then we have

$$\left| \theta'' - \theta' \right| \leqq \frac{\pi}{n_i + 1}, \tag{57}$$

$$F(\theta') - F(\theta) = F(\theta') - F(\theta'')$$
$$= \frac{v_i}{2\pi}\omega\left(\frac{\pi}{n_i+1}\right)\cdot(n_i+1)(\theta'-\theta'')\cos\left((n_i+1)\xi - \frac{\pi}{4}\right),$$

ξ lying between θ' and θ'', and consequently

$$|F(\theta') - F(\theta)| < \frac{1}{2}\cdot\frac{\omega\left(\frac{\pi}{n_i+1}\right)}{\frac{\pi}{n_i+1}}|\theta''-\theta'|$$
$$= \frac{1}{2}\cdot\frac{\omega\left(\frac{\pi}{n_i+1}\right)}{\frac{\pi}{n_i+1}}\cdot\frac{|\theta''-\theta'|}{\omega(|\theta''-\theta'|)}\cdot\omega(|\theta''-\theta'|);$$

or, on account of (57) and the fact that $\omega(\gamma)/\gamma$ does not increase with γ,

$$(58)\qquad |F(\theta') - F(\theta)| < \tfrac{1}{2}\omega(|\theta''-\theta'|) \leqq \tfrac{1}{2}\omega(|\theta'-\theta|).$$

Next suppose that θ belongs to the interval (μ_i, n_i) and θ' to the interval (μ_j, n_j), and let θ_i be the end point of (μ_i, n_i) nearest to θ', and θ_j the end point of (μ_j, n_j) nearest to θ_i; then, since $F(\theta_i) = F(\theta_j) = 0$, we have

$$|F(\theta') - F(\theta)| = |F(\theta') - F(\theta_j) + F(\theta_j) - F(\theta_i) + F(\theta_i) - F(\theta)|$$
$$\leqq |F(\theta') - F(\theta_j)| + |F(\theta_i) - F(\theta)|.$$

Applying (58) and considering that $\omega(\gamma)$ does not decrease as γ increases, we find

$$|F(\theta') - F(\theta_j)| < \tfrac{1}{2}\omega(|\theta'-\theta_j|) \leqq \tfrac{1}{2}\omega(|\theta'-\theta|),$$
$$|F(\theta_i) - F(\theta)| < \tfrac{1}{2}\omega(|\theta_i-\theta|) \leqq \tfrac{1}{2}\omega(|\theta'-\theta|),$$

so that finally, for any values of θ and θ' in the interval $(0, \pi)$

$$|F(\theta') - F(\theta)| < \omega(|\theta'-\theta|).$$

Now $|\theta'-\theta| \leqq \gamma$, since $|\theta'-\theta|$ is one of the sides in the right-angled spherical triangle of which γ is the hypothenuse, and consequently

$$|F(\theta') - F(\theta)| < \omega(\gamma).$$

We now complete the proof by showing that

$$|r_{n_i}\{F(0)\}| \geqq K''\Omega\left(\frac{1}{n_i}\right)\sqrt{n_i} \qquad (i = 1, 2, 3, \cdots).$$

On account of (52) and (55), we have

$$\left|r_{n_i}\left\{\frac{1}{2\pi}\omega\left(\frac{\pi}{n_i+1}\right)F_{n_i}(0)\right\}\right| > \frac{c_3}{2\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right),$$

so that in case $v_i = 1$ we have, in view of the definition of v_i,

$$\left| r_{n_i}\left\{ \sum_{j=1}^{i} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(0) \right\}\right| \geqq \left| r_{n_i}\left\{ \frac{1}{2\pi}\, \omega\left(\frac{\pi}{n_i+1}\right) F_{n_i}(0) \right\}\right|$$

$$- \left| r_{n_i}\left\{ \sum_{j=1}^{i-1} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(0) \right\}\right| > \frac{c_3}{2\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right)$$

$$- \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) = \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_{i+1}+1}\right).$$

In case $v_i = 0$, the definition of v_i gives

$$\left| r_{n_i}\left\{ \sum_{j=1}^{i} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(0) \right\}\right| = \left| r_{n_i}\left\{ \sum_{j=1}^{i-1} \frac{1}{2\pi} v_j \omega\left(\frac{\pi}{n_j+1}\right) F_n(0) \right\}\right|$$

$$> \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right),$$

so that we always have

$$\left| r_{n_i}\left\{ \sum_{j=1}^{i} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(0) \right\}\right| > \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right). \tag{59}$$

From (56) and (59) it follows that

$$\begin{aligned} \left| r_{n_i}\left\{ F(\theta) \right\}\right| &\geqq \left| r_{n_i}\left\{ \sum_{j=1}^{i} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(\theta) \right\}\right| \\ &\quad - \left| r_{n_i}\left\{ \sum_{j=i+1}^{\infty} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(\theta) \right\}\right|, \end{aligned} \tag{60}$$

and since

$$\left| \sum_{j=i+1}^{\infty} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(\theta) \right| \leqq \frac{1}{2\pi}\,\omega\left(\frac{\pi}{n_{i+1}+1}\right)$$

for all values of θ, it follows from the definition of ρ_n and the inequality (53) that

$$\begin{aligned} &\left| r_{n_i}\left\{ \sum_{j=i+1}^{\infty} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(\theta) \right\}\right| \\ &\leqq \frac{1}{2\pi}\,\omega\left(\frac{\pi}{n_{i+1}+1}\right)\cdot \frac{1}{4\pi}\int_S |\, s_{n_i}(\cos\gamma)\,|\, d\sigma' = \frac{1}{2\pi}\,\omega\left(\frac{\pi}{n_{i+1}+1}\right)\cdot \rho_{n_i} \\ &< \frac{c_4}{2\pi}\,\omega\left(\frac{\pi}{n_{i+1}+1}\right)\sqrt{n_i+1}. \end{aligned} \tag{61}$$

Making $\theta = 0$, we conclude from (60), (59) and (61) that

$$|\, r_{n_i}\{ F(0) \}\,| > \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) - \frac{c_4}{2\pi}\,\omega\left(\frac{\pi}{n_{i+1}+1}\right)\sqrt{n_i+1}.$$

The second inequality in (54) shows that the right-hand expression is greater than

$$\frac{c_3}{8\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) \geqq \frac{c_3}{8\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{1}{n_i}\right),$$

since

$$\frac{\pi}{n_i+1} \geqq \frac{1}{n_i},$$

and from the last inequality in (54) we finally obtain

$$\left| r_{n_i}\left\{ F(0) \right\} \right| > \frac{c_3}{8\pi}\,\Omega\left(\frac{1}{n_i}\right)\sqrt{n_i+1} \qquad (i = 1, 2, 3, \cdots)$$

which completes the proof of our theorem.

In the case of the Legendre series, Jackson has shown (l. c., I, theorem III) that there exists a K'' independent of n and a function $f(x) = f(\cos\theta)$ satisfying (5) such that

$$|\, r_n \{ f(0) \} \,| > K''\,\omega\left(\frac{1}{n}\right)\log n$$

for an infinity of values of n. Apart from the introduction of $\sqrt{n}$ instead of $\log n$ necessitated by the asymptotic expression for ρ_n, our theorem III differs from Jackson's result in two respects: first, by the introduction of the function $\Omega(\gamma)$ converging toward zero with γ more strongly than $\omega(\gamma)$, and second, by the fact that while the two conditions

$$|f(\cos\theta') - f(\cos\theta)| \overset{=}{<} \omega(|\cos\theta' - \cos\theta|) \qquad \text{(Jackson)},$$

$$|f(\cos\theta') - f(\cos\theta)| \overset{=}{<} \omega(|\theta' - \theta|) \qquad \text{(present paper)}$$

are essentially equivalent in the interior of $0 < \theta < \pi$, this is no longer the case in the vicinity of $\theta = 0$, since

$$\lim_{\theta,\,\theta'=0} \frac{\cos\theta' - \cos\theta}{\theta' - \theta} = 0.$$

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LINEAR ASSOCIATIVE ALGEBRAS AND ABELIAN EQUATIONS*

BY

L. E. DICKSON

1. In the recent development of the theory of linear associative algebras the coördinates of whose numbers range over the field of ordinary complex numbers, an important rôle is played by the so-called simple algebras, the only ones being the matric algebras with r^2 units. For the more general case of algebras the coördinates of whose numbers range over any field (Körper) K of finite or infinite order, a corresponding importance is to be attached to simple algebras; an algebra is simple † if and only if it is the direct product of a matric algebra and a primitive algebra (one in which every element has an inverse). The problem of primitive algebras is the chief outstanding problem in the theory of algebras over a field K. Aside from real quaternions the only known primitive linear associative algebras are fields. We proceed to exhibit a primitive algebra L with r^2 units over K.

Let $\kappa(x) = 0$ be a uniserial abelian equation of degree r with respect to the field K, i. e., let the equation have as its coefficients numbers in K, be irreducible in K, and have as its roots

$$(1) \quad i, \ \theta(i), \ \theta^2(i) \equiv \theta[\theta(i)], \ \cdots, \ \theta^{s+1}(i) \equiv \theta^s[\theta(i)], \ \cdots, \ \theta^{r-1}(i),$$

where $\theta(x)$ is a polynomial with coefficients in K, while

$$(2) \qquad \theta^r(i) = i.$$

Then the algebra in question is the linear associative algebra L over K (i. e., the coördinates of its numbers are arbitrary elements of K) with the r^2 units

$$(3) \qquad i^s j^k \qquad (s, k = 0, 1, \cdots, r-1),$$

where

$$(4) \qquad ji = \theta(i) j,$$

$$(5) \qquad j^r = g \qquad (g \text{ in } K).$$

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* Presented to the Society, April 14, 1906. Cf. the abstract in the Bulletin of the American Mathematical Society, vol. 12 (1905–6), p. 442. But §§ 9–22 were added March, 1913.

† J. H. M. Wedderburn, Proceedings of the London Mathematical Society, ser. 2, vol. 6 (1907), p. 99.

Using the associative law, we readily deduce (7) and hence

$$(6) \qquad l(i)\,j^s \cdot f(i)\,j^k = l(i)\,f[\theta^s(i)] \begin{cases} \cdot\, j^{s+k} & (\text{if } s+k<r) \\ \cdot\, gj^{s+k-r} & (\text{if } s+k \geqq r), \end{cases}$$

where $l(i)$ and $f(i)$ are any polynomials in i of degree $< r$ with coefficients in K. Conversely, it is readily verified that the associative law holds when multiplication is defined by the preceding relation and the supplementary law that the product of two polynomials in i is found as in ordinary algebra with a subsequent reduction of the degree to $r-1$ by use of the equation $\kappa(i) = 0$ of degree r. We may of course write i_s for i^s and j_k for j^k in our formulæ. Hence to any uniserial abelian equation* of degree r with respect to K there corresponds a linear associative algebra L over K with r^2 units.

For $r = 2$, L is a generalization of quaternions discussed in § 9; we may take $i^2 = c$; then† L is primitive if and only if g is not the norm $x^2 - cy^2$ of any number $x + yi$ in the field $K(i)$.

As stated in the abstract of 1906 (see the first reference above), an algebra L with r^2 units is primitive when g and the coefficients of $\kappa(x)$ are suitably chosen numbers of the field K. The argument in my manuscript of that date is as follows: Take $r = 3$ and denote the general number of the algebra by $M = R + Sj + Tj^2$, where R, S, T are numbers of $K(i)$. For $M \neq 0$, the equation $XM = M'$ uniquely determines a number X of the algebra if and only if

$$\Delta \equiv \begin{vmatrix} R & gT_1 & gS_2 \\ S & R_1 & gT_2 \\ T & S_1 & R_2 \end{vmatrix}$$

vanishes only when $R = S = T = 0$, where $R_1 = R(\theta)$, $R_2 = R(\theta^2)$,

$$S_1 = S(\theta),\ S_2 = S[\theta^2(i)], \qquad T_1 = T(\theta),\ T_2 = T[\theta^2(i)].$$

If $T = 0$, $S \neq 0$, then $\Delta = 0$ implies that $g = n(-R/S)$, where the norm SS_1S_2 of S is denoted by $n(S)$. If $T \neq 0$, T has an inverse in $K(i)$, so that it suffices to consider numbers M with $T = 1$. Then

$$\Delta = g^2 + g(SS_1S_2 - RS_1 - R_1S_2 - R_2S) + RR_1R_2.$$

* Note the special case of a binomial equation $x^r - h = 0$. If a primitive rth root ϵ of unity occurs in the field K, and h is not the rth power of an element of K, and if r is a prime, $x^r - h$ is known to be irreducible in K. The algebra is then defined by

$$i^r = h, \quad j^r = g, \quad ji = \epsilon ij.$$

If we set $h = g = 1$, we lose the irreducibility property, but obtain the algebra ($i = e_2$, $j = e_1$) considered by Wedderburn, Proceedings of the Royal Society of Edinburgh, vol. 26 (1906), pt. 1, p. 48.

† Dickson, these Transactions, vol. 13 (1912), p. 66.

If $S = 0$, then $\Delta = 0$ implies that $g = 0$ or $n(-g/R)$, according as $R = 0$, or $R \neq 0$. Henceforth, let $S \neq 0$. To simplify Δ, set

$$R = SS_2 Q, \qquad g = G n(S).$$

Then Δ has the factor $n^2(S)$ and $\Delta = 0$ implies that

$$G^2 + G(1 - Q - Q_1 - Q_2) + Q Q_1 Q_2 = 0.$$

The product of this by $G - 1$ may be written in the form

$$G n(Q - 1) = n(Q - G).$$

Thus, if $Q \neq 1$, G (and hence g) is the norm of a number of $K(i)$. If $Q = 1$, then $(G - 1)^2 = 0$. Hence *if g is not the norm of any number of $K(i)$, the algebra L is primitive.*

For a general value of r the last theorem was found in November, 1913, by Professor Wedderburn; his brief and elegant proof will appear in an early number of these Transactions.

It will be proved that any linear associative algebra having the properties specified in § 2 contains a subalgebra of type L. It is later proved that there exist such algebras, not identical with their subalgebras L. Some remarkable algebras are obtained in this way.

2. Let A be any linear associative algebra the coördinates of whose numbers range over any given field F and for which the following three properties hold:

(*a*) There exists in A a number i satisfying an equation $\phi(x) = 0$ of degree n with coefficients in F and irreducible in F.

(*b*) Any number of A which is commutative with i is in the field $F(i)$.

(*c*) There exists in A a number j, not in $F(i)$, such that $ji = \theta j + \sigma$, where θ and σ are in $F(i)$.

Note that the field $F(i)$ is composed of the totality of polynomials in i with coefficients in F. Indeed, in view of the associative law, any two powers of i are commutative; hence any two polynomials in i are commutative. That the quotient of two polynomials in i can be expressed as a polynomial in i is proved as in the theory of algebraic numbers.*

Conditions (*a*)–(*c*) hold when A is the system of quaternions with coördinates in the field of real numbers. Conditions (*a*) and (*b*) are satisfied for any linear associative algebra D over F in which each right-hand and left-hand division is always possible and unique. In fact, any number i of D satisfies an equation with coefficients in F and irreducible in F. Choose i so that the degree n of the equation is a maximum. If a number v of D is

* Cf. Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, 1905, p. 15.

commutative with i, it is in the field $F(i)$. For if not, a standard theorem on algebraic numbers shows that the field $F(i, v)$ would contain a number satisfying an equation of degree exceeding n and irreducible in F.

For a linear associative algebra over F satisfying condition (a), the number of units is a multiple of n; they may be taken to be

$$i^s,\ i^s j,\ i^s j_1,\ \cdots \qquad (s = 0, 1, \cdots, n-1).$$

In case the number of units is $2n$, ji is necessarily a linear combination of the i^s, $i^s j$ and hence is of the form $\theta j + \sigma$ in (c).

These remarks serve to indicate the origin of conditions (a)–(c). All three conditions are satisfied by any algebra of type L (§ 8).

3. From the relation in (c), we find by induction that

$$ji^s = \theta^s j + (\theta^{s-1} + \theta^{s-2} i + \cdots + \theta i^{s-2} + i^{s-1})\sigma,$$

for $s > 1$. We are here using ordinary powers of θ and not symbolical powers as in (1). Multiply the members of the equation by c_s and sum, where

$$\phi(x) = \sum_{s=0}^{n} c_s x^s$$

is the function in (a) with the root i. Thus

$$0 = \phi(\theta) j + c_1 \sigma + \sum_{s=2}^{n} (\theta^{s-1} + \theta^{s-2} i + \cdots + \theta i^{s-2} + i^{s-1}) c_s \sigma.$$

But j is not in the field $F(i)$. Hence $\phi(\theta) = 0$. Suppose first that $\theta = i$. If $\sigma = 0$, then $ji = ij$, j not in $F(i)$, contrary to (b). Hence $\sigma \neq 0$. The preceding relation thus gives

$$0 = c_1 + \sum_{s=2}^{n} s i^{s-1} c_s = \phi'(i).$$

Thus i would be a double root of $\phi(x) = 0$, contrary to its irreducibility. Hence $\theta \neq i$. Thus $Ji = \theta J$ if

$$J = j + \frac{\sigma}{(\theta - i)}.$$

Hence after a suitable choice of j we have relation (4) and

$$ji^s = [\theta(i)]^s j \qquad (s = 1, \cdots, n).$$

If $f(x)$ is any polynomial with coefficients in F,

$$jf(i) = f(\theta) j, \quad \cdots, \quad j^s f(i) = f[\theta^s(i)] j^s, \tag{7}$$

the last relation following by induction and the use of the notation defined in (1). In particular, if $f(x)$ is the function $\phi(x)$ which vanishes for $x = i$,

we see from the first relation that $\theta(i)$ is a root of $\phi(x) = 0$. Hence $\phi[\theta(x)] = 0$ has the root i in common with the irreducible equation $\phi(x)=0$ and therefore the root $\theta(i)$. Thus $\theta^2(i)$ is a root of $\phi(x) = 0$. We see in this way that each of the numbers (1) is a root of $\phi(x) = 0$. Hence there must be an equality

$$\theta^{k+r}(i) = \theta^k(i) \qquad (r > 0).$$

If $k = 0$, (2) holds. If $k > 0$, $x = \theta(i)$ is a root of

$$\theta^{k+r-1}(x) = \theta^{k-1}(x). \tag{8}$$

Since this equation with coefficients in the field F has a root in common with the equation $\phi(x) = 0$, irreducible in F, it has also the root $x = i$. Then (8) for $x = i$ gives a relation like the initial equality but with lower symbolic powers. Proceeding in this manner, we ultimately obtain relation (2).

Let r be the least positive integer for which (2) holds. If $r = 1$, (4) would be in contradiction with (b). The argument just used shows that the numbers (1) are all distinct.

Theorem 1. *The algebra A contains a number j not in the field $F(i)$ and satisfying relation* (4). *Among the roots of $\phi(x) = 0$ occur the r $(r > 1)$ distinct numbers* (1); *they form a closed cycle since relation* (2) *holds.*

4. In the last equation (7) take $s = r, f(x) = x$. By (2),

$$j^r i = \theta^r(i) j^r = ij^r.$$

Hence, by (b), j^r is in the field $F(i)$:

$$j^r = g(i), \tag{9}$$

where $g(i)$ is a polynomial with coefficients in F. Henceforth we shall assume that $g(i) \neq 0$.

For $\rho = 1$ and $\rho = 2$ we know that the numbers

$$i^s, \quad i^s j, \quad i^s j^2, \quad \cdots, \quad i^s j^{\rho-1} \qquad (s = 0, 1, \cdots, n-1;\ 0 < \rho < r) \tag{10}$$

are linearly independent with respect to the field F. Assume that the ρn numbers (10) are linearly independent. Then these and the n numbers $i^s j^\rho$ are linearly independent. For, if not, there is a polynomial $\tau(i) \neq 0$ such that τj^ρ is a linear function of the numbers (10) with coefficients in F. Since τ has an inverse in the field $F(i)$,

$$j^\rho = R_0 + R_1 j + \cdots + R_{\rho-1} j^{\rho-1}, \tag{11}$$

where each R_k is in $F(i)$. Multiply each term of (11) by i on the right and apply (7). We get

$$\theta^\rho(i) j^\rho = R_0 i + R_1 \theta(i) j + \cdots + R_{\rho-1} \theta^{\rho-1}(i) j^{\rho-1}.$$

Eliminating j^ρ by means of (11) and noting that the numbers (10) are linearly independent with respect to F, we see that

$$\theta^\rho(i) R_k = R_k \theta^k(i) \quad [k = 0, 1, \cdots, \rho - 1;\ \theta^0(i) \equiv i].$$

Since $k < \rho < r$, $\theta^\rho(i) \neq \theta^k(i)$ by Theorem I. Hence each $R_k = 0$. Then $j^\rho = 0$ by (11), so that $j^r = 0$, contrary to the assumption on (9).

Theorem 2. *The algebra A contains a linear associative subalgebra S over F with the rn units*

$$i^s, \quad i^s j, \quad i^s j^2, \quad \cdots, \quad i^s j^{r-1} \quad (s = 0, 1, \cdots, n-1) \tag{12}$$

linearly independent with respect to the field F. The product of any two of these units can be expressed as a linear function of the units by means of $\phi(i) = 0$ and relations (4) *and* (9).

5. In view of the first relation (7),

$$\begin{gathered} ji = \theta(i) j, \qquad j\theta(i) = \theta^2(i) j, \\ j\theta^2(i) = \theta^3(i) j, \quad \cdots, \qquad j\theta^{r-1}(i) = ij, \end{gathered} \tag{13}$$

where the final equation has been simplified by use of (2). Hence if s_k is an elementary symmetric function of the numbers (1),

$$js_k = s_k j. \tag{14}$$

Let K be the field composed of all rational functions with coefficients in F of these elementary symmetric functions:

$$K = F(s_1, s_2, \cdots, s_r). \tag{15}$$

Hence any number of K is commutative with j and, being a polynomial in i, also with i. Thus any number of K is commutative with every number of the algebra S.

If a number N of S is commutative with every number of S, then N is in the field K. For, by (*b*), N is in the field $F(i)$. Next, if $0 \leqq s < r$, $Nj^s = j^s N$ requires, by (7), that

$$N(i) j^s = N[\theta^s(i)] j^s.$$

Multiply each member by j^{r-s} on the right. Since the terms of (9) are not zero, we get

$$N(i) = N[\theta^s(i)] \qquad (s = 0, \cdots, r-1).$$

If we exclude the case of fields F having a modulus dividing r, we obtain, by addition and division by r, $N(i)$ expressed as a symmetric function of the numbers (1) and hence as a number of K. But we can easily give a proof valid for all fields F. By means of the equation of degree r having the roots

(1), we can express $N(i)$ as a polynomial in i of degree $r-1$ with coefficients in K. Thus

$$N(x) = a_0 + a_1 x + \cdots + a_{r-1} x^{r-1} \qquad (a\text{'s in } K)$$

holds for $x = i$. But the left member is unaltered when x is replaced by any one of the r numbers (1). Since an equation of degree $r-1$ with r distinct roots is an identity, $N(i) = a_0$, $a_1 = 0, \cdots$.

Theorem 3. *A number of the subalgebra S over F with the units* (12) *is commutative with every number of S if and only if it is in the field K defined by* (15).

Corollary. The right member of (9) is in the field K.

6. Let $\psi(x) = 0$ be the equation with coefficients in K and irreducible in K which has the root i. Since j is commutative with every number of K, in view of (14), relations (7) hold for any polynomial $f(x)$ with coefficients in K. Taking $\psi(x)$ as $f(x)$, we conclude that every $\theta^s(i)$ is a root of $\psi(x) = 0$. Thus the latter has the r distinct roots (1). But these r numbers are the roots of an equation $\kappa(x) = 0$ with coefficients in K, in view of the definition of K. Hence $\kappa(x)$ is irreducible in K.

Theorem 4. *The r numbers* (1) *are the roots of an equation $\kappa(x) = 0$ of degree r with coefficients in the field K and irreducible in K.*

7. Any number of algebra S, i. e., any linear function of its units (12) with coefficients in F, can be expressed by means of the equation $\kappa(i) = 0$ of degree r as a linear function of the r^2 numbers (3) with coefficients in the field K. These r^2 numbers are linearly independent with respect to K; the proof is as in § 4 with n replaced by r, F by K, $F(i)$ by $K(i) \equiv F(i)$. Hence the algebra S over F can be exhibited as the algebra L over K defined in § 1.

Theorem 5. *Any linear associative algebra A over the field F with the properties* (*a*), (*b*), (*c*), *and such that $g(i) \neq 0$ in* (9), *has a subalgebra S over F which can be exhibited as the algebra L over K with r^2 units* (3) *linearly independent with respect to K.*

8. If N is a number of the algebra L which is commutative with i, then N is a polynomial in i with coefficients in the field K. Indeed,

$$N = R_0 + R_1 j + \cdots + R_{r-1} j^{r-1},$$

where each R_s is a polynomial in i with coefficients in K. By (7),

$$j^s i = \theta^s(i) j^s,$$

$$Ni = R_0 i + R_1 \theta(i) j + \cdots + R_{r-1} \theta^{r-1}(i) j^{r-1}.$$

Equating the latter to iN and noting that the units (3) are linearly independent with respect to K, and that the numbers (1) are distinct, we find that R_1, $R_2, \cdots, R_{r-1}$ are zero.

Theorem 6. *Any algebra of type L has the properties (a), (b), (c), with n replaced by r and F by K.*

Algebras A With $r = 2$.

9. For $r = 2$, the irreducible equation satisfied by i may be given the form* $i^2 = c$ by applying a linear transformation on i. The algebra L then has the units $1, i, j, k$, where

$$
\begin{gathered}
i^2 = c, \qquad j^2 = g, \qquad ij = k = -ji, \\
k^2 = -cg, \qquad jk = -gi = -kj, \qquad ki = -cj = -ik.
\end{gathered} \tag{16}
$$

The relations in the second line were derived from those in the first line by use of the associative law. Right-hand and left-hand division† is always possible and unique in this algebra L if and only if c is not the square of a number in K and g is not expressible in the form $x^2 - cy^2$, with x and y in K. We shall assume henceforth that these conditions are satisfied. Then L is a generalized‡ quaternion algebra over K.

Consider an algebra A related to this L as in Theorem 5. Then K is of rank $n/2$ with respect to its subfield F. When K is so related to F, algebra L can evidently be exhibited as an algebra S of $2n$ units over F. We seek algebras A with such a subalgebra S but distinct from S. Since right- and left-hand division is always possible in S, the number of units m of A is a multiple of $2n$.

10. For the simplest case $n = 2$, $K \equiv F$ and A has L as a subalgebra. Let A have the least possible number 8 of units:

$$1, \quad i, \quad j, \quad k, \quad f, \quad if, \quad jf, \quad kf.$$

Then $fi = Q + Rf$, where Q and R are in L. Multiply by i on the right. Thus $R^2 = c$, $Qi + RQ = 0$. Hence $R = \pm i$. If $R = i$, then $Q = ej + lk$, and i is commutative with $f + jl/2 + ke/(2c)$, contrary to (b). Hence $R = -i$, $Qi = iQ$, $Q = e + li$. Taking $f - l/2 - ie/(2c)$ as a new unit f, we have

$$fi = -if. \tag{17}$$

Next, $fj = v + wf$, where v and w are in L. Multiply by j on the right. Thus $w^2 = g$, $w = \pm j$, $vj \pm jv = 0$.

---

* We exclude fields having modulus 2.

† These Transactions, vol. 13 (1912), p. 66. Division is always uniquely possible in the more general algebra (9), p. 67, since $\Delta(X) = \Delta'(X) = \sigma^2$.

‡ The ordinary quaternion algebra if $c = g = -1$ and if K is the field of all real numbers.

For the upper signs, $v = ri + sk$. Taking $f - kr/(2g)$ as a new f, we see that (17) remains true and that

$$fj = sk + jf. \tag{18}$$

Multiply (17) on the right by j. Thus $fk = -scj - kf$. Multiply (17) on the left by f. Thus f^2 is commutative with i. Hence, by (b),

$$f^2 = a + bi \qquad (a, b \text{ in } F). \tag{19}$$

Multiply (18) on the left by f. Thus $bk = -s^2 cj - bk$, $b = s = 0$,

$$fi = -if, \qquad fj = jf, \qquad fk = -kf, \qquad f^2 = a. \tag{20}$$

The general element is $Q + Rf$, Q and R in L. We have

$$(Q + Rf)(q + rf) = Qq + Rr^* a + (Qr + Rq^*) f, \tag{20'}$$

where r^* is derived from r by changing the signs of the coefficients of i and k. Since $fr = r^* f$, $(r\rho)^* = r^* \rho^*$, the algebra with the multiplication table (20′) is associative. It is not an algebra A over F since every element is commutative with jf. Indeed, $Qj = jQ^*$.

For the lower signs, $v = e + lj$, so that

$$fj = e + lj - jf.$$

Multiply (17) on the right by j. We get

$$fk = -ei - lk + kf. \tag{21}$$

Now the multiplication table (16) for L is unaltered in form if we set

$$j = k_1, \qquad k = cj_1, \qquad g = -cg_1.$$

Then, by (21),

$$fj_1 = -\frac{e}{c} i - lj_1 + j_1 f.$$

Dropping the subscripts, we have the former case $fj = v + jf$.

Theorem 7. *There does not exist an algebra of type A with 8 units.*

Nor is there any algebra $A > L$ in which every number satisfies a quadratic equation with coefficients in F. For, if so, we may set $f^2 =$ constant. Then $(f \pm i)^2$ is a linear function of $f \pm i$, so that $fi + if$ is a linear function of $f + i$ and also of $f - i$ and hence is a constant:

$$fi + if = c_1, \qquad fj + jf = c_2, \qquad fk + kf = c_3,$$

$$fk = fij = (-if + c_1) j = i(jf - c_2) + c_1 j = kf - c_2 i + c_1 j.$$

Hence $2fk$ and thus f is in L. Thus* $A \equiv L$.

* We may thus give a remarkably simple proof of the famous theorem that a linear associative algebra (over the field of reals) in which division is always possible and unique is either the field of reals, the field of ordinary complex numbers or the system of real quaternions. Indeed, it is very easily shown that if there be more than two units, the algebra has a subalgebra of the quaternion type [making use of these Transactions, vol. 13 (1912), p. 64, first lines of § 6]. See § 11 of *Linear Algebras* (in press), Cambridge Tracts.

Algebras A With 16 Units.

11. For the next case $n = 4$, there exist algebras A with the minimum number 16 of units. We may take K to be the field $F(\xi)$, where

$$\xi^2 = \nu \qquad (\nu \text{ not a square in } F). \tag{22}$$

As the 8 units of algebra S we may take

$$1, \quad \xi, \quad i, \quad \xi i = i\xi, \quad j, \quad \xi j = j\xi, \quad k, \quad \xi k = k\xi. \tag{23}$$

The products of these by f on the right may be taken as the remaining 8 units of algebra A. Then $f\xi = Q + Rf$, where Q and R are in S. Multiplying by ξ on the right, we see that

$$R^2 = \nu \equiv \xi^2, \qquad (\xi + R)\,Q = 0.$$

If $\xi + R \neq 0$, then $Q = 0$, $R = \xi$, $f\xi = \xi f$, so that A is an algebra of 8 units over the field $F(\xi)$ and hence, if existent, is a quaternary algebra L over a larger field (§ 10). We therefore take $R = -\xi$. Taking $f - Q/(2\xi)$ as a new f, we have

$$f\xi = -\xi f. \tag{24}$$

Thus ξ is commutative with $q + rf$, where q and r are in S, if and only if $r = 0$. But ξ is commutative with f^2. Hence

$$f^2 = \lambda + \mu j \qquad [\lambda, \mu \text{ in } F(i)]. \tag{25}$$

12. The following notations will be employed. If $u = a + b\xi$, where a and b are in the field F, write $u' = a - b\xi$. If $\alpha = r + si$, where r and s are in the field $F(\xi)$, write $\bar{\alpha} = r - si$. Hence

$$fu = u'f, \qquad j\alpha = \bar{\alpha}j, \qquad k\alpha = \bar{\alpha}k \qquad [u \text{ in } F(\xi),\ \alpha \text{ in } F(i)]. \tag{26}$$

13. We must have

$$fi = Q + Rf, \qquad Q = \alpha + \beta j, \qquad R = \gamma + \delta j \qquad [\alpha, \cdots, \delta \text{ in } F(i)].$$

Then

$$fi^2 = c'f = Qi + RQ + R^2 f, \qquad c' = R^2, \qquad Qi + RQ = 0,$$

$$c' = \gamma^2 + \delta\bar{\delta}g, \qquad \delta(\gamma + \bar{\gamma}) = 0, \tag{27}$$

$$g\bar{\beta}\delta = -\alpha(i + \gamma), \qquad \bar{\alpha}\delta = \beta(i - \gamma). \tag{28}$$

First, let $\delta = 0$. If either α or β is not zero, then

$$\gamma = \pm i, \qquad c' = \gamma^2 = c,$$

and i satisfies the equation $i^2 = c$, with c in F, contrary to $n = 4$. Hence $\alpha = \beta = 0, fi = \gamma f$. Now $\gamma = x + yi$, x, y, and c' being in $F(\xi)$. Thus $c' = \gamma^2$ gives $xy = 0$. If $y = 0$, then $i^2 = c = x'^2$, and $i = \pm x'$ would be n $F(\xi)$. Hence $x = 0$,

$$\text{(I)} \qquad fi = yif, \qquad c' = y^2 c \qquad [\, y \text{ in } F(\xi)\,].$$

If a number of A is commutative with i it is easily seen (see § 19) to be in $F(i)$, so that property (b) holds.

Next, let $\delta \neq 0$. Then by (27_2), $\gamma = \sigma i$, σ in $F(\xi)$. If $\beta \neq 0$, we find by eliminating α from (28), and using (27_1), that $c' = c$. Hence $\beta = 0$, $\alpha = 0$, and

$$\text{(I}'\text{)} \qquad fi = (\sigma i + \delta j) f, \qquad c' = c\sigma^2 + g\delta\bar{\delta} \quad [\, \sigma \text{ in } F(\xi),\ \delta \neq 0 \text{ in } F(i)\,].$$

14. We must have $fj = q + rf$, q and r in S. Then

$$\text{(29)} \qquad fj^2 = g'f = qj + rq + r^2 f, \qquad g' = r^2, \qquad qj = -rq.$$

Set $q = B + Cj$, $r = D + Ej$, where $B, \cdots, E$ are in $F(i)$. Then

$$\text{(30)} \qquad g' = D^2 + gE\bar{E}, \qquad E(D + \bar{D}) = 0,$$

$$\text{(31)} \qquad gC + DB + gE\bar{C} = 0, \qquad B + DC + E\bar{B} = 0.$$

First, let $E = 0$. Set $D = w + zi$, w and z in $F(\xi)$. Since $D^2 = g'$ is in $F(\xi)$, $wz = 0$. If $z = 0$, then $j^2 = g = w'^2$, and $j = \pm w'$ would be in $F(\xi)$. Hence $z \neq 0$, $D = zi$. Eliminating B from (31), we get $C(g - D^2) = 0$. If $C \neq 0$, then $j = \pm zi$. Hence $C = 0$, $B = 0$, and

$$\text{(II)} \qquad fj = zif, \qquad g' = z^2 c \qquad [\, z \text{ in } F(\xi)\,].$$

Next, let $E \neq 0$. Then, by (30), $D = \omega i$, ω in $F(\xi)$, and

$$\text{(32)} \qquad g' = c\omega^2 + gE\bar{E}.$$

Multiply (31_1) by i and substitute the resulting value of $c\omega B$ into the product of (31_2) by ωc. By use of (32), we get

$$iC(g' - g) = 0.$$

If $g' \neq g$, then $C = 0$ and, by (31), $B = 0$, since the case $E\bar{E} = 1$, $\omega = 0$, is excluded by (32). Hence

$$\text{(II}'\text{)} \qquad fj = (\omega i + Ej) f, \qquad g' \neq g, \qquad E \neq 0 \ [\text{subject to (32)}].$$

For $g' = g$, it is simpler to employ conditions (29). Then $r^2 = j^2$, $r = \pm j$.

If $r = j$, set $f_1 = if$. Then (I) and (I′) hold with f replaced by f_1 and δ by $-\delta$. Also

$$f_1 j = i(q + rf) = iq - jif = iq - jf_1.$$

Hence it suffices to treat the case $r = -j$. Then $fj + jf = q$ is commutative with ξ. But, by (24), $(fj + jf)\xi = -\xi(fj + jf)$. Hence

$$\text{(II}''\text{)} \qquad fj = -jf, \quad g' = g.$$

15. It remains to combine one of the cases (I), (I′) with one of the cases (II), (II′), (II″). The combination (I) and (II) is readily excluded. Multiply (I) on the right by j and apply (II). We get

$$fk = yi \cdot zif = yzcf = fy'z'c', \qquad k = y'z'c',$$

whereas k is not in $F(\xi)$.

16. Let (I) and (II″) hold. By the latter, $f^2 j = jf^2$. Hence by (25) and (26_2), $\lambda = \bar{\lambda}$, $\mu = \bar{\mu}$, so that λ and μ are in $F(\xi)$. By (I),

$$f^2 i = yy' if^2, \qquad y^2 y'^2 = 1.$$

Using (25), we see that $\mu = 0$ or $\lambda = 0$, according as $yy' = +1$ or -1. For the present, let the first alternative hold:

$$\text{(33)} \qquad yy' = +1, \qquad f^2 = \lambda,$$

where λ is in $F(\xi)$. Then f is commutative with λ. Hence by (26_1), λ is in F.

Writing* f_1, f_2, f_3 for if, jf, kf, we may express any number of the algebra as a linear combination of $1, i, j, k, f, f_1, f_2, f_3$, with coefficients in the field $F(\xi)$. The associative law then uniquely determines the multiplication table for these eight units:

	i	j	k	f	f_1	f_2	f_3
i	c	k	cj	f_1	cf	f_3	cf_2
j	$-k$	g	$-gi$	f_2	$-f_3$	gf	$-gf_1$
k	$-cj$	gi	$-gc$	f_3	$-cf_2$	gf_1	$-gcf$
f	yf_1	$-f_2$	$-yf_3$	λ	$y\lambda i$	$-\lambda j$	$-y\lambda k$
f_1	ycf	$-f_3$	$-ycf_2$	λi	$c\lambda y$	$-\lambda k$	$-yc\lambda j$
f_2	$-yf_3$	$-gf$	gyf_1	λj	$-y\lambda k$	$-g\lambda$	$gy\lambda i$
f_3	$-ycf_2$	$-gf_1$	$ygcf$	λk	$-yc\lambda j$	$-g\lambda i$	$gyc\lambda$

* The rest of this section and § 17 are not essential in the later work.

Let N be unity or one of the seven numbers $i, \cdots, f_3$ at the head of the table. Let N_1 be one of the same eight numbers. Let x and u be any numbers in $F(\xi)$. Then

$$(34) \qquad (xN)(uN_1) = (xU)(NN_1) \qquad \begin{pmatrix} U = u \text{ if } N = 1, i, j, k; \\ U = u' \text{ if } N = f, f_1, f_2, f_3 \end{pmatrix}.$$

By means of the table and (34), we can express the product of any two linear functions of $1, i, \cdots, f_3$ with coefficients in $F(\xi)$ as a third such function. Multiplication thus uniquely defined is associative; it is far simpler to prove this fact by means of the condensed notations of § 18.

17. The preceding algebra can be given a simpler form by using a functional notation in addition to the notations in § 12. If $A = u + vi$, where u and v are in $F(\xi)$, write $A^* = u' + yv'i$. Then by (26), (I), (II″),

$$(35) \qquad fA = A^* f, \qquad jA = \bar{A} j, \qquad \phi A = \bar{A}^* \phi,$$

where $\phi = jf$. Any number of the algebra is of the form

$$N = X + Yj + Zf + W\phi \qquad [X, \cdots, W \text{ in } F(i)].$$

Then

$$(36) \qquad N(A + Bj + Cf + D\phi) = P + Qj + Rf + S\phi,$$

where

$$(37) \qquad \begin{aligned} P &= XA + Y\bar{B}g + ZC^*\lambda - W\bar{D}^*\lambda g, \\ Q &= XB + Y\bar{A} - ZD^*\lambda + W\bar{C}^*\lambda, \\ R &= XC + Y\bar{D}g + ZA^* - W\bar{B}^* g, \\ S &= XD + Y\bar{C} - ZB^* + W\bar{A}^*. \end{aligned}$$

18. A further simplification is effected by writing the general number of the algebra in the form $U + Vf$, where U and V are in S (i. e., are generalized quaternions), on writing $U\dagger = X^* - Y^* j$ if $U = X + Yj$, X and Y being in $F(i)$. Then $fU = U\dagger f$. The law of multiplication is now

$$(38) \qquad (U + Vf)(G + Hf) = UG + VH\dagger\lambda + (UH + VG\dagger)f.$$

This multiplication is seen at once to be associative since the element $f^2 = \lambda$ of F is commutative with every U and since

$$(39) \qquad (UV)\dagger = U\dagger V\dagger, \qquad (U\dagger)\dagger = U.$$

The latter relations follow at once from

$$(40) \quad (X+Y)^* = X^* + Y^*, \quad (XY)^* = X^* Y^*, \quad (X^*)^* = X, \quad (\bar{X})^* = \bar{X}^*,$$

where X, Y are any numbers in $F(i)$. Relations (40) are easily verified, use being made of (I), (33), $(u')' = u$, $(uv)' = u'v'$.

19. Let i be commutative with $U + Vf$. Then $Ui = iU$, so that U is in $F(i)$, and $yVi = iV$. Multiply the latter by i on the left. Then

$$cV = i^2 V = yiVi = y(yVi)i = y^2 cV.$$

If $y^2 = 1$, then $c' = c$ by (I) and i is a root of the equation $i^2 = c$ with coefficients in F, contrary to $n = 4$. Hence $V = 0$. Thus any number of the algebra commutative with i is in the field $F(i)$.

20. Since $yy' = 1$, the condition in (I) may be written $c'y' = cy$. Hence cy is in the field F, so that $c = dy'$, d in F. Set $y = a - b\xi$, a and b being in F. Then $yy' = 1$ gives

$$a^2 - \nu b^2 = 1. \tag{41}$$

Now i is a root of $x^2 = c$ and hence of

$$x^2 = d(a + b\xi), \qquad (x^2 - da)^2 = d^2 b^2 \nu.$$

By using (41), we see that i is a root of

$$x^4 - 2dax^2 + d^2 = 0. \tag{42}$$

Theorem 8. *If a, b, d, λ, g, ν are numbers in the field F such that* (41) *holds and such that equations* (42) *and $\xi^2 = \nu$ are irreducible in F, and if we set $y = a - b\xi$, $c = dy'$, the linear algebra over F with the* 16 *units* $1, i, j, k, f, f_1, f_2, f_3$ *and their products by ξ on the left, with multiplication defined by* (34) *and the table in* § 16, *or more compactly by* (38), *is a linear associative algebra with the properties* (*a*), (*b*), (*c*) *of* § 2.

The conditions on a, b, d are easily discussed. For example, let F be the field of all rational numbers. Take $\nu = -1$, $d = \pm 1$, $a = 0$, $b = 1$. Then (41) holds and (42) becomes the equation $x^4 + 1 = 0$ for the primitive eighth roots of unity and is known to be irreducible in the field of rational numbers.

Then $y = -\xi$, $c = \pm\xi$, and K is the field of the numbers $r + s\xi$, where r and s are rational and $\xi^2 = -1$. While it is not an essential condition on our algebra, we may choose g so that division* is unique in the algebra L.

21. We have treated the first one (33) of two alternatives. Next, let

$$yy' = -1, \qquad f^2 = \mu j.$$

* Evidently c is not the square of $r + s\xi$, since then $2s^2 = +1$ or -1. Next, there is a rational number g not of the form $x^2 - cy^2$, where $x = r + s\xi$, $y = l + m\xi$. Indeed, $x^2 - \xi y^2 = A + B\xi$, $A = r^2 - s^2 + 2lm$, $B = 2rs - l^2 + m^2$. Take $B = 0$. If $r = 0$, then $A = \pm 2l^2 - s^2$. If $r \neq 0$, we eliminate s and get $4r^2 A = \{2r^2 + (l+m)^2\}\{2r^2 - (l-m)^2\}$. If $2r^2 - v^2$ is divisible by the prime p, but not by p^2, then $p = 8k \pm 1$. For $2r^2 + v^2$ the condition is $p = 8k + 1$ or $8k + 3$. The primes not in either set are $p = 8k + 5$. Hence we may take as g a number divisible by an odd power of a prime $8k + 5$.

By $ff^2 = f^2 f$, we get $\mu' = -\mu$. Thus $\mu = t\xi$,

$$yy' = -1, \quad f^2 = t\xi j \qquad (t \text{ in } F). \tag{43}$$

Define A^* as in § 17, and $U\dagger$ as in § 18. As before,

$$fA = A^* f, \qquad fU = U\dagger f, \qquad (AB)^* = A^* B^*, \qquad (UV)\dagger = U\dagger V\dagger, \tag{44}$$

but we now have

$$(A^*)^* = \bar{A}, \qquad (U\dagger)\dagger j = jU, \qquad [(X + Yj)\dagger]\dagger = \bar{X} + \bar{Y}j, \tag{45}$$

of which the last two equations are equivalent. Multiplication

$$(U + Vf)(G + Hf) = UG + VH\dagger t\xi j + (UH + VG\dagger) f \tag{46}$$

is easily verified to be associative. Also § 19 holds here. By (I),

$$c' y' \xi' = cy\xi, \qquad c = dy' \xi \qquad (d \text{ in } F)$$

As in § 20, $a^2 - \nu b^2 = -1$, while i is a root of

$$x^4 - 2bd\nu x^2 + \nu d^2 = 0. \tag{47}$$

The first condition is satisfied if $a = b = 1$, $\nu = 2$, and for $d = 1$ equation (47) becomes $x^4 - 4x^2 + 2 = 0$, which is irreducible in the field of rational numbers. *Hence we have an algebra for which (a), (b), (c) hold.*

Theorem 9. *If a, b, d, t, g, ν are numbers in the field F such that $a^2 - \nu b^2 = -1$ and such that (47) and $\xi^2 = \nu$ are irreducible in F, if $y = a - b\xi$, $c = dy'\xi$, and if U and V are numbers of the algebra (16) over $F(\xi)$, then the numbers $U + Vf$, subject to the law of multiplication (46), define a linear associative algebra with 16 units and having the properties (a), (b), (c) of § 2.*

22. Finally, let (I) and (II′) hold. Multiply (I) by j on the right and replace the fj introduced by its value in (II′). Thus

$$fk = y\omega c + yEkf.$$

Multiply this by k and eliminate the fk introduced. Then

$$-c' g' f = y^2 (Ek)^2 f + y\omega c (1 + yE) k.$$

If $\omega \neq 0$, then $yE = -1$ and, by the coefficients of f, $cg = c' g'$. Now y and hence E is in $F(\xi)$, so that $\bar{E} = E$. Hence $y^2 E\bar{E} = 1$. Multiply this by c and apply (I). Thus $c' E\bar{E} = c$. But (32) then gives $\omega = 0$.

Hence $\omega = 0$. Consider the case $yE = -1$. By (I), $yy' = \pm 1$. Thus $E = \mp y'$, $g' = gy'^2$. Let $J = k$, $K = cj$. Then

$$fi = yif, \qquad fJ = -Jf, \qquad iJ = K, \qquad J^2 = G, \qquad K^2 = -cG,$$

where $G = -cg = G'$. Hence the algebra is equivalent to that in § 16.

As it is not our purpose to enumerate all the algebras A with 16 units, we do not consider the remaining values of E, nor an algebra in which (I′) holds.

Theorems 8 and 9 show that there exist linear associative algebras A with the properties (a), (b), (c), and distinct from their sub-algebras S. Hence the class of algebras A is more extensive than the class of primitive algebras L.

SOME THEOREMS CONCERNING GROUPS WHOSE ORDERS ARE POWERS OF A PRIME*

BY

WILLIAM BENJAMIN FITE

Professor Burnside has called attention to certain limitations on the nature of the derived groups of groups of prime power order.† One of these limitations is that such a derived group cannot be non-abelian and have a cyclic central. Theorem I of the present paper gives a slight generalization of this limitation, and an obvious modification of the argument used in the proof of this theorem establishes the important result that the second central of a group whose order is a power of an odd prime cannot be cyclic.

The results given at the end of the paper are continuations of some obtained by Professor Burnside in the article just cited.

Throughout the paper frequent use is made of the theorem that in a group whose order is a power of a prime the operations which correspond to the invariant operations of the ith cogredient are commutative with all of the ith commutators of the group.‡

We shall represent by G a group of class k and order p^m, where p is a prime; by G' the first cogredient of G; by H_i the ith central§ of G; by K_j the jth commutator subgroup; and by $K_{j,\,i}$ the subgroup of K_j that is contained in H_i.

Suppose that $K_{j,\,2}$ is cyclic and generated by the operation t of order p^a. If $k > 3$, every operation of $K_{j,\,3}$ of order p must be contained in $K_{j,\,2}$ since the operations of this latter group are invariant in K_j and its operations of order p are invariant in G. Hence $K_{j,\,3}$ has only one subgroup of order p. It follows at once that $K_{j,\,3}$ is cyclic if p is odd. If $p = 2$, $K_{j,\,2}$ contains an operation of order 2^2 that is invariant in $K_{j,\,3}$. But in the non-cyclic group of order a power of 2 with only one subgroup of order 2 no operation of

* Presented to the Society at the Cleveland meeting, January 1, 1913.

† Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912), pp. 241, 242.

‡ Fite, these Transactions, vol. 7 (1906), p. 62.

§ I follow here the notation of Burnside's *Theory of Groups*, first edition, p. 62, rather than that of the second edition, p. 120. Cf. de Séguier, *Éléments de la théorie des groupes abstraits*, p. 87.

order 2^2 is invariant. We conclude therefore that in this case also $K_{j,\,3}$ is cyclic.

Suppose then that $K_{j,\,3}$ is generated by the operation t_1 of order $p^{\alpha+\beta}$. If we denote by $K'_{j,\,i}$ the subgroup of the jth commutator subgroup of G' that is contained in the $(i-1)$th central of G', it follows from what has been proved that $K'_{j,\,4}$ must be cyclic. Hence the quotient group $K_{j,\,4}/K_{j,\,3}$ must be cyclic. If it is of order p^γ and t_2 is an operation of $K_{j,\,4}$ that corresponds to a generator of it, $t_2^{p^\gamma}$ is a power of t_1 whose exponent is relatively prime to p, and therefore $K_{j,\,4}$ is cyclic. An obvious continuation of this argument shows that K_j is cyclic. This establishes the following

Theorem I. *The jth commutator subgroup of a group of prime power order is cyclic if those of its operations that are contained in the second central of the group form a cyclic subgroup.*

Since $K_{1,\,2}$ is contained in the central of K_1, Theorem I includes as a special case Professor Burnside's theorem referred to in the introduction.

In case p is odd an argument closely analogous to the one used in the proof of Theorem I shows that if H_2 were cyclic G itself would be cyclic. But if G were cyclic there would be no H_2. This proves

Theorem II. *The second central of a group of order p^m, where p is an odd prime, cannot be cyclic.*

If $p = 2$, the preceding argument does not apply, since the operations of H_2 are not all necessarily invariant in H_3 and therefore an operation of H_3 of order 2 may give a commutator of order 2^2. We can however proceed as follows.

If $k > 3$, H_3 contains a commutator t_1 that is not in H_2. Now $\{H_2, t_1\}$ is an abelian group with only one subgroup of order 2, since t_1 is commutative with every one of its commutators. Hence this group is cyclic. If H_3 contained an operation s of order 2 that is not in $\{H_2, t_1\}$, s would transform t_1 into itself multiplied by an invariant commutator of G of order 2. Hence s would be commutative with t, and accordingly every operation of G would transform it into itself multiplied by an invariant commutator of G. But this is not consistent with the fact that s is not in H_2. We conclude therefore that H_3 contains only one subgroup of order 2. It contains the operation t_1 and every one of its operations transforms t_1 into itself multiplied by an operation of H_1. But this would be impossible in case H_3 were non-cyclic. We conclude therefore that H_3 is cyclic.

The continuation of the argument up to the point of showing that H_{k-1} must be cyclic is the same as in the case of an odd prime. But beyond this point the argument breaks down, since G/H_{k-1} has no commutator besides the identity. As a matter of fact, there are groups of order 2^m whose second centrals are cyclic—for example, the diedral groups of these orders when $m > 3$.

Suppose now that K_1 is cyclic and generated by the commutator t of order

p^λ, where p is odd. If $k > 3$, G/H_{k-2} is metabelian* with a cyclic commutator subgroup. If p^r is the order of this subgroup, G/H_{k-3} must contain an invariant commutator of order p^r and the invariant commutators of G/H_{k-3} must form a cyclic subgroup of order p^s, where $s \geqq r$.

Since any operation of G/H_{k-3} that corresponds to an invariant operation of G/H_{k-2} is commutative with every commutator of G/H_{k-3}, no operation of H_{k-1} can transform t into itself multiplied by a commutator that is not in H_{k-3}. But some operation of G must transform t into $t^{1+xp^{s_1}}$, where x is relatively prime to p, and $r \leqq s_1 < r + s$, since otherwise t would be contained in H_{k-2} and G/H_{k-2} would be abelian. But this is impossible since G is of class k.

Suppose then that A is so selected that

$$A^{-1}\, tA = t^{1+xp^{s_1}},$$

where x is relatively prime to p, and s_1 is a minimum. Then

$$A^{-1}\, t^{p^{r-1}}\, A = t^{p^{r-1}+xp^{r+s_1-1}}, \qquad A^{-1}\, t^{p^r}\, A = t^{p^r+xp^{r+s_1}}.$$

Now since t^{p^r} is contained in H_{k-2}, while $t^{p^{r-1}}$ is not, it follows that

$$r + s_1 \geqq r + s, \qquad r + s_1 - 1 < r + s.$$

Hence $s_1 = s$, and therefore

$$A^{-p^r}\, tA^{p^r} = t^{(1+xp^s)^{p^r}}.$$

But

$$(1 + xp^s)^{p^r} \equiv 1 + xp^{r+s} \pmod{p^{2s+1}}.$$

Moreover $r + s < 2s + 1 \leqq \lambda$, since the lowest power of t that is contained in H_{k-3} is the p^{r+s}th power and since H_{k-3} must contain a commutator of order p^s. Hence

$$(1 + xp^s)^{p^r} \not\equiv 1 \pmod{p^\lambda},$$

and A^{p^r} is not commutative with t. Since however G/H_{k-2} is metabelian and contains no commutator of order greater than p^r, A^{p^r} must be contained in H_{k-1}. We conclude therefore that *if K_1 is cyclic and $k > 3$, not all of the operations of K_1 are invariant in H_{k-1}.*

This argument with a slight modification is valid when $p = 2$, provided that $s > 1$; but when $s = 1$ it breaks down and the conclusion does not hold, as may be seen from the groups of order 2^5.

We have assumed that $k > 3$. The existence of metabelian groups with cyclic commutator subgroups shows that the conclusion does not hold when

* I use the term *metabelian* in the sense defined by me in these Transactions, vol. 3 (1902), p. 331. Burnside uses the term in a different sense. See his *Theory of Groups*, second edition, p. 57.

$k = 2$. When $k = 3$ every operation of H_2 is commutative with t. But the following conclusion from this argument for $k > 3$ does hold when $k = 3$; namely,* *if K_1 is cyclic, H_{k-1} cannot coincide with K_1.*

The commutator formed by any two operations, t_1 and t_2, of K_1 is contained in H_{k-3}, since G / H_{k-3} is of class three and the commutator subgroup of a group of the third class is abelian. The index of $K_{1,\ k-3}$ under K_1 is at least p^2. If it is exactly p^2, we can assume that t_2 is equal to the product of some operation of H_{k-2} and t_1. Hence† the commutator formed by t_1 and t_2 is in H_{k-4} and the index of the commutator subgroup of K_1 under K_1 is at least p^3.

If t_1 and t_2 are operations of K_i they are ith commutators or products of ith commutators and correspond to invariant operations of the ith cogredient of G / H_{k-2i-1}, since they are contained in H_{k-i}. Hence the commutator formed by t_1 and t_2 is contained in H_{k-2i-1}. Now if K_i is non-abelian the index of $K_{i,\ k-2i-1}$ under K_i is at least p^{i+1}. If it is exactly p^{i+1}, we can show by an argument similar to the one used in the special case just considered that the commutator formed by t_1 and t_2 is contained in H_{k-2i-2}. Hence the index of the commutator subgroup of K_i under K_i is at least p^{i+2} if K_i is non-abelian.

If the jth derived group is contained in H_x, the $(j+1)$th derived group is contained in $H_{x-(j+1)}$, provided that $x > j + 1$. Hence the ith derived group is contained in $H_{k-i(i+1)/2}$, and therefore in a group of order p^m whose ith derived group is not the identity we must have $k > i\,(i+1)/2$. Hence $m \geqq 2 + i\,(i+1)/2$. For $i > 4$ this exceeds the lower limit for m given by Professor Burnside.

COLUMBIA UNIVERSITY,
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* de Séguier, *loc. cit.*, p. 127. This follows also from Theorem II.

† This result is given by Burnside in the article cited in the introduction.

LIMITS IN TERMS OF ORDER, WITH EXAMPLE OF LIMITING ELEMENT NOT APPROACHABLE BY A SEQUENCE*

BY

RALPH E. ROOT

1. Introduction.

The importance of the rôle of the order relation in the theory of the linear continuum has been recognized since the appearance, in 1872, of Dedekind's *Stetigkeit und irrationale Zahlen*,† and it received added importance at the hands of Cantor‡ in 1895. A later contribution to the theory was made by O. Veblen§ in 1905, and a systematic account of the theory of the continuum as a type of order was given by E. V. Huntington || the same year. The work of these writers is directed toward a complete characterization of the linear continuum in terms of order alone. A set or class of elements, otherwise undefined, is assumed to fulfil conditions, stated in terms of order, sufficiently restrictive to admit of effective use of the class in the rôle of range of the continuous independent variable.

Meanwhile certain classes have been recognized as being, in effect, the range of an independent variable, while not fulfilling all the conditions of the linear continuum. The desirability of utilizing analogies that exist between theories that pertain to these classes and the theory of the continuum is obvious. To this end it is important to generalize, as far as may be, the notions of point sets so as to be applicable to such classes.

The thesis of M. Fréchet¶ was a notable effort toward this form of generalization. He presupposes a definition of limit of a sequence of elements, the definition fulfilling certain mild conditions, and defines in terms of this the notion of limiting element of a subclass, and develops a theory applicable to any class for which a limit relation can be defined fulfilling the prescribed

* Presented to the Society (Chicago), March 21, 1913.

† English translation by W. W. Beman, in a volume titled Dedekind's *Essays on the Theory of Numbers*, 1901.

‡ Mathematische Annalen, vol. 46 (1895), p. 481.

§ These Transactions, vol. 6 (1905), p. 165.

|| Annals of Mathematics, ser. 2, vol. 6 (1905), p. 151, and vol. 7 (1905), p. 15.

¶ Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 6.

conditions. He later adopts an analogue of the distance function (*voisinage*), the restrictions of which give a much more extensive theory.

E. H. Moore, in his paper at the Rome Congress in 1908,* and in his later published work on general analysis,† introduces a form of generalization of different character, considering properties of classes whose elements are real valued functions of a variable absolutely unconditioned. This general variable admits of a wide range of special applications.

E. R. Hedrick‡ contributes to the theory of such general classes by defining, in terms of the limit relation used by Fréchet, an "enclosable property" which replaces the voisinage or distance function used by the latter to define a special type of limit. In a paper by T. H. Hildebrandt§ in which the *voisinage* of Fréchet is replaced by a K relation of a type used by E. H. Moore, it is shown that most of the theorems proved by Fréchet may be based on milder assumptions than he used. The papers by Fréchet, Hedrick and Hildebrandt each consists of two stages, the first stage pertaining to a range possessing a mildly conditioned definition of limit of a sequence, the second stage pertaining to a range conditioned in such manner as to involve a notion closely analogous to that of "neighborhood," and introducing in some form the notion of "uniformity on the range."

In a former paper‖ by the present writer postulates are stated directly in terms of "neighborhood," but are free from any requirement of uniformity on the range. This absence of uniformity facilitates the treatment of ideal limiting elements, while invariance of postulates under the process of composition of classes is utilized to simplify the treatment of multiple and iterated limits. The paper may, in some sense, be regarded as an intermediate stage between the two stages of each of the three papers just mentioned.

The object of the present paper is to use in this field of generalization some of the simple properties of ordered classes. A triadic relation of the type of "betweenness" serves in the definition of segment, which, with certain mild postulates that are fulfilled by any simply ordered class, is effective in the development of a theory of limits and continuity. The postulates possess the invariance under composition and the freedom from uniformity that are

* *On a Form of General Analysis, with Application to Linear Differential and Integral Equations*, Atti del IV° Congresso Internazionale dei Matematici, Roma, 1908, vol. 2, p. 98.

† *Introduction to a Form of General Analysis*, published in a volume titled *The New Haven Mathematical Colloquium*, 1910.

‡ *On Properties of a Domain for which any Derived Set is Closed*, these Transactions, vol. 12 (1911), pp. 285–294.

§ *A Contribution to the Foundations of Fréchet's Calcul Fonctionnel*, American Journal of Mathematics, vol. 34 (1912), p. 237.

‖ *Iterated Limits in General Analysis*, American Journal of Mathematics, Jan. and Apr., 1914.

emphasized in the previous paper, and avoid some of the restrictions of that paper.

The theory developed in sections 2–7 may be characterized as a "neighborhood theory," being based on a definition of limiting element in terms of segment, a type of neighborhood. In section 8 we indicate the character of a "sequential theory"* based on the same postulates, which emerges by the adoption of a definition of limiting element in terms of "limit of a sequence," as is done in the first stage of the paper by Fréchet. This form of definition is used also by Hedrick, and in the first part of the paper by Hildebrandt. Finally, in section 9, we define a "sequential property" for classes of elements, the presence of which, in addition to Postulates I–III, is a necessary and sufficient condition for the equivalence of the two theories considered. It is shown that simply ordered classes exist which do not possess this property.† A fourth postulate, sufficient to secure the sequential property, is discussed in relation to certain fundamental assumptions that have occurred in discussions of the linear continuum and of general analysis.

2. Definitions in a system $(\mathfrak{Q}; B)$.

The system $(\mathfrak{Q}; B)$ with which we are concerned consists of a class $\mathfrak{Q}$ of elements q of any kind whatever, and a relation B defined for triads of elements of $\mathfrak{Q}$. The relation B is said to be defined for a class $\mathfrak{Q}$ if for every three elements q_1, q_2, and q_3 (distinct or not) the question "does the relation $B_{q_1 q_2 q_3}$ hold" has a definite answer, yes or no. A simple instance of such a system is obtained if we take for $\mathfrak{Q}$ the class of all points on a line, and let B denote the ordinary relation of "betweenness," so that $B_{q_1 q_2 q_3}$ may be read "q_2 is between q_1 and q_3."

The following definitions are found convenient in developing a theory pertaining to a system $(\mathfrak{Q}; B)$.

Definition 1. If q_1 and q_2 are such that $B_{q_1 q q_2}$ holds for some q, the class of all elements q such that $B_{q_1 q q_2}$ holds is called a *segment.*

Such a segment is denoted by $\mathfrak{S}_{q_1 q_2}$ or, in case clearness permits, simply by $\mathfrak{S}$.

* T. H. Hildebrandt (*loc. cit.*, p. 239) uses the term "sequential continuity" with respect to functions on a range where limiting element has a definition of this form.

† In the Bulletin of the American Mathematical Society, vol. 17, (1911), p. 538, we gave a brief forecast of the principal features of the paper cited above on iterated limits. The closing sentence of that forecast states that the treatment is available for any class having linear order. The statement is exact, but the application then in view would give a theory for linear sets substantially equivalent to the "sequential" theory for such sets indicated in the present paper. This result is attained by assigning to each element not approachable by a sequence a neighborhood consisting of the element itself, thus making it an isolated element also in the neighborhood theory. In the last section of the present paper we indicate a restriction on linear sets sufficient to render the neighborhood theory of this paper a special case under the general theory of the paper cited above.

DEFINITION 2. An element q is a *limiting element of a subclass* $\mathfrak{P}$ of $\mathfrak{Q}$ if every segment containing q contains an element p of $\mathfrak{P}$ distinct from q.

DEFINITION 3. A subclass $\mathfrak{P}$ is *closed* if every limiting element of $\mathfrak{P}$ is in $\mathfrak{P}$.

DEFINITION 4. A subclass $\mathfrak{P}$ is *dense in itself* if every element of $\mathfrak{P}$ is a limiting element of $\mathfrak{P}$.

DEFINITION 5. A subclass $\mathfrak{P}$ is *compact* if every infinite subclass of $\mathfrak{P}$ has a limiting element.*

DEFINITION 6. An element q is *interior* to a subclass $\mathfrak{P}$ if there is a segment containing q and contained in $\mathfrak{P}$.

DEFINITION 7. An element q is a *limit of a sequence* $\{ q_n \}$ if for every segment $\mathfrak{S}$ containing q there is an $n_{\mathfrak{S}}$ such that, for $n > n_{\mathfrak{S}}$, q_n is in $\mathfrak{S}$.

DEFINITION 8. The *derived class* of a class $\mathfrak{P}$ is the class of all limiting elements of $\mathfrak{P}$.

DEFINITION 9. The *frontier class* of a class $\mathfrak{P}$ is the class of all elements in $\mathfrak{P}$ or the derived class of $\mathfrak{P}$ that are not interior to $\mathfrak{P}$.

3. THEOREMS REQUIRING NO RESTRICTIONS ON THE SYSTEM.

The theorems of the present section hold for any system of the type $(\mathfrak{Q}; B)$ in the absence of any restrictions whatever.† The proofs, being entirely obvious, are omitted.

THEOREM I. *If $\mathfrak{L}$ is the derived class of $\mathfrak{P}$, then the class $\mathfrak{P} + \mathfrak{L}$ is closed.*

THEOREM II. *The frontier class of any class is closed.*

THEOREM III. *If q is a limiting element of a subclass $\mathfrak{P}$ it is a limiting element of every class containing $\mathfrak{P}$.*‡

THEOREM IV. *A class $\mathfrak{P}$ is compact if and only if every subclass of $\mathfrak{P}$ is compact.*§

THEOREM V. *A class formed of a finite number of compact classes is compact.*

THEOREM VI. *A sequence formed by repeating a single element has that element as limit.*

THEOREM VII. *If a sequence has a limit every infinite subsequence has the same limit.*

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* Fréchet, loc. cit., p. 6.

† It is common, under similar circumstances, to state theorems as corollaries of definitions. We desire to emphasize the fact that the office of a definition is to conserve economy in the use of language, and that it is, in reality, a part of the statement of every proposition which involves the term it defines. The validity of a theorem depends on the definitions of its terms only in the same sense that it depends on its own hypothesis.

‡ A proposition of this form is used by F. Riesz as a postulate in his paper on *Stetigkeitsbegriff und abstracte Mengenlehre*, Atti del IV° Congresso Internazionale dei Matematici, Roma, 1908, vol. 2, pp. 18–24. Propositions of the form of his other postulates occur as theorems in section 7.

§ This theorem and the next are given by Fréchet, *loc. cit.*, p. 7. Theorems VI and VII are similar in form to postulates used by Fréchet. Further reference to his work is made in section 7.

It should be understood that the limit of a sequence need not be unique. We are able to prove this property only after the adoption of Postulate III.

THEOREM VIII. *If q is interior to $\mathfrak{P}$, all but a finite number of terms of any sequence whose limit is q are interior to $\mathfrak{P}$.**

THEOREM IX. *If there exists a sequence of distinct elements of $\mathfrak{P}$ having q as limit, q is a limiting element of $\mathfrak{P}$.*

We note that the converse of this proposition holds only in a very restricted system. It is because of this that we find a divergence of the "neighborhood" and the "sequential" theories. We discuss this question more fully in the last two sections.

4. THEOREMS IN A COMPOSITE SYSTEM.

From two systems $(\mathfrak{Q}'; B')$ and $(\mathfrak{Q}''; B'')$ of the type under consideration we derive a system $(\mathfrak{Q}; B)$ of the same type. The class $\mathfrak{Q}$ is the product class $\mathfrak{Q}'\,\mathfrak{Q}''$, consisting of all elements q of the form $q = q'\,q''$, where q' is in $\mathfrak{Q}'$ and q'' is in $\mathfrak{Q}''$. The element q is not to be regarded as a product, but simply as the pair $q'\,q''$. For three elements q_1, q_2, q_3 the relation $B_{q_1 q_2 q_3}$ holds if and only if $q_1 = q_1'\,q_1''$, $q_2 = q_2'\,q_2''$ and $q_3 = q_3'\,q_3''$, and the relations $B'_{q_1' q_2' q_3'}$ and $B''_{q_1'' q_2'' q_3''}$ hold. Thus $(\mathfrak{Q}; B)$ is a definite system of the required type, and is called the composite system of the systems $(\mathfrak{Q}'; B')$ and $(\mathfrak{Q}''; B'')$. Similarly any number of systems of this type give rise to a definite composite system.†

A system $(\mathfrak{Q}; B)$ may fulfil the following postulate:

POSTULATE I. *If q is an element of $\mathfrak{Q}$ there exists a segment containing q.*

If the relation B is interpreted as ordinary betweenness, and fulfils conditions sufficient to render $\mathfrak{Q}$ a simply ordered class, then the addition of this postulate is equivalent to the assumption that there is no first or last element. This is not, in effect, any restriction; for any simply ordered class may be regarded as a subclass of a class having no extreme elements. Moreover, if we so modify the relation B that $B_{q q q_1}$ and $B_{q_1 q q}$ shall hold in case q is an extreme element, while it remains otherwise as ordinary betweenness, then we find that Postulate I is no restriction on a simply ordered class.

If each of two or more systems fulfil Postulate I their composite system does also. And if a composite system fulfils this postulate so must every component system, providing that $\mathfrak{Q}$ in the composite system contains at least one element. Excluding the trivial case just provided against, we may say that Postulate I is invariant under composition of systems.

The theorems of this section are, for the sake of clearness and brevity, stated for a system $(\mathfrak{Q}; B)$, composite of two systems, $(\mathfrak{Q}'; B')$ and $(\mathfrak{Q}''; B'')$.

* E. R. Hedrick, loc. cit., p. 286.

† Compare T. H. Hildebrandt, loc. cit., p. 250.

The notations $\mathfrak{P}$, $\mathfrak{R}$, etc., indicate subclasses of $\mathfrak{Q}$, and with suitable accents they denote subclasses of $\mathfrak{Q}'$ and $\mathfrak{Q}''$. The manner of extending to the case of a composite system of several systems is entirely obvious.

THEOREM X. *A class $\mathfrak{P}$ is a segment if and only if there exist segments $\mathfrak{S}'$ and $\mathfrak{S}''$ such that $\mathfrak{P}$ is the product class $\mathfrak{S}'\mathfrak{S}''$.*

This proposition is obviously true, even for systems in which Postulate I is not assumed to hold.

THEOREM XI. *If $\mathfrak{P}$ is the product class $\mathfrak{P}'\mathfrak{P}''$ and if $q = q'q''$, then a necessary and sufficient condition that q shall be a limiting element of $\mathfrak{P}$ is that q' is a limiting element of $\mathfrak{P}'$ and q'' is in $\mathfrak{P}''$, or q'' is a limiting element of $\mathfrak{P}''$ and q' is in $\mathfrak{P}'$, or q' and q'' are limiting elements respectively of $\mathfrak{P}'$ and $\mathfrak{P}''$.*

That the condition is sufficient is easily seen, without the use of the postulate. The necessity of the condition may be shown to involve the postulate as follows: Let q' be in a segment containing no other element, and suppose q'' is not in any segment; then q is not in any segment and is therefore a limiting element of every $\mathfrak{P}$, but q' is not a limiting element of any $\mathfrak{P}'$. By considering $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, where $\mathfrak{P}'$ does not contain q', we see that the condition is not necessary. The proof of the necessity of the condition is very simple if Postulate I is fulfilled.

THEOREM XII. *If $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, then $\mathfrak{P}$ is closed if both $\mathfrak{P}'$ and $\mathfrak{P}''$ are closed, and, in case $\mathfrak{P}$ contains at least one element, both $\mathfrak{P}'$ and $\mathfrak{P}''$ are closed if $\mathfrak{P}$ is closed.*

The first part of this theorem involves the necessity of the condition in Theorem XI, and therefore the use of the postulate, but the second part of the theorem involves only the sufficiency of the condition, and can be proved for an unrestricted system.

THEOREM XIII. *If $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, then $\mathfrak{P}$ is dense in itself if and only if one of the component classes, $\mathfrak{P}'$ or $\mathfrak{P}''$, is dense in itself.*

The proof may be arranged, without use of the postulate, as follows: Suppose $\mathfrak{P}'$ is dense in itself. An element p is of the form $p'p''$, and a segment $\mathfrak{S}$ containing p is of the form $\mathfrak{S}'\mathfrak{S}''$, where $\mathfrak{S}'$ contains p' and $\mathfrak{S}''$ contains p''. $\mathfrak{S}'$ contains a p_1' distinct from p', and $p_1 = p_1'p''$ is in $\mathfrak{S}$ and distinct from p. Thus every p is seen to be a limiting element of $\mathfrak{P}$, and $\mathfrak{P}$ is dense in itself. Now suppose neither $\mathfrak{P}'$ nor $\mathfrak{P}''$ is dense in itself, then there exists p' in a segment $\mathfrak{S}'$ containing no other element of $\mathfrak{P}'$, and there exists p'' in a segment $\mathfrak{S}''$ containing no other element of $\mathfrak{P}''$, thus the element $p = p'p''$ is in the segment $\mathfrak{S} = \mathfrak{S}'\mathfrak{S}''$ which contains no other element of $\mathfrak{P}$. This element p is not a limiting element of $\mathfrak{P}$, so that $\mathfrak{P}$ is not dense in itself.

THEOREM XIV. (*a*) *If $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, then if $\mathfrak{P}$ is compact both $\mathfrak{P}'$ and $\mathfrak{P}''$ are compact, provided that each contains at least one element.*

(*b*) *If $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ and $\mathfrak{P}'$ and $\mathfrak{P}''$ are both compact, then every infinite product class in $\mathfrak{P}$ has a limiting element.*

The proof of (a) requires the use of Postulate I, but (b) holds for any system. It will be observed that the compactness of the two component classes does not imply the compactness of $\mathfrak{P}$, except in case of a more restricted system.*

THEOREM XV. *If $\mathfrak{P} = \mathfrak{P}' \mathfrak{P}''$ and $q = q' q''$, then q is interior to $\mathfrak{P}$ if and only if q' and q'' are interior to $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively.*

This theorem clearly does not involve Postulate I.

THEOREM XVI. *If $q = q' q''$, then a necessary and sufficient condition that q shall be the limit of the sequence $\{ q_n \}$ is that the sequences $\{ q_n' \}$ and $\{ q_n'' \}$, where $q_n = q_n' q_n''$ for every n, shall have respectively the limits q' and q''.*

As in Theorem XI, the necessity of the condition involves the postulate, while the sufficiency can be proved for an unrestricted system. The proof, being very simple, is omitted.

5. LIMITS AND CONTINUITY OF FUNCTIONS.

Functions defined on subclasses of $\mathfrak{Q}$ are denoted by the letters μ, φ, θ, etc. A function μ is said to be defined on a subclass $\mathfrak{P}$ if to every element p there corresponds a definite function value μ_p, which is a real number, or $+\infty$ or $-\infty$. With respect to a given function μ on a given class $\mathfrak{P}$ we consider, relative to a definite limiting element l of $\mathfrak{P}$, three symbols,

$$\lim_{p \to l} \mu_p, \quad \overline{\lim_{p \to l}}\, \mu_p, \quad \underline{\lim_{p \to l}}\, \mu_p,$$

which denote respectively the limit, the upper limit and the lower limit of μ as p approaches l. In this section 5, the symbol $p \to l$ under "lim" is omitted; it is to be understood in every case that we mean the limit as p approaches l.

DEFINITION 10. Following are the conditions under which the *limit*, the *upper limit* and the *lower limit* of μ on $\mathfrak{P}$ exist at a given limiting element l.

(a) $\lim \mu_p = a$ is equivalent to the condition: For every e there exists a segment $\mathfrak{S}_e$ containing l such that for every p in $\mathfrak{S}_e$ distinct from l we have $| \mu_p - a | \leqq e$.†

(b) $\overline{\lim}\, \mu_p = a$ is equivalent to the two conditions: (1) For every e there exists an $\mathfrak{S}_e$ containing l such that if p is in $\mathfrak{S}_e$ and distinct from l then $\mu_p \leqq a + e$. (2) For every $\mathfrak{S}$ containing l and every e there exists a p in $\mathfrak{S}$ distinct from l such that $\mu_p \geqq a - e$.

(c) $\overline{\lim}\, \mu_p = +\infty$ is equivalent to the condition: For every $\mathfrak{S}$ containing l and every a there is a p in $\mathfrak{S}$ distinct from l such that $\mu_p > a$.

* See Theorem VIII (d) of chapter II of the paper on *Iterated Limits in General Analysis*, loc. cit. In section 9 of the present paper conditions sufficient to validate such a theorem are stated for a system ($\mathfrak{Q}$; B).

† The letter a denotes a real number, while e invariably denotes a positive real number.

(*d*) $\overline{\lim}\, \mu_p = -\infty$ and $\lim \mu_p = -\infty$ are each equivalent to the condition: For every a there exists an $\mathfrak{S}_a$ containing l such that if p is in $\mathfrak{S}_a$ and distinct from l then $\mu_p < a$.

(*e*) Conditions for $\underline{\lim}\, \mu_p = a$, $\underline{\lim}\, \mu_p = -\infty$ are analogous to conditions in (*b*) and (*c*), while conditions for $\underline{\lim}\, \mu_p = +\infty$ and $\lim \mu_p = +\infty$ are analogous to (*d*).

THEOREM XVII. *If a function μ defined on $\mathfrak{P}$ is such that for a given limiting element l of $\mathfrak{P}$ there exists an a such that for every p distinct from l we have $|\mu_p| \leqq a$, then there exist uniquely $\bar{a}$ and $\underline{a}$ such that*

$$\overline{\lim_{p \to l}}\, \mu_p = \bar{a}, \quad \underline{\lim_{p \to l}}\, \mu_p = \underline{a}.$$

For each $\mathfrak{S}$ containing l let $\bar{a}_{\mathfrak{S}}$ be the least upper bound of μ_p for all elements p in $\mathfrak{S}$ distinct from l. It is not difficult to see that the greatest lower bound of $\bar{a}_{\mathfrak{S}}$ for all segments $\mathfrak{S}$ containing l is the $\bar{a}$ required. Similarly the existence of $\underline{a}$ may be shown. The proof depends on Postulate I.

If in the hypothesis of this theorem we remove the restriction that μ is bounded on the range obtained by omitting the element l from $\mathfrak{P}$, then the least upper bound of μ_p on a given $\mathfrak{S}$, excluding $p = l$, may be $+\infty$ or $-\infty$, so that the greatest lower bound of these least upper bounds may be $+\infty$ or $-\infty$. But in any case the latter quantity definitely exists, and is the upper limit of μ at l. Since a similar procedure gives the lower limit of μ at l, we have the following corollary.

COROLLARY. *For every μ defined on $\mathfrak{P}$ and for every limiting element l of $\mathfrak{P}$ the upper limit of μ at l and the lower limit of μ at l exist.*

But making use of Postulate I it is not possible to prove that $\overline{\lim}\, \mu_p \geqq \underline{\lim}\, \mu_p$, and the existence of a limit for μ at a limiting element l neither implies nor is implied by the equality of the upper and lower limits of μ at l. Furthermore, the limit of a function at a limiting element, when it exists, is not necessarily unique. These points may be established by the following examples.

EXAMPLE 1. Let $\mathfrak{Q}$ consist of the points (x, y) of a plane, and let $B_{q_1 q_2 q_3}$ hold if the three points are on a line and q_2 is between q_1 and q_3 in the usual sense. Postulate I is fulfilled. Let $\mathfrak{P}$ be the class of points in the first quadrant, and let μ be defined on $\mathfrak{P}$ so that for $p = (x, y)$ we have $\mu_p = y/x$, then for $l = (0, 0)$ we have $\overline{\lim}\, \mu_p = 0$ and $\underline{\lim}\, \mu_p = +\infty$, while for every positive a we have the conditions fulfilled for $\lim \mu_p = a$, and also we have $\lim \mu_p = +\infty$.

EXAMPLE 2. With the same system $(\mathfrak{Q}; B)$ and the same class $\mathfrak{P}$ as above, take μ on $\mathfrak{P}$ so that for $p = (x, y)$ we have μ_p equal to unity if $x = y$, equal to one half if x is rational and greater than y, equal to zero if x is irrational and greater than y, equal to 2 if x is rational and less than y, and equal to 3 if x

is irrational and less than y. We now have a unique limit, unity, at the origin, while the upper limit is one half and the lower limit is 2.

EXAMPLE 3. Again with the same system and the same class $\mathfrak{P}$, take μ on $\mathfrak{P}$ so that for $p = (x, y)$ we have μ_p equal to unity for x rational, equal to zero for x irrational and equal to or greater than y, and equal to 2 for x irrational and less than y. The upper limit and the lower limit at the origin have the same value, unity, while the limit does not exist.

We have, however, the theorem:

THEOREM XVIII. *If μ is defined on $\mathfrak{P}$, then μ can have no limit at a given limiting element l of $\mathfrak{P}$ which is greater than the lower limit of μ at l or less than the upper limit of μ at l.*

As an immediate corollary we have

COROLLARY. *If the upper and lower limits are equal, then if a limit exists it is unique and equal to the upper and lower limits.*

For further theorems we require a second postulate, which may conveniently take the following form:

POSTULATE II. *If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are segments containing q, then there is a common subclass of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ which is a segment containing q.*

This condition is clearly invariant under composition of systems, provided that the composite class contains at least one segment. Several theorems result from this postulate.

THEOREM XIX. *If μ is defined on $\mathfrak{P}$, then for every limiting element l of $\mathfrak{P}$ we have $\underline{\lim}_{p \to l} \mu_p \leq \overline{\lim}_{p \to l} \mu_p$.*

THEOREM XX. *For a given μ on $\mathfrak{P}$ and a given limiting element l of $\mathfrak{P}$, a necessary and sufficient condition for the existence of a limit of μ at l is the equality of the upper and lower limits of μ at l.*

COROLLARY. *If a limit of μ at l exists, it is unique and equal to the upper and lower limits of μ at l.*

THEOREM XXI. *If μ is defined on $\mathfrak{P}$ and l is a limiting element of $\mathfrak{P}$, then the following two conditions are necessary and sufficient for the existence of a finite limit of μ at l:*

1. *In every segment containing l there is a p distinct from l such that μ_p is finite.*

2. *For every e there is an $\mathfrak{S}_e$ containing l such that, if p_1 and p_2 are in $\mathfrak{S}_e$ and distinct from l, then $|\mu_{p_1} - \mu_{p_2}| \leqq e$.*

The necessity of these conditions is obvious, and the sufficiency is easily established by showing that the upper limit and the lower limit cannot be distinct.

DEFINITION 11. The function μ is *continuous* on $\mathfrak{P}$ if μ_p is finite for every p and if for every limiting element l of $\mathfrak{P}$ that lies in $\mathfrak{P}$ we have $\lim \mu_p = \mu_l$.

DEFINITION 12. The function μ is *extensibly continuous* on $\mathfrak{P}$ if μ is con-

tinuous on $\mathfrak{P}$ and if for every limiting element l of $\mathfrak{P}$ not in $\mathfrak{P}$ we have $\lim \mu_p$ existing and finite.*

THEOREM XXII. *If $\mathfrak{L}$ is the derived class of $\mathfrak{P}$ and μ is defined on $\mathfrak{P}$ and φ is defined on $\mathfrak{P} + \mathfrak{L}$ such that for every p we have $\mu_p = \varphi_p$ and for every l not in $\mathfrak{P}$ we have $\overline{\lim}\, \mu_p = \varphi_l$, then if μ is extensibly continuous on $\mathfrak{P}$, φ is extensibly continuous on $\mathfrak{P} + \mathfrak{L}$.*

Since $\mathfrak{P} + \mathfrak{L}$ is closed, on this class extensible continuity is not different from ordinary continuity. For every l and e there exists an $\mathfrak{S}_e$ containing l such that for p in $\mathfrak{S}_e$ we have $|\varphi_p - \varphi_l| \leqq e/2$. For any l_1 in $\mathfrak{S}_e$ there is an $\overline{\mathfrak{S}}_e$ containing l_1 such that for p in $\overline{\mathfrak{S}}_e$ we have $|\varphi_{l_1} - \varphi_p| \leqq e/2$, and since $\mathfrak{S}_e$ and $\overline{\mathfrak{S}}_e$ have a common p, we see from these conditions that for any q of $\mathfrak{P}$ or $\mathfrak{L}$ in $\mathfrak{S}_e$ we have $|\varphi_q - \varphi_l| \leqq e$.

6. FUNCTIONS ON A COMPOSITE RANGE.

For theorems on iterated limits we consider functions on a composite range. It is sufficient for our purpose to consider a system $(\mathfrak{Q}; B)$, composite of two systems $(\mathfrak{Q}'; B')$ and $(\mathfrak{Q}''; B'')$, each fulfilling Postulates I and II. The propositions of the present section relate to a function μ defined on a subclass $\mathfrak{P}$ of $\mathfrak{Q}$ of the form $\mathfrak{P}'\mathfrak{P}''$, where $\mathfrak{P}'$ and $\mathfrak{P}''$ are subclasses of $\mathfrak{Q}'$ and $\mathfrak{Q}''$ respectively. The notations l, l', l'' invariably represent limiting elements respectively of $\mathfrak{P}$, $\mathfrak{P}'$ and $\mathfrak{P}''$. The definite significance of these notations is presupposed in the statement of each theorem.

THEOREM XXIII. *If $l = l'l''$, then*

$$(a) \qquad \lim_{p''\to l''} \underline{\lim}_{p'\to l'} \mu_{p'p''} \leqq \lim_{p''\to l''} \overline{\lim}_{p'\to l'} \mu_{p'p''} \leqq \overline{\lim}_{p''\to l''} \overline{\lim}_{p'\to l'} \mu_{p'p''},$$

$$(b) \qquad \lim_{p''\to l''} \underline{\lim}_{p'\to l'} \mu_{p'p''} \leqq \overline{\lim}_{p''\to l''} \underline{\lim}_{p'\to l'} \mu_{p'p''} \leqq \overline{\lim}_{p''\to l''} \overline{\lim}_{p'\to l'} \mu_{p'p''},$$

$$(c) \qquad \underline{\lim}_{p\to l} \mu_p \leqq \lim_{p''\to l''} \underline{\lim}_{p'\to l'} \mu_{p'p''} \leqq \overline{\lim}_{p''\to l''} \overline{\lim}_{p'\to l'} \mu_{p'p''} \leqq \overline{\lim}_{p\to l} \mu_p.$$

(*d*) *If* $\lim \mu_p$ *exists then*

$$\lim_{p''\to l''} \underline{\lim}_{p'\to l'} \mu_{p'p''} = \lim_{p''\to l''} \overline{\lim}_{p'\to l'} \mu_{p'p''} = \lim_{p\to l} \mu_p.$$

Propositions (*a*) and (*b*) are sufficiently obvious applications of Theorem XIX. Proposition (*d*) is a corollary of (*c*), which may be established as follows: We show that the assumption that the lower limit is greater than the iterated lower limit leads to a contradiction. If we make this assumption, two cases

* In the paper on *Iterated Limits in General Analysis*, loc. cit., we show that in many applications this condition is equivalent to uniform continuity. See especially chapter IV, theorems V, XIII, XIV and XV.

may arise, the lower limit may be a finite number, or $+\infty$. Suppose

$$\varliminf_{p \to l} \mu_p = a, \qquad \varliminf_{p'' \to l''} \varliminf_{p' \to l'} \mu_{p' p''} = a - 4e.$$

Let φ be a function on $\mathfrak{P}''$ such that for every p'' we have

$$\varliminf_{p' \to l'} \mu_{p' p''} = \varphi_{p''}.$$

There is an $\mathfrak{S}_e$ containing l such that, if p is distinct from l in $\mathfrak{S}_e$, $\mu_p \geqq a - e$. This $\mathfrak{S}_e$ is of the form $\mathfrak{S}' \mathfrak{S}''$, and there exist p'' in $\mathfrak{S}''$ distinct from l'' such that $\varphi_{p''} \leqq a - 3e$ and p' in $\mathfrak{S}'$ distinct from l' such that $\mu_{p' p''} \leqq \varphi_{p''} + e$. We have, therefore, $\mu_{p' p''} \leqq a - 2e$, and the contradiction is found in the fact that $p = p' p''$ is in $\mathfrak{S}_e$. If we suppose the lower limit is $+\infty$ and the iterated lower limit is finite, similar reasoning leads to a contradiction.

THEOREM XXIV. (*a*) *If l' is such that for every l of the form $l = l' p''$ we have*

$$\lim_{p \to l} \mu_p = \varphi_{p''},$$

where φ is a finite valued function on $\mathfrak{P}''$, then φ is continuous on $\mathfrak{P}''$.

(*b*) *If l' is such that for every l of the form $l = l' q''$, q'' being in $\mathfrak{P}'' + \mathfrak{L}''$, we have*

$$\lim_{p \to l} \mu_p = \varphi_{q''},$$

where φ is a finite valued function on $\mathfrak{P}'' + \mathfrak{L}''$, then φ is extensibly continuous on $\mathfrak{P}'' + \mathfrak{L}''$.

(*c*) *If μ is continuous on $\mathfrak{P}$, then, for every p', $\mu_{p'}$ is continuous on $\mathfrak{P}''$ and, for every p'', $\mu_{p''}$ is continuous on $\mathfrak{P}'$.**

(*d*) *If μ is extensibly continuous on $\mathfrak{P}$, then, for every p', $\mu_{p'}$ is extensibly continuous on $\mathfrak{P}''$ and, for every p'', $\mu_{p''}$ is extensibly continuous on $\mathfrak{P}'$.*

These propositions may be regarded as corollaries of those of the theorem just preceding.

In the remaining theorems of this section we employ notions analogous to that of uniform convergence of a sequence of functions, and to that of a family of equally continuous functions.†

DEFINITION 13. If φ is defined and finite on $\mathfrak{R}''$, a subclass of $\mathfrak{P}''$, $\mu_{p'}$ is said to *approach φ uniformly on $\mathfrak{R}''$* as p' approaches l' if for every e there exists an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ distinct from l' and p'' is in $\mathfrak{R}''$ we have $|\mu_{p' p''} - \varphi_{p''}| \leqq e$. The notation for this condition is $\lim_{p' \to l'} \mu_{p'} = \varphi\,(\mathfrak{R}'')$.

* The notation $\mu_{p'}$ indicates a function on $\mathfrak{P}''$, the argument p' being fixed while p'' ranges over the class $\mathfrak{P}''$.

† Fréchet, loc. cit., p. 10.

DEFINITION 14. The function μ is *continuous on* $\mathfrak{P}'$ *uniformly on* $\mathfrak{P}''$ if for every l' in $\mathfrak{P}'$ we have $\lim\limits_{p' \to l'} \mu_{p'} = \mu_{l'}(\mathfrak{P}'')$.

DEFINITION 15. The function μ is *extensibly continuous on* $\mathfrak{P}'$ *uniformly on* $\mathfrak{P}''$ if it is continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ and if for every l' not in $\mathfrak{P}'$ there is a finite valued function φ on $\mathfrak{P}''$ such that we have $\lim\limits_{p' \to l'} \mu_{p'} = \varphi(\mathfrak{P}'')$.

THEOREM XXV. *If $l = l'\, l''$ and $\mathfrak{R}''$ is the class of all elements p'' in some fixed $\mathfrak{S}''$ containing l'', then if φ is a finite valued function on $\mathfrak{P}''$ such that*

$$\lim_{p' \to l'} \mu_{p'} = \varphi(\mathfrak{R}'')$$

we have

$$\underline{\lim}_{p \to l}\, \mu_p = \underline{\lim}_{p'' \to l''}\, \varphi_{p''}, \qquad \overline{\lim}_{p \to l}\, \mu_p = \overline{\lim}_{p'' \to l''}\, \varphi_{p''}, \tag{1}$$

provided that in case l' is in $\mathfrak{P}'$ we have the additional hypothesis

$$\underline{\lim}_{p'' \to l''}\, \varphi_{p''} \leqq \underline{\lim}_{p'' \to l''}\, \mu_{l'\, p''}, \qquad \overline{\lim}_{p'' \to l''}\, \mu_{l'\, p''} \leqq \overline{\lim}_{p'' \to l''}\, \varphi_{p''},$$

and in case l'' is in $\mathfrak{P}''$ the hypothesis

$$\underline{\lim}_{p'' \to l''}\, \varphi_{p''} \leqq \varphi_{l''} \leqq \overline{\lim}_{p'' \to l''}\, \varphi_{p''}.*$$

Denote the lower limits in (1) by L and L''. We prove $L = L''$ for the case when l' is not in $\mathfrak{P}'$ and l'' is not in $\mathfrak{P}''$. That L is not greater than L'' follows from Theorem XXIII, (*c*). Now suppose that L'' is the greater. Consider first the case when $L'' = a$, and let $L = a - 4e$. There exists an $\mathfrak{S}''_e$ containing l'' and contained in the fixed $\mathfrak{S}''$ such that if p'' is in $\mathfrak{S}''_e$ we have $\varphi_{p''} \geqq a - e$, and there exists $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ and p'' in $\mathfrak{S}''_e$ we have $|\mu_{p'\, p''} - \varphi_{p''}| \leqq e$, whence $\mu_{p'\, p''} \geqq a - 2e$. Also, in every $\mathfrak{S}$ containing l there is a p such that $\mu_p \leqq a - 3e$. We find a contradiction when $\mathfrak{S} = \mathfrak{S}'_e\, \mathfrak{S}''_e$. Consider now $L'' = +\infty$ and $L = \underline{a}$. For a given a there is an $\mathfrak{S}''_a$ containing l'' and contained in $\mathfrak{S}''$ such that if p'' is in $\mathfrak{S}''_a$ we have $\varphi_{p''} \geqq a$, and for a given e there is an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ and p'' is in $\mathfrak{S}''_a$ we have $|\mu_{p'\, p''} - \varphi_{p''}| \leqq e$, whence $\mu_{p'\, p''} \geqq a - e$. Also, in every $\mathfrak{S}$ containing l there is a p such that $\mu_p \leqq \underline{a} + e$. We find a contradiction when $\mathfrak{S} = \mathfrak{S}'_e\, \mathfrak{S}''_a$ and $a > \underline{a} + 2e$.

The equality of the upper limits for this case follows by analogy, and in case the restriction as to l is removed the additional hypotheses then provided clearly validate the conclusions.

Similar considerations lead also to the following theorem, which gives similar results from milder hypotheses.

* We omit the corresponding propositions involving uniform approach to $+\infty$ and to $-\infty$, which are stated in the American Journal of Mathematics (loc. cit., chapter III, theorem X), and are valid here.

THEOREM XXVI. *If $l = l' l''$ and l'' is not in $\mathfrak{P}''$, and if φ and θ are finite valued functions on $\mathfrak{P}''$, then*

(*a*) *A sufficient condition for*

$$\overline{\lim_{p \to l}}\, \mu_p \leqq \overline{\lim_{p'' \to l''}}\, \varphi_{p''}$$

is that there exist an $\mathfrak{S}''$ containing l'' such that for every e there is an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ and p'' is in $\mathfrak{S}''$ then $\mu_{p' p''} \leqq \varphi_{p''} + e$:

(*b*) *A sufficient condition for*

$$\underline{\lim_{p \to l}}\, \mu_p \geqq \underline{\lim_{p'' \to l''}}\, \theta_{p''}$$

is that there exist an $\mathfrak{S}''$ containing l'' such that for every e there is an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ and p'' is in $\mathfrak{S}''$ then $\mu_{p' p''} \geqq \theta_{p''} - e$.

THEOREM XXVII. *If $l = l' l''$ and $\mathfrak{S}''$ contains l'' and $\mathfrak{R}''$ consists of all elements p'' in $\mathfrak{S}''$, and if in every $\mathfrak{S}'$ containing l' there is a p' distinct from l' such that, for some a,*

$$\lim_{p'' \to l''} \mu_{p' p''} = a,$$

and if φ is a finite valued function on $\mathfrak{P}''$ such that

$$\lim_{p' \to l'} \mu_{p'} = \varphi(\mathfrak{R}''),$$

then there is an a_1 such that

$$\lim_{p'' \to l''} \varphi_{p''} = a_1.$$

For by the uniform approach to φ we have for every e an $\mathfrak{S}'_e$ containing l' such that if p' is in $\mathfrak{S}'_e$ distinct from l' and if p'' is in $\mathfrak{S}''$ then

$$(1) \qquad |\mu_{p' p''} - \varphi_{p''}| \leqq e/3.$$

Also there is a p'_1 in $\mathfrak{S}'_e$ such that, taking account of Theorem XXI, there exists an $\mathfrak{S}''_1$ containing l'' such that if p''_1 and p''_2 are in $\mathfrak{S}''_1$ and distinct from l'' we have

$$(2) \qquad |\mu_{p_1' p_1''} - \mu_{p_1' p_2''}| \leqq e/3.$$

Take $\mathfrak{S}''_e$ common to $\mathfrak{S}''$ and $\mathfrak{S}''_1$, then if p''_1 and p''_2 are in $\mathfrak{S}''_e$ we have from (1) and (2) the condition $|\varphi_{p_1''} - \varphi_{p_2''}| \leqq e$, and since condition (1) of Theorem XXI is obviously fulfilled by φ at l'', we have the desired conclusion.

The two following theorems are easily deduced from the preceding. The propositions are true either with or without the words in brackets.

THEOREM XXVIII. *If l' is such that in every $\mathfrak{S}'$ containing l' there is a p' distinct from l' such that $\mu_{p'}$ is [extensibly] continuous on $\mathfrak{P}''$, and if φ is a*

finite valued function on $\mathfrak{P}''$ *such that*

$$\lim_{p' \to l'} \mu_{p'} = \varphi(\mathfrak{P}''),$$

then φ *is* [*extensibly*] *continuous on* $\mathfrak{P}''$.

THEOREM XXIX. *If* μ *is* [*extensibly*] *continuous on* $\mathfrak{P}'$ *uniformly on* $\mathfrak{P}''$, *and if for every* p' $\mu_{p'}$ *is* [*extensibly*] *continuous on* $\mathfrak{P}''$, *then* μ *is* [*extensibly*] *continuous on* $\mathfrak{P}$.

7. THEOREMS REQUIRING A THIRD POSTULATE.

Consider a class $\mathfrak{Q}$ consisting of all integers, positive and negative. Let $B_{q_1 q_2 q_3}$ hold if q_1 and q_3 are integral multiples of 4 and q_2 is between them in the usual sense. The system $(\mathfrak{Q}; B)$ fulfils Postulates I and II, but clearly violates several conditions usually considered as fundamental in the theory of limits. For example, the proposition that no finite class has a limiting element does not hold, since any class containing two successive integers has one of these as a limiting element. Again, the proposition that a sequence can have but one limit does not hold, for the sequence 2, 2, 2 . . . has the limits 1, 2, and 3. These conditions and certain others that may be desirable in extending a theory of limits are fulfilled in a system $(\mathfrak{Q}; B)$ restricted by the following postulate.

POSTULATE III. *If* q_1 *and* q_2 *are distinct elements, each contained in some segment, then there exist segments,* $\mathfrak{S}_1$ *containing* q_1 *and* $\mathfrak{S}_2$ *containing* q_2, *such that* $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$ *contain no common element.*

This postulate, like the first two, is invariant under composition of systems if we exclude the trivial case in which the composite class contains no segment. Also it is obviously fulfilled by any simply ordered class, if the relation B plays the rôle of betweenness.

On the basis of the three postulates our definition of limiting element is found to fulfil the four conditions regarded as fundamental by F. Riesz.* The first is expressed in Theorem III, and the remaining conditions are embodied in the propositions of the following theorem.

THEOREM XXX. (*a*) *If* $\mathfrak{P} = \mathfrak{P}_1 + \mathfrak{P}_2$ *then every limiting element of* $\mathfrak{P}$ *is a limiting element of* $\mathfrak{P}_1$ *or of* $\mathfrak{P}_2$.

(*b*) *No finite class has a limiting element.*

(*c*) *Each limiting element of a class* $\mathfrak{P}$ *is uniquely determined by the totality of all subclasses of* $\mathfrak{P}$ *of which it is a limiting element.*

Proposition (*a*) is seen to depend only on Postulate II. As to (*b*), if $\mathfrak{P}$ consists of a finite set, $p_1, p_2, \cdots, p_n$, then for any q there exist, by Postulates I and III, segments $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n$ containing q such that if $p_i \neq q$ the seg-

* Loc. cit.

ment $\mathfrak{S}_i$ does not contain p_i. By Postulate II we secure a segment $\mathfrak{S}$ contained in all the $\mathfrak{S}_i$ but containing q. Since $\mathfrak{S}$ does not contain any p distinct from q, q is not a limiting element of $\mathfrak{P}$. To prove (*c*), let q_1 and q_2 be two distinct elements, then by Postulates I and III there exist $\mathfrak{S}_1$ containing q_1 and $\mathfrak{S}_2$ containing q_2 such that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ have no common elements. If q_1 and q_2 are limiting elements of $\mathfrak{P}$, they are limiting elements respectively of the two subclasses $\mathfrak{P} - \mathfrak{S}_2$ and $\mathfrak{P} - \mathfrak{S}_1$, but q_1 is not a limiting element of $\mathfrak{P} - \mathfrak{S}_1$, nor q_2 of $\mathfrak{P} - \mathfrak{S}_2$.

THEOREM XXXI. *A sequence cannot have two distinct limits.*

Theorems VI, VII and XXXI show that our definition of limit of a sequence fulfils the conditions used by M. Fréchet on page 5 of his thesis. We can now prove the two following theorems, the first stated after the manner of a theorem by Fréchet, the second after a theorem by Riesz.

THEOREM XXXII. *If every member of a sequence of subclasses of a compact class is closed, contains the succeeding member and contains at least one element, then there is an element common to all members of the sequence.*

THEOREM XXXIII. *If every member of a sequence of infinite subclasses of a compact class contains the succeeding member, the members of the sequence have a common limiting element.*

To prove this last theorem let $\{ \mathfrak{P}_n \}$ be such a sequence of subclasses and let $\{ q_n \}$ be a sequence of distinct elements such that, for every n, q_n is in $\mathfrak{P}_n$. The class $\mathfrak{P}$ of all elements in the sequence $\{ q_n \}$ is infinite and has at least one limiting element q. This q is a limiting element of every class of the sequence $\{ \mathfrak{P}_n \}$, by Theorem XXX; for, any given member of the sequence of classes contains all but a finite number of elements of the sequence of elements.

Theorem XXXII, which does not follow from the work of Fréchet because of the difference in definitions of limiting elements, is an immediate corollary of the theorem just proved.

The three postulates are sufficient also to secure the following form of the Heine-Borel theorem.

THEOREM XXXIV. *If $\{ \mathfrak{P}_n \}$ is a sequence of subclasses of $\mathfrak{Q}$ such that every element of a certain class $\mathfrak{P}$ which is compact and closed is interior to at least one of the $\mathfrak{P}_i$, then there is a finite set of the $\mathfrak{P}_i$ such that every element of $\mathfrak{P}$ is interior to at least one class of the finite set.*

Suppose the theorem is untrue. Then there exists a sequence $\{ p_m \}$ of distinct elements of $\mathfrak{P}$ such that p_m is not interior to $\mathfrak{P}_n$ if $n < m$. The class $\overline{\mathfrak{P}}$ of all elements of the sequence $\{ p_m \}$ has a limiting element p, which is in $\mathfrak{P}$ and therefore interior to some $\mathfrak{P}_i$, say $\mathfrak{P}_{n_1}$. Let $\overline{\mathfrak{P}} = \mathfrak{P}_1 + \mathfrak{P}_2$, where $\mathfrak{P}_1$ is the part interior to $\mathfrak{P}_{n_1}$, and $\mathfrak{P}_2$ is the remainder. $\mathfrak{P}_1$ is finite and has no limiting element. There is a segment $\mathfrak{S}$ containing p and contained in $\mathfrak{P}_{n_1}$,

therefore containing no element of $\mathfrak{P}_2$, so that p is not a limiting element of $\mathfrak{P}_2$, and consequently not of $\overline{\mathfrak{P}}$. Our supposition is therefore false.

The proof of the following theorem, which, in view of Theorem XXX (a), implies Theorem I, is obvious.

THEOREM XXXV. *The derived class of any class is closed.*

To the list of theorems rendered valid by the addition of the third postulate may be added the following propositions on functions.

THEOREM XXXVI. (a) *If $\mathfrak{P}$ is compact and μ is extensibly continuous on $\mathfrak{P}$, then μ is finitely bounded on $\mathfrak{P}$.*

(b) *If $\mathfrak{P}$ is compact, then there exist q_1 and q_2 such that the the least upper bound of μ on $\mathfrak{P}$ is either μ_{q_1} or $\overline{\lim}_{p\to q_1} \mu_p$, and the greatest lower bound of μ on $\mathfrak{P}$ is either μ_{q_2} or $\underline{\lim}_{p\to q_2} \mu_p$.*

(c) *If $\mathfrak{P}$ is compact and closed and μ is continuous on $\mathfrak{P}$, then there exist p_1 and p_2 such that the least upper bound of μ on $\mathfrak{P}$ is μ_{p_1} and the greatest lower bound of μ on $\mathfrak{P}$ is μ_{p_2}.*

Proof of (a). Suppose that μ is not bounded from $+\infty$, then it can easily be shown that there exists a sequence $\{ p_n \}$ of distinct elements such that

$$\lim_{n\to\infty} \mu_{p_n} = +\infty.$$

The elements of this sequence constitute a class having a limiting element, say l. Since μ is extensibly continuous, there is a segment $\mathfrak{S}$ containing l such that for some a and for every p in $\mathfrak{S}$ distinct from l we have $\mu_p < a$. Since $\mathfrak{S}$ must contain an infinite number of elements of the sequence $\{ p_n \}$, we reach a contradiction. Similarly it may be shown that μ is bounded from $-\infty$.

Proof of (b). We consider the cases when the least upper bound of μ on $\mathfrak{P}$ is finite and $+\infty$. In either case there exists an element p such that μ_p is this least upper bound, or there is a sequence $\{ p_n \}$ of distinct elements such that the limit of μ_{p_n} is this least upper bound. If the latter situation prevails, then this sequence gives rise to a limiting element l, and the assumption that the upper limit of μ at l is different from the least upper bound leads obviously to a contradiction. Thus if not a p, then an l will serve as the q_1 required. Similarly we may show the existence of a q_2 meeting the requirements of the theorem.

Proposition (c) is a corollary of (b).

THEOREM XXXVII. *If μ is finitely bounded on $\mathfrak{P}$, then, for a fixed positive a, the set of elements l at which*

$$\overline{\lim_{p\to l}}\, \mu_p - \underline{\lim_{p\to l}}\, \mu_p \geqq a$$

is closed.

Suppose every segment containing l contains an l_1 for which the upper and lower limits of μ differ by a quantity greater than or equal to a. Let the upper limit and lower limit at l be respectively $\bar{a}$ and $\underline{a}$, and suppose that $\bar{a} - \underline{a} < a$. Take e as one sixth of $a - \bar{a} + \underline{a}$, then there is an $\mathfrak{S}_e$ containing l such that if p is in $\mathfrak{S}_e$ and distinct from l we have $\underline{a} - e \leqq \mu_p \leqq \bar{a} + e$, and therefore if p_1 and p_2 are in $\mathfrak{S}_e$ and distinct from l we have $|\mu_{p_1} - \mu_{p_2}| \leqq a - 4e$. But if l_1 is in $\mathfrak{S}_e$ and such that the upper limit and lower limits of μ at l_1 are respectively $\bar{a}_1$ and $\underline{a}_1$, where $\bar{a}_1 - \underline{a}_1 \geqq a$, then we may take p_1 and p_2 in $\mathfrak{S}_e$ distinct from l such that $\mu_{p_1} \geqq \bar{a}_1 - e$ and $\mu_{p_2} \leqq \underline{a}_1 + e$, whence $|\mu_{p_1} - \mu_{p_2}| \geqq a - 2e$, and this affords a contradiction.

8. A sequential theory for a system $(\mathfrak{Q}; B)$.

In the theory of point sets a limiting point of a set may be defined as a point every neighborhood of which contains a point of the set distinct from itself, or as a point which is the limit of a sequence of distinct points of the set. The notion "limit of a sequence" being defined in terms of "neighborhood," the two definitions are known to be equivalent. In the domain of general analysis, where elements of arbitrary character are considered, it may happen that, even though the notions involved in both definitions of limiting element are well defined, and even though either definition may lead to a theory of very considerable extent, the two theories need not be equivalent.

The definition of limiting element we have employed in the foregoing sections is an instance of a "neighborhood" definition, and the theory developed is in terms of "segment," which is an instance of the general notion of neighborhood. The theory might be characterized as a "neighborhood theory" for a system $(\mathfrak{Q}; B)$. But since the notion "limit of a sequence" is defined (Def. 7), we might adopt a "sequential" definition of limiting element and develop a "sequential theory" for a system of this type. Fortunately, however, this theory becomes a special case of a general theory receiving consideration in papers by M. Fréchet,* E. R. Hedrick,† and T. H. Hildebrandt.‡ Our definition of limit of a sequence satisfies all conditions used by Fréchet and Hildebrandt in the purely sequential parts of their papers. Hildebrandt shows that for many of the theorems none of these conditions are necessary. It is obviously true, also, that in this sequential theory we should have the theorem "every derived class is closed," which in the neighborhood theory must be stated, "if every element of a sequence whose limit is q is a limit of a

* The first part of the paper by M. Fréchet referred to above is purely "sequential" in character.

† *On Properties of a Domain for which any Derived Set is Closed*, these Transactions, vol. 12 (1911), pp. 285–294.

‡ *A Contribution to the Foundations of Frechet's Calcul Fonctionnel*, American Journal of Mathematics, vol. 34 (1912), p. 237.

sequence of distinct elements of $\mathfrak{P}$, q is a limit of a sequence of distinct elements of $\mathfrak{P}$." This proposition, with the conditions on "limit" postulated by Fréchet, forms the basis for the first part of the paper by Hedrick.

It appears, then, that a considerable body of "sequential theory" is available for a system $(\mathfrak{Q}; B)$ subject to our three postulates. We show by the following two examples that for such a system theorems of like form in the sequential and neighborhood theories need not be equivalent theorems.

EXAMPLE 4. Let $\mathfrak{Q}$ be the class of all infinite sequences of real numbers. For any three such sequences, $q_1 = q_1^1, q_1^2, q_1^3, \cdots$, $q_2 = q_2^1, q_2^2, q_2^3, \cdots$; and $q_3 = q_3^1, q_3^2, q_3^3, \cdots$, let the relation $B_{q_1 q_2 q_3}$ hold if and only if, for every value of i, we have either $q_1^i < q_2^i < q_3^i$ or $q_3^i < q_2^i < q_1^i$. The three postulates are obviously fulfilled by this system $(\mathfrak{Q}; B)$. Let $\mathfrak{P}$ be the subclass consisting of all infinite sequences of positive real numbers. The element $q = 0, 0, 0, \cdots$ is a limiting element of $\mathfrak{P}$, but q is not the limit of any sequence of elements of $\mathfrak{P}$. For suppose the sequence $\{ p_n \}$ to have the limit q, then the element $p = p^1, p^2, p^3, \cdots$, where $p^i = p_i^i$, with any sequence of negative real numbers, clearly determines a segment such that no element of the sequence $\{ p_n \}$ is in the segment.

The system $(\mathfrak{Q}; B)$ of this example may be regarded as a composite system, composite of a denumerable infinitude of systems $(\mathfrak{Q}^i; B^i)$, in each of which $\mathfrak{Q}^i$ is the class of all real numbers and B^i is the ordinary relation of betweenness.

EXAMPLE 5. Let $\mathfrak{K}$ be a non-denumerable class of elements k to which a definite normal order has been assigned, i. e., let $\mathfrak{K}$ be well-ordered but not capable of a one-to-one correspondence with the class of natural numbers or a subclass of it.* Let $\mathfrak{P}$ consist of all denumerable subclasses of $\mathfrak{K}$, and let $\mathfrak{Q}$ consist of all elements of $\mathfrak{P}$ together with all real numbers except those between zero and one. For this class $\mathfrak{Q}$ we define first a relation of precedence $(<)$. With respect to the real numbers in $\mathfrak{Q}$ the relation precedes shall have the usual significance. Every element p of $\mathfrak{P}$ shall precede all positive real numbers in $\mathfrak{Q}$, and shall follow all other real numbers in $\mathfrak{Q}$. Of two distinct elements, p_1 and p_2, of $\mathfrak{P}$, that one precedes which contains the first element k, according to the normal order of $\mathfrak{K}$, which is in one and only one of the two subclasses p_1 and p_2 of $\mathfrak{K}$. The relation of precedence so defined is transitive, so that the conditions for linear order are fulfilled. It may also be shown that $\mathfrak{Q}$ is dense, and that the pseudo-archimedean postulate used by Veblen is fulfilled.†

Let $B_{q_1 q_2 q_3}$ hold if and only if $q_1 < q_2 < q_3$ or $q_3 < q_2 < q_1$, then it is clear that all three of our postulates are fulfilled by the system $(\mathfrak{Q}; B)$.

* Compare O. Veblen, *Definition in Terms of Order Alone in the Linear Continuum and in Well-ordered Sets*, these Transactions, vol. 6 (1905), p. 169.

† O. Veblen, loc. cit., p. 165.

Now the element $q_0 = 0$ is a limiting element of the subclass $\mathfrak{P}$. For, given any p, a denumerable subclass of $\mathfrak{K}$, we may take p_1 as a subclass of $\mathfrak{K}$ including every k in p and one additional k; then $p_1 < p$. But no sequence $\{ p_n \}$ of elements of $\mathfrak{P}$ can have the limit q_0; for the least common superclass of the subclasses p_i that occur in the sequence is denumerable, and an element p consisting of this and one k in addition will precede every p_i.

It thus appears that the equivalence of the "neighborhood" and the "sequential" definitions of limiting element involves serious restrictions on a system $(\mathfrak{Q}; B)$, in addition to our postulates, and even in addition to the assumption that $\mathfrak{Q}$ is simply ordered, dense, and satisfies the pseudo-archimedean postulate.

9. The sequential property.

We shall say that a subclass $\mathfrak{P}$ of $\mathfrak{Q}$ has the *sequential property* if every subclass of $\mathfrak{P}$ that has a limiting element q contains a sequence of distinct elements whose limit is q. This property is definite for any subclass of a class $\mathfrak{Q}$ for which the two notions, "limiting element of a subclass" and "limit of a sequence" are defined.* In particular, if the idea of neighborhood of an element is available for a class $\mathfrak{Q}$, then definitions of the form of definitions 2 and 7 of section 2 afford a definite sequential property.

Relative to a system of the type $(\mathfrak{Q}; B)$ we consider the following postulates.

Postulate IV. *For every q there exists a sequence $\{ \mathfrak{S}_n \}$ of segments containing q such that, for every segment $\mathfrak{S}$ containing q, there is an $n_{\mathfrak{S}}$ such that, if $n > n_{\mathfrak{S}}$, $\mathfrak{S}_n$ is a subclass of $\mathfrak{S}$.*

Postulate IV′. *For every q there exists a sequence $\{ \mathfrak{S}_n \}$ of segments containing q and having no other common element.*

Each of these postulates is invariant under composition, provided only that the composite system shall possess at least one element. If $\mathfrak{Q}$ is simply ordered and B is the ordinary relation of betweenness, then these two postulates are equivalent, and each is equivalent to the assumption that $\mathfrak{Q}$ has the sequential property. Without the assumption of linear order, it is easily seen that Postulate IV implies I and II, and that III and IV together imply IV′. The illustration at the beginning of section 7 is an instance of a system fulfilling I, II and IV, but not III or IV′. Example 4 fulfils I, II, III and IV′ but not IV. It may be shown that in any system fulfilling Postulates III and IV the class $\mathfrak{Q}$ has the sequential property, and therefore that for such a system the sequential theory and the neighborhood theory are equivalent.

Systems which satisfy Postulates III and IV, and therefore I, II and IV′, are of very frequent occurrence. In the definition of the linear continuum

* The question of a sequential property is obviously trivial in case the sequential definition of limiting element is employed. It is conceivable that various definitions of limiting element may be given, without the use of sequence or neighborhood.

given by O. Veblen, condition A (linear order) and condition E (postulate of uniformity), the latter so modified as to be effective without B, C, and D, and weakened by the omission of properties 1 and 3, are sufficient to secure such a system, where the relation B is ordinary betweenness.* This modified E may be replaced by the stronger "postulate of linearity" which is due to G. Cantor and occurs in E. V. Huntington's exposition of the modern theory of the continuum.† This postulate would require that there exist a denumerable subclass $\mathfrak{R}$ of $\mathfrak{Q}$ such that every segment must contain at least one element of $\mathfrak{R}$, and that $\mathfrak{Q}$ shall be dense.

Turning now to a consideration of the sequential property in relation to a general notion of "neighborhood," of which the notion segment is obviously a special case, we observe that this property is present in systems of much more general character than a system $(\mathfrak{Q}; B)$ fulfilling Postulates III and IV. In the paper on iterated limits in general analysis‡ we take the notion "neighborhood" as undefined, that is, a relation T between subclass and element is undefined but subject to postulates that validate a theory in which a class having the relation T to an element serves as a neighborhood of that element. The postulates, stated in the second chapter of the paper, pertain to a system $(\mathfrak{P}; \mathfrak{U}; T)$, in which $\mathfrak{P}$ is a class of elements, $\mathfrak{U}$ is a class of ideal elements, and T is a relation between subclasses of $\mathfrak{P}$ and elements of the class $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$. We obtain a special instance of a system of this type from a system $(\mathfrak{Q}; B)$ if we set $\mathfrak{P} = \mathfrak{Q}$, let $\mathfrak{U}$ be the null class, and let a class $\mathfrak{R}$ have the relation T to an element p if and only if $\mathfrak{R}$ is a segment containing p. Postulates III and IV on the system $(\mathfrak{Q}; B)$ are sufficient to secure the five postulates used for the system $(\mathfrak{P}; \mathfrak{U}; T)$, in fact, all but the third and fifth result from the type of $(\mathfrak{Q}; B)$, without the use of postulates. Now Theorem IV of the second chapter of the paper cited shows that every subclass has the sequential property; and the proof of this theorem involves only Postulates (I), (II), (III) and (V), while (II) may be dispensed with in case no ideal elements are present.

Other systems in terms of which a system $(\mathfrak{P}; \mathfrak{U}; T)$ may be set up in such manner as to satisfy Postulates (I)–(V), and which, therefore, secure the sequential property, are discussed in the fourth chapter of the same paper. These include cases of neighborhood defined in terms of *voisinage* as used by M. Fréchet,§ and in terms of suitably conditioned K relations of the type used by E. H. Moore‖ and T. H. Hildebrandt.¶

* O. Veblen, loc. cit., p. 166.

† E. V. Huntington, Annals of Mathematics, ser. 2, vol. 7 (1905), p. 15.

‡ Loc. cit.

§ Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 17.

‖ *Introduction to a Form of General Analysis*, part II, p. 125 et seq.

¶ Loc. cit., p. 242.

In his paper on properties of a domain for which any derived set is closed, E. R. Hedrick* reverses the process we have used and defines "neighborhood" in terms of "limit." Assuming a limit relation fulfilling the conditions prescribed by Fréchet, and using a sequential definition of limiting element, he postulates that the fundamental domain is compact and that the derived class of every subclass is closed. He later assumes the "enclosable property," which affords a set of subclasses subject to conditions that validate their use in the sense of neighborhood. The type of neighborhood thus available may be used to set up a new definition of limiting element, in which case the fundamental domain is found to have the sequential property, so that the sequential theory he secures for the domain thus conditioned is equivalent to a neighborhood theory based, in the usual manner, on the type of neighborhood his assumptions afford. In fact, a domain with the enclosable property leads, in a very obvious manner, to a system ($\mathfrak{P}$; $\mathfrak{U}$; T) fulfilling all five of the postulates we have used for such a system.

University of Missouri,
April, 1913.

* Loc. cit., p. 285.

THE SYMBOLICAL THEORY OF FINITE EXPANSIONS*

BY

O. E. GLENN

Introduction.

The first two sections of this paper are the result of an attempt to identify, and state in general terms, certain fundamental common properties of a large number of known types of finite expansions of polynomials in p variables. The number of such expansions which are known and available for such a synthesis is of course large, and only the most salient properties can be comprised under one general theory. The principles proved in these preliminary sections are subjected to verification when it is shown that they hold for a number of new expansions. These are derived in sections 3 to 6 inclusive. They are of considerable interest in themselves, apart from the general theory, to the fundamental laws of which they conform.

The Aronhold symbolical notation† for algebraic forms subject to linear transformations constitutes the basis of the general methods of the paper.

To state briefly our initial point of view; if a function f can be given an expression of the form

$$f = B_0 A_1^m + mB_1 A_1^{m-1} A_2 + \binom{m}{2} B_2 A_1^{m-2} A_2^2 + \cdots + B_m A_2^m,$$

it can be represented symbolically as an mth power, say

$$f = \Xi_A^m = (\Xi_1 A_1 + \Xi_2 A_2)^m = \Xi_1^m A_1^m + m \Xi_1^{m-1} \Xi_2 A_1^{m-1} A_2 + \cdots.$$

But with $\nabla_1 = \Xi_2 (\partial / \partial \Xi_1)$ this may be written

$$f = \Xi_1^m A_1^m + \frac{\nabla_1 \Xi_1^m}{\lfloor 1} A_1^{m-1} A_2 + \frac{\nabla_1^2 \Xi_1^m}{\lfloor 2} A_1^{m-2} A_2^2 + \cdots + \frac{\nabla_1^m \Xi_1^m}{\lfloor m} A_2^m.$$

Now the symbol Ξ_1^m equals B_0. If it prove possible to find a non-symbolical operator ∇ equivalent (§ 1) to ∇_1, then we shall have expressed f in the funda-

* Presented to the Society, April, 1911, and March, 1913.

† Aronhold, Journal für Mathematik, vol. 55 (1858), p. 97.

mental, non-symbolical form

$$f = B_0 A_1^m + \frac{\nabla B_0}{\underline{|1}} A_1^{m-1} A_2 + \frac{\nabla^2 B_0}{\underline{|2}} A_1^{m-2} A_2^2 + \cdots + \frac{\nabla^m B_0}{\underline{|m}} A_2^m .$$

This is what is actually done in what follows.

§ 1. Fundamental Lemma.

Definitions. The general term *function* will be used to indicate an algebraic form in p variables.

By an *expansion* of a function we shall mean a representation of it as a finite series proceeding by a law in powers of one or more arguments, with coefficients dependent upon the coefficients, and possibly upon the variables also (§3), of the original function.

Equivalent operators. In connection with the familiar symbolical representation of a binary form

$$f = a_x^n = (\alpha_1 x_1 + \alpha_2 x_2)^n = \alpha_1^n x_1^n + n\alpha_1^{n-1} \alpha_2 x_1^{n-1} x_2 + \cdots$$
$$= a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \cdots ,$$

we note that the expressions $\alpha_1^n (= a_0)$, $\alpha_1^{n-1} \alpha_2 (= a_1)$, $\cdots$ and linear combinations of these, are the only expressions in the symbolical α_i, which are *defined* in terms of the actual coefficients. We call an expression, as $I = \alpha_1^n + p\alpha_1^{n-2} \alpha_2^2$, the *symbolical equivalent* of the corresponding expression in the a's, $J = a_0 + pa_2$.

The differential operator $\alpha_2 (\partial / \partial\alpha_1)$ is such that when

$$\text{(i)} \qquad \alpha_2 \frac{\partial}{\partial\alpha_1} I = K ,$$

and I is a defined expression, then K is a defined expression. Let the non-symbolical equivalent of K be L, that of I being J as illustrated above. Then the non-symbolical operator

$$O = na_1 \frac{\partial}{\partial a_0} + (n-1) a_2 \frac{\partial}{\partial a_1} + \cdots + a_n \frac{\partial}{\partial a_{n-1}}$$

has the property that*

* For example when I is the particu'ar expression given above, equations (i), (ii) are respectively

$$\alpha_2 \frac{\partial}{\partial\alpha_1} (\alpha_1^n + p\alpha_1^{n-2} \alpha_1^n) = n\alpha_1^{n-1} \alpha_2 + p(n-2) \alpha_1^{n-3} \alpha_2^3 ,$$

$$O(a_0 + pa_2) = na_1 + p(n-2) a_3 .$$

The operands on the left are equivalent, and the right-hand members are also equivalent expressions.

(ii) $$O\,J = L.$$

Two operators, one symbolical and the other non-symbolical, related like $\alpha_2\,(\partial/\partial\alpha_1)$ and O will be called equivalent or isomorphic operators. We shall refer to this important property of equivalence of two operators as the property (A), and we now state it in such general terms as will completely define it for all the symbolical representations of functions that are employed in the sequel.

(A) Let there be given a definite symbolical representation of a function f. A symbolical derivative operator on an expression in the defined combinations of the symbols, which produces a result containing the symbols in defined combinations only, has corresponding to it a non-symbolical derivative operator which carries the non-symbolic equivalent of the first expression into that of the second.

Finite Expansions. Any finite expansion of a function f can be made the basis of a symbolical representation of that function. For, if there exists an expansion of f in powers of a given argument A,

$$f = B_0 + \binom{m}{1} B_1 A + \binom{m}{2} B_2 A^2 + \cdots + B_m A^m,$$

then f may be represented as the mth power of a purely symbolical binomial expression

(1) $$\Xi_1 + \Xi_2 A.$$

That is, if we write $\Xi_{(A)}$ as an abbreviation of this non-homogeneous (as to A) expression, then

(1′) $$f = \Xi_{(A)}^m = (\Xi_1 + \Xi_2 A)^m = \Xi_1^m + m\Xi_1^{m-1}\Xi_2 A + \cdots = B_0 + mB_1 A + \cdots.$$

Then the only defined expressions in the symbols are

$$\Xi_1^{m-k}\Xi_2^k\,(= B_k) \qquad (k = 0, 1, \cdots, m),$$

and linear combinations of these, where, according to our definition of an expansion, the B_k are non-symbolical, determinate expressions in the coefficients and variables of f.

Symbolical basis. In the case of every finite expansion with which we shall deal, (i) the property (A) holds for both operators

$$\nabla_1 = \Xi_2 \frac{\partial}{\partial \Xi_1}, \qquad \nabla_2 = \Xi_1 \frac{\partial}{\partial \Xi_2},$$

and (ii) the non-symbolical equivalents of ∇_i can actually be found, expressed, not in terms of the unknown B_k, but in terms of the coefficients of the known function f, and the known argument A.

When (i), (ii) have actually been verified we say that the existence of a symbolical basis of the expansion, viz.,

$$\Xi_{(A)} = \Xi_1 + \Xi_2 A,$$

has been verified.

Fundamental lemma. For an expansion for which a symbolical basis exists we now prove a lemma. Evidently

$$\nabla_1 \Xi_1^{m-k} \Xi_2^k = (m-k) \Xi_1^{m-k-1} \Xi_2^{k+1} \qquad (k = 0, 1, \cdots, m).$$

Hence (1′) gives

$$(2') \qquad f = \Xi_1^m + \nabla_1 \Xi_1^m A + \frac{\nabla_1^2 \Xi_1^m}{\lfloor 2} A^2 + \cdots.$$

Write ∇ for the non-symbolical equivalent of ∇_1, according to the property (A), and set $\Phi = B_0 (= \Xi_1^m)$. Then from (2′) we have

Lemma. *There exists a function Φ and a differential operator ∇, capable of being represented symbolically by the linear operator*

$$\nabla_1 = \Xi_2 \frac{\partial}{\partial \Xi_1},$$

such that the explicit expansion of f is

$$(2) \qquad f = \Phi + \frac{\nabla \Phi}{\lfloor 1} A + \frac{\nabla^2 \Phi}{\lfloor 2} A^2 + \cdots + \frac{\nabla^m \Phi}{\lfloor m} A^m.$$

If a symbolical basis be assumed in homogeneous form, or, what amounts to the same thing, if f be assumed to have a homogeneous expansion in two arguments A_1, A_2 for which a symbolical basis exists in the form

$$\Xi_A = \Xi_1 A_1 + \Xi_2 A_2,$$

then

$$(3') \quad f = \Xi_A^m = \Xi_1^m A_1^m + m \Xi_1^{m-1} \Xi_2 A_1^{m-1} A_2 + \cdots = \Phi_1 A_1^m + \cdots + \Phi_2 A_2^m.$$

This is the same as (2) when $A_1 = 1$. But, in general, we have

$$\frac{\nabla_1^r \Xi_1^m}{\lfloor r} = \frac{\nabla_2^{m-r} \Xi_2^m}{\lfloor m-r} = \frac{1}{2}\left(\frac{\nabla_1^r \Xi_1^m}{\lfloor r} + \frac{\nabla_2^{m-r} \Xi_2^m}{\lfloor m-r}\right) = \binom{m}{r} \Xi_1^{m-r} \Xi_2^r.$$

Thus the third expression can be used in place of the general coefficient in (3′). Let ∇_i' be the non-symbolical equivalent of ∇_i $(i = 1, 2)$. Then we get from (3′) an expansion formula more general, and more symmetrical than (2), viz.,

$$(3) \quad f = \Phi_1 A_1^m + \frac{1}{2}\left(\frac{\nabla_1' \Phi_1}{\lfloor 1} + \frac{\nabla_2'^{m-1} \Phi_2}{\lfloor m-1}\right) A_1^{m-1} A_2 + \cdots$$
$$+ \frac{1}{2}\left(\frac{\nabla_1'^{m-1} \Phi_1}{\lfloor m-1} + \frac{\nabla_2' \Phi_2}{\lfloor 1}\right) A_1 A_2^{m-1} + \Phi_2 A_2^m.$$

Such an expansion will be completely given by two non-symbolical operators ∇_i' and two pre-determined functions Φ_i $(i = 1, 2)$.

The importance of a general formula like (3) evidently depends upon the number of expansions which can be found to conform to its laws. We proceed to point out the fact that many of the known finite expansions are special cases of (2) or (3).

(*a*) Any binary form $f(x_1, x_2) = a_0 x_1^m + m a_1 x_1^{m-1} x_2 + \cdots$, when transformed by

$$x_1 = \lambda_1 x_1' + \mu_1 x_2', \qquad x_2 = \lambda_2 x_1' + \mu_2 x_2',$$

gives a result $f_1(x_1', x_2')$ which is an expansion of f in the arguments x_i'. It is then a case of (3) where now $A_i = x_i'$, and

$$\Phi_1 = f(\lambda_1, \lambda_2), \qquad \Phi_2 = f(\mu_1, \mu_2),$$

$$\nabla_1' = \mu_1 \frac{\partial}{\partial \lambda_1} + \mu_2 \frac{\partial}{\partial \lambda_2}, \qquad \nabla_2' = \lambda_1 \frac{\partial}{\partial \mu_1} + \lambda_2 \frac{\partial}{\partial \mu_2}.$$

(*b*) Any covariant of $f(x_1, x_2)$, say $C = C_0 x_1^w + \cdots$, considered as an expansion in x_1, x_2 is a case of (3). Here $\Phi_1 = C_0$, the seminvariant leading coefficient; $\Phi_2 = C_w$. Or $x_1^{-w} C$ is a case of (2), where $A = x_1 / x_2$. In either case

$$\nabla_1' = m a_1 \frac{\partial}{\partial a_0} + (m-1) a_2 \frac{\partial}{\partial a_1} + \cdots + a_m \frac{\partial}{\partial a_{m-1}},$$

$$\nabla_2' = a_0 \frac{\partial}{\partial a_1} + 2 a_1 \frac{\partial}{\partial a_2} + \cdots + m a_{m-1} \frac{\partial}{\partial a_m}.$$

(*c*) If $\varphi = (a_0, a_1, \cdots)(x_1, x_2)^m$ goes into $(a_0', a_1', \cdots)(x_1', x_2')^m$ under the special transformation $x_1 = x_1' + A x_2'$, $x_2 = x_2'$, and $I(a_0, a_1, \cdots)$ is any rational integral function* of $a_0, a_1, \cdots$, then the expansion of $I(a_0', a_1', \cdots)$ in powers of A is a case of (2), where $f = I(a_0', \cdots)$, $\Phi = I(a_0, a_1, \cdots)$, and ∇ is the preceding ∇_2'.

(*d*) Taylor's expansion of a non-homogeneous polynomial $f(x)$ in powers of $x - a$ is a case of (2), where

$$\Phi = f(a), \qquad \nabla = \frac{\partial}{\partial a}.$$

(*e*) As a proposition inverse to the fundamental lemma (2), or (3), any formal identity containing three terms may be employed as the symbolical basis of finite expansions. In fact, if the identity is

$$\Xi_A = \Xi_1 A_1 + \Xi_2 A_2,$$

* Elliott, *Algebra of Quantics*, p. 115.

then functions $f = \Xi_A^m$ and corresponding expansions in the arguments A_i exist for all values of m. Gordan's* series may be developed (see § 6) from this point of view, from the formal identity

$$a_x b_y - a_y b_x = (ab)(xy).$$

This is also the principle upon which Hermite's theory of typical representation† and associated forms is based.

§ 2. Extension of the fundamental lemma.

The extension of (2) to the case of the general homogeneous expansion in p arguments can be readily made. We set

$$(4') \quad f = \Xi_A^m = (\Xi_1 A_1 + \Xi_2 A_2 + \cdots + \Xi_p A_p)^m = \Xi_1^m A_1^m + \cdots = \Phi_1 A_1^m + \cdots,$$

and assume that the property (A) is satisfied by all of the symbolical operators of the set

$$\nabla_i = \Xi_{i+1} \frac{\partial}{\partial \Xi_i} \qquad (i = 1, 2, \cdots, p-1).$$

Then it is easy to verify, by actually performing the indicated differentiations, that (4′) may be written

$$f = \sum \frac{\nabla_1^{i_1} \nabla_2^{i_2} \cdots \nabla_{p-1}^{i_{p-1}} \Xi_1^m}{\underline{|i_1}\, \underline{|i_2} \cdots \underline{|i_{p-1}}} A_1^{m-i_1} A_2^{i_1-i_2} \cdots A_p^{i_{p-1}},$$

where the set $(m - i_1, i_1 - i_2, \cdots, i_{p-2} - i_{p-1}, i_{p-1})$ ranges over the partitions of m into p parts, assuming each partition once and once only. Hence if, according to the property (A), ∇_i' is the non-symbolical equivalent of ∇_i, then there exists a function Φ and a set of $p-1$ operators ∇_i' such that the explicit expansion of f is (§ 5)

$$(4) \qquad f = \sum \frac{\nabla_1'^{i_1} \nabla_2'^{i_2} \cdots \nabla_{p-1}'^{i_{p-1}} \Phi}{\underline{|i_1}\, \underline{|i_2} \cdots \underline{|i_{p-1}}} A_1^{m-i_1} A_2^{i_1-i_2} \cdots A_p^{i_{p-1}}.$$

§ 3. On expressing one polynomial in terms of another.

Non-homogeneous variables will be used throughout this section. We let

$$f(x) = a_x^m = a_0 x^m + a_1 x^{m-1} + \cdots + a_m \quad (a_0 = 1, m = \mu\nu),$$

$$\varphi(x) = \xi_x^\nu = \xi_0 x^\nu + \xi_1 x^{\nu-1} + \cdots + \xi_\nu \equiv \xi_0 (x - s_1)(x - s_2) \cdots (x - s_\nu).$$

* Gordan, *Vorlesungen über Invariantentheorie*, vol. 2, § 117.

† Hermite, Journal für Mathematik, vol. 52 (1855), p. 23.

It may be shown by a form of the Euclidean process of successive division that there exists an expansion of f in powers of ξ_x^ν,

$$(5) \qquad f(x) = \Phi_1 + \Phi_2\, \xi_x^\nu + \Phi_3\, (\xi_x^\nu)^2 + \cdots + \Phi_\mu\, (\xi_x^\nu)^{\mu-1} + \Phi_{\mu+1}\, (\xi_x^\nu)^\mu,$$

where Φ_i is a polynomial in x of order $\nu - 1$, and in particular, say

$$\Phi_1 = \gamma_0\, x^{\nu-1} + \gamma_1\, x^{\nu-2} + \cdots + \gamma_{\nu-1}.$$

We proceed to the determination of (the coefficients of) Φ_1 and the operator ∇ in (2). These will then give the whole expansion, and it will appear that (5) has a symbolical basis of the form

$$(6) \qquad \Xi_{(\phi)} = \Xi_{1f(x)} + \Xi_{2f(x)}\, \varphi.$$

Substitution of the roots of $\varphi(x)$ for x in (5) gives at once

$$(7) \qquad f(s_i) = \gamma_0\, s_i^{\nu-1} + \gamma_1\, s_i^{\nu-2} + \cdots + \gamma_{\nu-1} \quad (i = 1, 2, \cdots, \nu).$$

These are linear equations in the γ_j $(j = 0, 1, \cdots, \nu - 1)$. Their solution is possible provided $D \neq 0$, where D is the discriminant of $\varphi(x)$. This solution yields

$$(8) \qquad \gamma_j = \frac{\begin{vmatrix} s_1^{\nu-1} & s_1^{\nu-2} & \cdots & s_1^{\nu-j} & f(s_1) & s_1^{\nu-j-2} & \cdots & s_1 & 1 \\ s_2^{\nu-1} & s_2^{\nu-2} & \cdots & s_2^{\nu-j} & f(s_2) & s_2^{\nu-j-2} & \cdots & s_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_\nu^{\nu-1} & s_\nu^{\nu-2} & \cdots & s_\nu^{\nu-j} & f(s_\nu) & s_\nu^{\nu-j-2} & \cdots & s_\nu & 1 \end{vmatrix}}{\sqrt{D}}$$

$$(j = 0, 1, 2, \cdots, \nu - 1).$$

Thus γ_j is symmetric in the roots s_i, rational in the coefficients ξ_k $(k = 0, 1, \cdots, \nu)$ and linear in $a_0\,(=1), a_1, a_2, \cdots, a_m$. It is necessary to evaluate the symmetric functions γ_j of the roots s_i in order to determine the operator ∇ required by the theory of the symbolical basis, and (2). I did this originally by means of the Euclidean process mentioned above, i. e., $f(x)$ was divided by φ^μ, the remainder by $\varphi^{\mu-1}$, and so forth.* The results for the general case are given below, without details of derivation. In these formulas, Ω indicates the following operator:

$$\Omega = a_0 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_2} + \cdots + a_{m-1} \frac{\partial}{\partial a_m}.$$

* This work was subsequently checked by inductive steps based upon ordinary computation by symmetric functions.

I obtain

$$
\begin{aligned}
F = \gamma_0 = \Omega^{\nu-1} f(-\xi_1) &+ \sum_{i=2}^{k_{11}} \xi_i \, \Omega^{\nu+i-2} f'(-\xi_1) \\
&+ \sum_{i=2}^{k_{21}} \sum_{j=2}^{k_{22}} \frac{\xi_i \, \xi_j}{\lfloor 2} \, \Omega^{\nu+i+j-3} f''(-\xi_1) \\
&+ \sum_{i=2}^{k_{31}} \sum_{j=2}^{k_{32}} \sum_{k=2}^{k_{33}} \frac{\xi_i \, \xi_j \, \xi_k}{\lfloor 3} \, \Omega^{\nu+i+j+k-4} f'''(-\xi_1) \qquad (9) \\
&+ \cdots\cdots\cdots\cdots \\
&+ \sum_{i=2}^{k_{t1}} \sum_{j=2}^{k_{t2}} \cdots \sum_{n=2}^{k_{tt}} \frac{\xi_i \, \xi_j \cdots \xi_n}{\lfloor t} \, \Omega^{\nu+i+j+\cdots+n-t-1} f^{(t)}(-\xi_1) \\
&\qquad (k_{rs} \leqq \nu, \; k_{s1} + k_{s2} + \cdots + k_{ss} \leqq (\mu - 1)\nu + 1).
\end{aligned}
$$

Then there results

$$
\begin{aligned}
\Phi = \Phi_1 = F x^{\nu-1} &+ [-(\xi_2 \, \Omega + \xi_3 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^{\nu-1}) F + a_{m-\nu+2}] \, x^{\nu-2} \\
&+ [-(\xi_3 \, \Omega + \xi_4 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^{\nu-2}) F + a_{m-\nu+3}] \, x^{\nu-3} \\
&+ \cdots\cdots\cdots\cdots \qquad (10) \\
&+ [-(\xi_{\nu-1} \, \Omega + \xi_\nu \, \Omega^2) F + a_{m-1}] \, x \\
&+ [-\xi_\nu \, \Omega F + a_m].
\end{aligned}
$$

Furthermore, we find during the course of the derivation of these results by the division process that an operator ∇ in (2) exists and is precisely [see (6)]

$$
\nabla = \Xi_{2f(x)} \frac{\partial}{\partial \, \Xi_{1f(x)}} = -\frac{\partial}{\partial \xi_\nu}. \qquad (11)
$$

Hence expansion (5) takes the form

$$
\begin{aligned}
f(x) = \Phi - \frac{\partial \Phi}{\partial \xi_\nu} \xi_x^\nu &+ \frac{\partial^2 \Phi}{\partial (\xi_\nu)^2} \frac{(\xi_x^\nu)^2}{\lfloor 2} - \cdots + (-1)^{\mu-1} \frac{\partial^{\mu-1} \Phi}{\partial (\xi_\nu)^{\mu-1}} \frac{(\xi_x^\nu)^{\mu-1}}{\lfloor \mu - 1} \qquad (12) \\
&+ (-1)^\mu \frac{\partial^\mu \Phi}{\partial (\xi_\nu)^\mu} \frac{(\xi_x^\nu)^\mu}{\lfloor \mu}.
\end{aligned}
$$

It is thus analogous to the Taylor expansion, to which it reduces when $\nu = 1$.

We note in passing the following interesting property of the function F:

$$
-(\xi_1 \, \Omega + \xi_2 \, \Omega^2 + \cdots + \xi_\nu \, \Omega^\nu) F + a_{m-\nu+1} \equiv F.
$$

§ 4. Homogeneous expansions.

As an illustration of formula (3), we consider the expansion of a homogeneous binary form in terms of two other binary forms of lower order. Let

$$f_M = a_x^M = \prod_{i=1}^{M} (r_2^{(i)} x_1 - r_1^{(i)} x_2) = a_0 x_1^M + \cdots$$

be the form to be expanded. Let the arguments of the expansion be

$$A_1 = f_{1n} = \alpha_x^n = \prod_{j=1}^{n} (\alpha_2^{(j)} x_1 - \alpha_1^{(j)} x_2) = \alpha_0 x_1^n + \cdots,$$

$$A_2 = f_{2n} = \beta_x^n = \prod_{j=1}^{n} (\beta_2^{(j)} x_1 - \beta_1^{(j)} x_2) = \beta_0 x_1^n + \cdots.$$

Then the expansion sought will be of the type

$$f_M = \varphi_{0p} f_{1n}^m + \varphi_{1p} f_{1n}^{m-1} f_{2n} + \cdots + \varphi_{mp} f_{2n}^m, \tag{13}$$

where φ_{ip} is a homogeneous binary form of order p.

It is assumed at the outset that f_M, f_{in} are perfectly general forms, i. e., that their coefficients are independent variables. These forms are also understood to be given in the sense that the coefficient forms φ_{ip} are to be determined from them. Hence the totality of coefficients of the φ_{ip} $(i = 0, 1, \cdots, m)$ must be equal in number to those of f_M,

$$M + 1 = (m + 1)(p + 1). \tag{14}$$

Also

$$M = mn + p.$$

Thus

$$p = n - 1, \qquad M = n(m + 1) - 1.$$

When M and p have these values the solution of $n(m + 1)$ linear equations will determine the coefficients of φ_{in-1} $(i = 0, 1, \cdots, m)$, and consequently the whole expansion. Hence the expansion exists provided these linear equations are consistent. It will be shown that the determinant representing the condition for consistency is a power of the resultant of f_{1n}, f_{2n}, multiplied by a constant.

For this determinant D is of order $n(m + 1)$, and is of the form of the eliminant of the set of $m + 1$ forms of order mn,

$$f_{1n}^m, \quad f_{1n}^{m-1} f_{2n}, \quad \cdots, \quad f_{2n}^m.$$

If $D = 0$, then there exists a linear relation of the type

$$\lambda_1 f_{1n}^m + \lambda_2 f_{1n}^{m-1} f_{2n} + \cdots + \lambda_{m+1} f_{2n}^m = 0, \tag{15}$$

where λ_{m+1} is of order $n-1$ in (x), and so f_{1n}, f_{2n} have at least one common root. Conversely if f_{1n}, f_{2n} have a common root then $D=0$, for otherwise $f_{n(m+1)-1}$ would have a root of multiplicity m.

An expansion (13) *of* $f_{n(m+1)-1}$ *exists provided the resultant* $R(\alpha, \beta)$ *of* f_{1n}, f_{2n} *does not vanish.*

To determine our expansion (13), let

$$(16)\quad \varphi_{in-1} = p_{i0}\, x_1^{n-1} + p_{i1}\, x_1^{n-2}\, x_2 + \cdots + p_{in-1}\, x_2^{n-1} = p_{ix}^{n-1} \quad (i = 0, 1, \cdots m).$$

Substituting $\beta^{(k)}$ for x in (13) we get

$$(17)\qquad p_{0\beta^{(k)}}^{n-1}\,(\alpha_{\beta^{(k)}}^{n})^{m} = a_{\beta^{(k)}}^{n(m+1)-1} \qquad (k = 1, 2, \cdots n).$$

Also owing to the symmetry of (13) in (α, β),

$$(18)\qquad p_{m\alpha^{(k)}}^{n-1}\,(\beta_{\alpha^{(k)}}^{n})^{m} = a_{\alpha^{(k)}}^{n(m+1)-1} \qquad (k = 1, 2, \cdots, n).$$

Equations (17) form a set of n linear equations for the determination of the n coefficients of p_{0x}^{n-1}. In order to exhibit the solutions as briefly as may be let Δ_β be the discriminant of β_x^n and Δ_α that of α_x^n. Then

$$\pm\,\Delta_\beta = \left|\, \beta_1^{(1)^{n-1}},\ \beta_1^{(2)^{n-2}}\beta_2^{(2)},\ \beta_1^{(3)^{n-3}}\beta_2^{(3)^2},\ \cdots,\ \beta_2^{(n)^{n-1}} \,\right|^2 \div \prod_{k=1}^{n} \beta_2^{(k)^{2(n-1)}}.$$

Also let

$$\Sigma_j\,(\alpha, \beta, a) = \left|\, \alpha_{\beta^{(1)}}^{mn}\beta_1^{(1)^{n-1}},\ \alpha_{\beta^{(2)}}^{mn}\beta_1^{(2)^{n-2}}\beta_2^{(2)},\ \cdots,\ \alpha_{\beta^{(j)}}^{mn}\beta_1^{(j)^{n-j}}\beta_2^{(j)^{j-1}},\right.$$
$$\left. a_{\beta^{(j+1)}}^{n(m+1)-1},\ \alpha_{\beta^{(j+2)}}^{mn}\beta_1^{(j+2)^{n-j-2}}\beta_2^{(j+2)^{j+2}},\ \cdots,\ \alpha_{\beta^{(n)}}^{mn}\beta_2^{(n)^{n-1}} \,\right|,$$

then we get

$$(19)\qquad p_{0j} = \frac{1}{R(\alpha, \beta)^{m}} \cdot \frac{\Sigma_j\,(\alpha, \beta, a)}{\prod_{k=1}^{n} \beta_2^{(k)^{n-1}}\sqrt{\Delta_\beta}} \quad (j = 0, 1, \cdots, n-1).$$

From symmetry, or directly from (18), we have also

$$(20)\qquad p_{mj} = \frac{1}{R(\beta, \alpha)^{m}} \cdot \frac{\Sigma_j\,(\beta, \alpha, a)}{\prod_{k=1}^{n} \alpha^{(k)^{n-1}}\sqrt{\Delta_\alpha}} \quad (j = 0, 1, \cdots, n-1).$$

The coefficients of $R(\alpha, \beta)^{-m}$ in (19), (20) are rational, *integral* and symmetric in the roots of f_{1n}, f_{2n} and hence rational in the quantities $\alpha_0, \alpha_1, \cdots$; $\beta_0, \beta_1, \cdots$.

Referring now to expansion formula (3) we note that the functions Φ_i $(i=1, 2)$ are determined in the present case by (19), (20).

The operators ∇_1', ∇_2' required by (3) to complete the determination of (13)

are the Aronhold operators obtained from α_x^n, β_x^n; i. e.,

$$(21)\qquad \begin{aligned} -\nabla_1' &= \alpha_0 \frac{\partial}{\partial \beta_0} + \alpha_1 \frac{\partial}{\partial \beta_1} + \cdots + \alpha_n \frac{\partial}{\partial \beta_n}, \\ -\nabla_2' &= \beta_0 \frac{\partial}{\partial \alpha_0} + \beta_1 \frac{\partial}{\partial \alpha_1} + \cdots + \beta_n \frac{\partial}{\partial \alpha_n}. \end{aligned}$$

These with $\Phi_1 = \varphi_{0n-1}$, $\Phi_2 = \varphi_{mn-1}$, $A_i = f_{in}$, give in explicit form the desired expansion of $f_{n(m+1)-1}$; that is, it takes the form (3):

$$(22)\quad f_{n(m+1)-1} = \Phi_1 f_{1n}^m + \frac{1}{2}\left(\frac{\nabla_1' \Phi_1}{\underline{|1}} + \frac{\nabla_2'^{\,m-1} \Phi_2}{\underline{|m-1}}\right) f_{1n}^{m-1} f_{2n} + \cdots + \Phi_2 f_{2n}^m.$$

It is of some interest to enumerate the conditions, necessary and sufficient, in order that series (22), taken in non-homogeneous form with $x_2 = 1$, say, should have constant coefficients. We get a minimum set of such conditions by equating to zero all of the coefficients of φ_{in-1} $(i = 0, 1, \cdots, m)$ except the last.

Let (19) be written in the form

$$p_{0j} = \frac{Q_j(\alpha, \beta)}{R(\alpha, \beta)^m} \qquad (j = 0, 1, \cdots, n-1).$$

Then the aforesaid conditions are given by

$$(23)\qquad C_{ij} \equiv \nabla_1'^{\,i} Q_j = 0 \qquad \begin{pmatrix} i = 0, 1, \cdots, m \\ j = 0, 1, \cdots, n-2 \end{pmatrix}.$$

Their number is

$$N = (n-1)(m+1).$$

It follows that a series free from conditions $(N = 0)$, having constant coefficients, exists only when the arguments f_{in} are linear $(n = 1)$.

By constructing equations analogous to (14) it is easy to show that a ternary form f_m can be expressed as a ternary expansion in three ternary arguments f_{in} $(i = 1, 2, 3)$,

$$f_m = \varphi_{m00} f_{1n}^m + \cdots,$$

with the coefficient forms φ all of order $n-1$, and with the expansion free from conditions, only when the arguments are linear.

§ 5. Expansions with linear arguments.

Consider next the problem of expanding a p-ary form a_x^m in terms of powers of p linear p-ary forms α_{ix} $(i = 1, 2, \cdots, p)$. Expansion formula (4) applies, and

$$(24)\qquad a_x^m = \sum_{i_j=1}^{m} \frac{\nabla_1'^{\,i_1} \nabla_2'^{\,i_2} \cdots \nabla_{p-1}'^{\,i_{p-1}} \Phi_1}{\underline{|i_1}\,\underline{|i_2} \cdots \underline{|i_{p-1}}} \alpha_{1x}^{m-i_1} \alpha_{2x}^{i_1-i_2} \cdots \alpha_{px}^{i_{p-1}}.$$

In this the coefficient Φ_j is given by the set

$$i_1 = i_2 = \cdots = i_{j-1} = m; \qquad i_j = i_{j+1} = \cdots = i_{p-1} = 0 \quad (j = 1, 2, \cdots, p).$$

To get Φ_1 we substitute for (x) in (24) the coördinates of the point of intersection of the $p - 1$ hyperplanes

$$\alpha_{ix} = \alpha_{i1} x_1 + \alpha_{i2} x_2 + \cdots + \alpha_{ip} x_p = 0 \quad (i = 2, 3, \cdots, p)$$

This gives

$$x_j = -\frac{(\alpha_2 \alpha_3 \cdots \alpha_{j-1} \alpha_p \alpha_{j+1} \cdots \alpha_{p-1}) x_p}{(\alpha_2 \alpha_3 \cdots \alpha_p)} \quad (j = 1, 2, \cdots, p),$$

and by substitution

$$\Phi_1 = \frac{(a\alpha_2 \alpha_3 \cdots \alpha_p)^m}{(\alpha_1 \alpha_2 \cdots \alpha_p)^m}.$$

For this expansion there exists a symbolical basis in the form of (4′), as may readily be verified by considering (24) to be the transformed of a_x^m by the inverse of

$$X_i = \alpha_{ix} \qquad (i = 1, 2, \cdots, p).$$

The operators for this case are the Aronhold operators

$$-\nabla_i' = \alpha_{i1} \frac{\partial}{\partial \alpha_{i+11}} + \alpha_{i2} \frac{\partial}{\partial \alpha_{i+12}} + \cdots + \alpha_{ip} \frac{\partial}{\partial \alpha_{i+1p}} \quad (i = 1, 2, \cdots, p-1),$$

the whole expansion (24) being given by Φ_1 and ∇_i'.

It can be shown that corresponding results hold for forms whose coefficients belong to a field or reduced residue system* [modd p, $P(x)$], where p is a prime number and $P(x)$ an irreducible quantic modulo p of order n. Thus when $m = p = 2$, we have, provided $(\alpha\beta) \not\equiv 0$ [modd p, $P(x)$], an expansion of the form

$$f \equiv \Phi_1 (\alpha_1 x_1 - \alpha_2 x_2)^2 + I (\alpha_1 x_1 - \alpha_2 x_2)(\beta_1 x_1 - \beta_2 x_2)$$
$$+ \Phi_2 (\beta_1 x_1 - \beta_2 x_2)^2 \text{ [modd } p, P(x)],$$

where the linear arguments have coefficients belonging to the residue system. Then

$$\Phi_1 \equiv (\alpha_1 \beta_2 - \alpha_2 \beta_1)^{p^n - 3} (a_0 \beta_2^2 + a_1 \beta_2 \beta_1 + a_2 \beta_1^2),$$

$$\Phi_2 \equiv (\alpha_1 \beta_2 - \alpha_2 \beta_1)^{p^n - 3} (a_0 \alpha_2^2 + a_1 \alpha_2 \alpha_1 + a_2 \alpha_1^2)$$
$$\text{[modd } p, P(x)],$$

* Dickson, *Linear Groups*, p. 7.

and with

$$\nabla_1' = -\left(\alpha_1 \frac{\partial}{\partial \beta_1} + \alpha_2 \frac{\partial}{\partial \beta_2}\right), \qquad \nabla_2' = -\left(\beta_1 \frac{\partial}{\partial \alpha_1} + \beta_2 \frac{\partial}{\partial \alpha_2}\right),$$

we have

$$I = \tfrac{1}{2}(\Delta_1' \Phi_1 + \nabla_2' \Phi_2) \equiv -(\alpha\beta)^{p^n-3}[2a_0\,\alpha_2\,\beta_2 + a_1(\alpha_1\beta_2 + \alpha_2\beta_1) + 2a_2\,\alpha_1\,\beta_1] \qquad [\text{modd } p, P(x)].$$

The form $f = a_x\, b_x$ is apolar to $\varphi = (\alpha_1 x_1 - \alpha_2 x_2)(\beta_1 x_1 - \beta_2 x_2) = \alpha_x \beta_x$ provided $I \equiv 0$. In fact the latter congruence is a necessary and sufficient condition that the two forms f, φ respectively be the following linear combinations of two squares in the field:

$$f \equiv s\alpha_x^2 + t\beta_x^2; \qquad \varphi \equiv \sigma a_x^2 + \tau b_x^2.$$

§ 6. The Clebsch-Gordan Expansion.

Gordan's series* is a case of expansion (2). It may be of interest to give a sketch of the derivation of this series from the present point of view, for two reasons: first to show how the preceding general theory applies, and secondly because our methods give a very compact treatment of the details of the derivation, which is believed to be novel.

We employ as symbolical basis the formal identity

$$(E) \qquad a_x\, b_y - a_y\, b_x = (ab)(xy).$$

From it we have

$$a_x^m\, b_y^n \equiv [a_y\, b_x + (ab)(xy)]^m\, b_y^{n-m} \qquad (n \geqq m),$$

or

$$(25) \qquad \begin{aligned} a_x^m\, b_y^n = {} & a_y^m\, b_x^m\, b_y^{n-m} + \binom{m}{1} a_y^{m-1}\, b_x^{m-1}\, b_y^{n-m}(ab)(xy) + \cdots \\ & + \binom{m}{k} a_y^{m-k}\, b_x^{m-k}\, b_y^{n-m}(ab)^k(xy)^k + \cdots + (ab)^m(xy)^m\, b_y^{n-m}. \end{aligned}$$

This is one form of Gordan's series. Here we have (§ 1)

$$\Phi = a_y^m\, b_x^m\, b_y^{n-m}, \qquad \nabla = \frac{\partial^2}{\partial x_1\, \partial y_2} - \frac{\partial^2}{\partial x_2\, \partial y_1}.$$

But to reduce the series to the ordinary form we use the following polar formula:

$$(26) \qquad (a_x^m,\, b_x^n)_{y^{n-k}}^k = \frac{(ab)^k}{\binom{m+n-2k}{n-k}} \sum_{h=0}^{m-k} \binom{m-k}{m-k-h}\binom{n-k}{n-m+h} a_y^{m-k-h}\, a_x^h\, b_x^{m-k-h}\, b_y^{n-m+h}.$$

* Gordan, *Ueber Combinanten*, Mathematische Annalen, vol. 5 (1872), p. 95.

Since the sum of the numerical coefficients in the polar of a product is unity, (26) can be written

$$(27)\qquad \begin{aligned}(a_x^m, b_x^n)_{y^{n-k}}^k &= (ab)^k a_y^{m-k} b_x^{m-k} b_y^{n-m} \\ &+ \frac{(ab)^k}{\binom{m+n-2k}{n-k}} \sum_{h=1}^{m-k} \binom{m-k}{m-k-h}\binom{n-k}{n-m+h} a_y^{m-k-h} b_x^{m-k-h} b_y^{n-m} \\ &\qquad \times (a_x^h b_y^h - a_y^h b_x^h).\end{aligned}$$

The first term of this is, aside from the factor $\binom{m}{k}$, the coefficient of $(xy)^k$ in (25). Hence this coefficient is the $(n-k)$th transvectant of a_x^m and b_x^n, minus terms which contain the factor $(ab)^{k+1}(xy)$. Hence (27) may be used as a recursion formula; and, starting with $k = m$, we get

$$(28)\qquad \begin{aligned}(ab)^m b_y^{n-m} &= (a_x^m, b_x^n)_{y^{n-m}}^m, \\ (ab)^{m-1} a_y b_x b_y^{n-m} &= (a_x^m, b_x^n)_{y^{n-m+1}}^{m-1} - \frac{1}{n-m+2}(a_x^m, b_x^n)_{y^{n-m}}^m (xy), \\ (ab)^{m-2} a_y^2 b_x^2 b_y^{n-m} &= (a_x^m, b_x^n)_{y^{n-m+2}}^{m-2} - \frac{4}{n-m+4}(a_x^m, b_x^n)_{y^{n-m+1}}^{m-1}(xy) \\ &\quad + \frac{2}{(n-m+2)(n-m+3)}(a_x^m, b_x^n)_{y^{n-m}}^m (xy)^2.\end{aligned}$$

And, in general, by induction

$$(29)\qquad \begin{aligned}(ab)^{m-k} a_y^k b_x^k b_y^{n-m} &= \alpha_0 (a_x^m, b_x^n)_{y^{n-m+k}}^{m-k} + \alpha_1 (a_x^m, b_x^n)_{y^{n-m+k-1}}^{m-k+1}(xy) + \cdots \\ &\quad + \alpha_j (a_x^m, b_x^n)_{y^{n-m+k-j}}^{m-k+j}(xy)^j + \cdots + \alpha_k (a_x^m, b_x^n)_{y^{n-m}}^m (xy)^k.\end{aligned}$$

Consequently it follows at once from (25) that

$$(30)\qquad \begin{aligned}a_x^m b_y^n &= C_0 (a_x^m, b_x^n)_{y^n}^0 + C_1 (a_x^m, b_x^n)_{y^{n-1}}^1 (xy) \\ &\quad + \cdots + C_j (a_x^m, b_x^n)_{y^{n-j}}^j (xy)^j + \cdots + C_m (a_x^m, b_x^n)_{y^{n-m}}^m (xy)^m\end{aligned}$$

The constants C_i may now be determined in the usual manner by operating on both sides of (30) with the proper power of Δ, and noting that when $y \backsim x$ in any polar it goes back into the original polarized form. The following known formula* may be used for this purpose:

$$(31)\qquad \Delta^i (xy)^j \alpha_x^{n-j} \alpha_y^{m-j} = \frac{\underline{|j}\ \ \underline{|m+n-j+1}}{\underline{|j-i}\ \ \underline{|m+n-j-i+1}} (xy)^{j-i} \alpha_x^{n-j} \alpha_y^{m-j} \qquad (i \leqq j).$$

* Grace and Young, *Algebra of Invariants*, p. 54.

The coefficients may also be derived by multiplying the respective members of the column (28) by

$$\binom{m}{m}(xy)^m,\ \binom{m}{m-1}(xy)^{m-1},\ \cdots,\ \binom{m}{2}(xy)^2,\ \binom{m}{1}(xy),\ 1;$$

and adding the results. Either method gives the well-known result

$$a_x^m\, b_y^n = \sum_{i=0}^{m} \frac{\binom{m}{i}\binom{n}{i}}{\binom{m+n-i+1}{i}} (xy)^i\, (a_x^m,\, b_x^n)_{y^{n-i}}^{i}\,. \tag{32}$$

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LEBESGUE INTEGRALS CONTAINING A PARAMETER, WITH APPLICATIONS*

BY

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1. **Introduction.**—In a recent memoir Lebesgue† has studied the following problem. Let $f(t)$ belong to one of a certain group of families of functions. What conditions should be imposed on another function $\phi(t, n)$ involving the parameter n, in order that the Lebesgue integral of the product of f and ϕ may exist and approach zero, as n becomes infinite, for all functions f of the family in question? I have studied elsewhere‡ the same problem when the functions involved are functions of several variables instead of one. The object of the present paper is to continue Lebesgue's work by considering other families of which most are narrower than his, and to make certain applications of the results. I have been led to do this because of an attempt to apply Lebesgue's theorems to certain series of orthogonal functions. It was found that the conditions were too severe, and the need was felt of simpler conditions, even at the expense of restricting the classes of functions for which the results would be valid. The conditions that should be imposed on ϕ in order that the integral of $f\phi$ shall remain limited (instead of approaching zero) are given in Theorem 4.

It is shown here how the theory may be applied to the series of Fourier and of Legendre to discover the degrees of their convergence and to find the rates at which their coefficients approach zero. In the case of the functions of Legendre certain inequalities are derived which may prove useful. The paper concludes with some theorems on the evaluation of definite integrals and on the integration of series.

2. The first theorem is taken from the paper of Lebesgue already mentioned. I copy it here, with certain slight changes, for which authority will be found in my earlier paper, because I shall need to refer to it frequently in the following pages.

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* Presented to the Society April 27, 1912.

† Lebesgue, Annales de la Faculté de Toulouse, ser. 3, vol. 1 (1909), pp. 25–117.

‡ These Transactions, vol. 14 (1913), pp. 42–64.

THEOREM 1: *Necessary and sufficient conditions that the integral*

$$\int_k^l f(t)\,\phi(t, n)\,dt,$$

where f and ϕ are defined in the interval (k, l) and n is a parameter belonging to a sequence having plus infinity as its only limit point, shall exist for sufficiently large values of n as an absolutely convergent L-integral, and approach zero for all functions f belonging to one of the families specified below, may be stated in the several cases as follows:*

(F_1) *In the case of the family F_1 of absolutely L-integrable functions,* (1°) *M shall exist so that $|\phi|$ is less than M if $n > n_M$† at all points of (k, l), except perhaps at a null set, and* (2°) *the integral of ϕ over each interval (a, b) in (k, l) shall approach zero as n becomes infinite.*

(F_2) *In the case of the family F_2 of functions whose squares are L-integrable,* (1°) *M shall exist so that in (k, l) the integral of ϕ^2 is less than M if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_3) *In the case of the family F_3 of limited L-integrable functions,* (1°) *the integral of ϕ shall be "equi-absolutely" continuous,‡ and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_4) *In the case of the family F_4 of simply discontinuous§ functions,* (1°) *M shall exist so that the integral of $|\phi|$ is less than M if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_{4a}) *In the case of the family F_{4a} of continuous functions,* (1°) *condition* (1°) *for F_4 shall be satisfied,* (2°) *for each a in (k, l) the integral of $(t - a)\,\phi$ from a to l, and* (2a°) *the integral of ϕ from k to l shall approach zero as n becomes infinite.*

(F_5) *In the case of the family F_5 of functions having limited variation,* (1°) *ϕ shall be absolutely L-integrable,* (1a°) *M shall exist so that the integral of ϕ from k to t is less than M uniformly for all values of t if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_{5a}) *In the case of the family F_{5a} of continuous functions having limited variation,* (1°, 1a°, 2°, 2a°) *conditions* (1°) *and* (1a°) *for F_5, and* (2°) *and* (2a°) *for F_{4a}, shall be satisfied.*

COROLLARY: *The conditions for F_1, F_2, and F_3 remain sufficient if, in place of* (2°), *it be required that $\phi(t, n)$ approach zero in (k, l), except perhaps at a null set of points.*

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* I. e., integral in the sense of Lebesgue.

† I use n_M in place of the usual n_0 to denote a fixed number.

‡ In the sense of Vitali, Rendiconti del Circolo Matematico di Palermo, vol. 23 (1907), p. 139. See also my other paper, loc. cit., p. 44.

§ I. e., having at most discontinuities of the first kind.

For I have shown in another paper* (by the use of a theorem of Vitali's) that termwise integration of the sequence $\{\phi(t, n)\}$ is permissible, if condition (1°) for the family in question is satisfied. Therefore, if ϕ approaches zero, so does its integral.

3. The following theorem is of interest because of its bearing on the development of an arbitrary function in series of orthogonal functions. There follow quite readily from it, for example, certain known facts concerning the convergence of Fourier's and Legendre's developments at the point x.

THEOREM 2: *In order that the integral of Theorem* 1 *shall exist for sufficiently large values of* n, *and approach zero for all functions* f *belonging to the family* F_6 *of functions which are continuous in* (k, l), *and have at a previously chosen point* x *a derivative, it is necessary and sufficient that* (1°) *the integral of* ϕ *from* k *to* l *approach zero as* n *becomes infinite, and that* (2°) *the conditions of Theorem* 1 *relative to* F_{4a} *be satisfied when* ϕ *is replaced by* $(t-x)\phi$.

Proof. Except for $t = x$, we have

$$f(t) = f(x) + (t-x)\frac{f(t)-f(x)}{t-x}$$

identically, and therefore

$$(1)\quad \int_k^l f\phi\,dt = f(x)\int_k^l \phi\,dt + \int_k^l \frac{f(t)-f(x)}{t-x}[(t-x)\phi]\,dt = \mathrm{I} + \mathrm{II}.$$

It is necessary and sufficient that (1) approach zero. I approaches zero if (1°) is fulfilled. Since, however, (1°) is clearly necessary (for if it is not satisfied the function $f = 1$ contradicts the theorem), it follows that it is necessary as well as sufficient that I and II separately approach zero. In II $[f(t)-f(x)]/(t-x)$ is, if properly defined at x, continuous, since $f(t)$ is continuous, and at x its derivative, that is, the limit of this fraction, exists. By (2°) we may therefore apply to II the conditions of Theorem 1 relative to F_{4a}, ϕ being replaced by $(t-x)\phi$, and it follows that II approaches zero, and the conditions stated are sufficient. Moreover the conditions in 2° are necessary, for, if any one of them is not satisfied, there exists by Theorem 1 a continuous function $q(t)$ such that

$$\int_k^l q(t)(t-x)\phi\,dt$$

does not approach zero. In that case the function $f = q(t)(t-x)$ contradicts the theorem, for it is continuous in (k, l) and has at x a derivative, viz:

$$\lim_{t=x}\frac{q(t)(t-x)-q(x)(x-x)}{t-x} = q(x).$$

* These Transactions, loc. cit., pp. 63, 64.

COROLLARY 1: *If condition* (1°) *be omitted, the theorem applies to the family of functions which have the properties of* F_6 *and, in addition, vanish at* x.

COROLLARY 2: *If, in the theorem, it is desired that the integral approach* $f(x)$ *instead of zero, it is necessary and sufficient that* (2°) *hold, and that in* (1°) *the integral be required to approach unity instead of zero.*

4. We now consider the same problem relative to the families $G_1', \cdots, G_6'; G_1'', \cdots, G_6''; \cdots$, defined as follows:

DEFINITION 1: The function f belongs to $G_j^{(i)}$ if

$$f(t) = C_i + \int_k^t f' \, dt,$$

where f' is a function belonging to $G_j^{(i-1)}$, and C_i is a constant, and $G_j^{(0)} = F_j$ (i. e., $f^{(i)}$ is in F_j). We do not here assert that f' is necessarily the derivative.

DEFINITION* 2: The function f belongs to $G_j^{(i)}$ if its $(i-1)$th derivative exists,† and is an indefinite integral, and its ith derivative (which will then exist except perhaps at a null set) may be defined, or redefined, in a null set so that it becomes a function of F_j.

Of these definitions the first will be the more convenient to use in our theoretical work, the second the easier to apply in practice. Before proceeding to discuss the problem, I will show that the two definitions are equivalent. The following theorems‡ will be used.

(*a*) *If* ϕ *is continuous in* (k, l), *it equals the derivative of its indefinite integral.*

(*b*) *The same is true, except perhaps at a null set, if* ϕ *is absolutely L-integrable in* (k, l).

(*c*) *If* f *is an indefinite integral, its derivative exists, except perhaps at a null set, and* f *is the indefinite integral of its derivative.*

(*d*) If df/dt *exists and is absolutely L-integrable in* (k, l), *then*

$$f(t) - f(k) = \int_k^t \frac{df}{dt} \, dt.$$

Assuming, first, Definition 1, let

$$f(t) = C_1 + \int_k^t f' \, dt, \tag{1}$$

* In particular, f satisfies this definition for $G_{4a}^{(i)}$, $G_{5a}^{(i)}$, or $G_6^{(i)}$, if its ith derivative is a member of F_{4a}, F_{5a}, or F_6, respectively. But a similar statement does not hold for $G_j^{(i)}$ when $j = 1, 2, 3$, or 4. In fact, we cannot even say in these cases that, if df/dt exists, except perhaps at a null set, and after being defined, or redefined, in a null set becomes a member of F_j, then f belongs to $G_j^{(i)}$. For example, if $f = 0$ in $(0 \leqq t \leqq 1)$ and 1 in $(1 < t \leqq 2)$, df/dt exists in $(0, 2)$, except at 1, and if it be defined there in any manner it becomes a member of F_4, but f is not an integral.

† If a derivative is infinite at a point, I say it does not exist at that point.

‡ (*a*) is well known. For (*b*) and (*c*) see de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, 2d ed., vol. 1 (1909), p. 267; for (*d*) see page 263 of the same volume.

$f^{(i)}$ being in F_j. By (a), since by the definition f' is continuous when $i > 1$,

$$\frac{df}{dt} = f'(t) = C_2 + \int_k^t f'' \, dt,$$

$$\frac{d^2 f}{dt^2} = f''(t) = C_3 + \int_k^t f''' \, dt, \tag{2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^{i-1} f}{dt^{i-1}} = f^{(i-1)}(t) = C_i + \int_k^t f^{(i)} \, dt.$$

Then, since $f^{(i)}$ is given absolutely L-integrable, by (b), except perhaps at a null set,

$$\frac{d^i f}{dt^i} = f^{(i)}(t). \tag{3}$$

We may define, or redefine, $d^i f / dt^i$ in this null set so that (3) is true at every point of (k, l). Then this new function belongs to the same family as $f^{(i)}$, namely to F_j.

Now let us suppose Definition 2 to be satisfied. Since $d^{i-1} f / dt^{i-1}$ is an indefinite integral, $d^i f / dt^i$ exists by (c), except perhaps at a null set, and, no matter how it is defined in this or in any other null set,

$$\frac{d^{i-1} f}{dt^{i-1}} - C_1 = \int_k^t \frac{d^i f}{dt^i} \, dt.$$

By successive applications of (d), it follows that

$$\frac{d^{i-2} f}{dt^{i-2}} - C_2 = \int_k^t \frac{d^{i-1} f}{dt^{i-1}} \, dt,$$

etc. Finally,

$$f(t) - C_i = \int_k^t \frac{df}{dt} \, dt,$$

and df / dt belongs to $G_j^{(i-1)}$, since $d^i f / dt^i$ may be defined, or redefined, in a null set so that it belongs to F_j. This statement means that f satisfies the requirements of Definition 1.

We have now the following general theorem, which holds for all positive integral values of i.

THEOREM 3: *In order that the integral of Theorem* 1 *shall exist, for sufficiently large values of* n, *and approach zero as* n *becomes infinite, for all functions belonging to the family* * $G_j^{(i)}$, *it is necessary and sufficient that* (1°) *the conditions of the corresponding theorem for the family* $G_j^{(i-1)}$ *be satisfied* ($G_j^{(0)}$ *being* F_j) *when*

* Defined at the beginning of this section.

ϕ is replaced by its integral over (k, t), and that* (2°) *the integral of ϕ over (k, l) approach zero as n becomes infinite.*

Proof. Using Definition 1 and integrating by parts,† we have

$$(1) \qquad \int_k^l f\phi\, dt = f(l) \int_k^l \phi\, dt - \int_k^l \left(\int_k^t \phi\, dt \right) f'\, dt = \mathrm{I} + \mathrm{II}.$$

By (2°), I approaches zero. By (1°) and the corresponding theorem for $G_j^{(i-1)}$, II approaches zero. Hence the conditions are *sufficient.* To show that they are necessary we note first that the function $f = 1$ proves that (2°) is necessary, and that therefore, in (1), I must approach zero. It follows that II must approach zero. Now, if (1°) is not satisfied, there exists, by the theorem for $G_j^{(i-1)}$, a function $q(t)$ belonging to $G_j^{(i-1)}$ such that

$$(2) \qquad \int_k^l \left(\int_k^t \phi\, dt \right) q\, dt$$

does not approach zero. Let $f = \int_k^t q\, dt$. Then f belongs to $G_j^{(i)}$ and (1) holds, q replacing f'; but by (2) II does not approach zero, a condition which we have just found to be necessary.

Corollary 1: *If condition* (2°) *be omitted, the theorem applies to that family of functions which have the properties of $G_j^{(i)}$ $(i > 0)$ and, in addition, vanish at l.*

Sufficiency.—In (1), II approaches zero as before, and I vanishes since $f(l) = 0$.

Necessity.—Suppose (1°) to be not satisfied, and consider the function $q(t)$ given in (2). Let

$$f(t) = \int_k^t q(t)\, dt - \int_k^l q(t)\, dt.$$

For this function we may write, analogous to (1),

$$\int_k^l f\phi\, dt = \left[f(t) \int_k^t \phi\, dt \right]_k^l - \int_k^l \left(\int_k^t \phi\, dt \right) q\, dt = -\int_k^l \left(\int_k^t \phi\, dt \right) q\, dt,$$

and, by (2), this does not approach zero.

Corollary 2: *In order that the integral from k to k' of $f\phi$ shall approach $f(l)$ for all functions f which belong to $G_j^{(i)}$ $(i > 0,\ j < 6)$ in the interval $(k \leqq l \leqq k')$, it is sufficient that the conditions of Corollary* 1 *be satisfied for each of the intervals (k, l), (k', l), and that the integral from k to k' of ϕ approach unity.*

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* This supposes that ϕ is absolutely L-integrable for sufficiently large values of n, a condition which is necessary, for if it is not satisfied the function $f = 1$ contradicts the theorem.

† Lebesgue, loc. cit., p. 46.

For, by the last condition the expressions

$$f(l) - \int_k^{k'} f\phi\, dt, \qquad \int_k^{k'} [f(l) - f(t)]\phi\, dt$$

have the same limit, and the function $f(l) - f(t)$ has the properties of $G_j^{(i)}$ in each of the intervals stated, and in addition vanishes at l. By the preceding corollary, therefore, the first expression approaches zero.

5. Sometimes it is desirable to know whether the integral of Theorem 1 remains limited,* not whether it approaches zero, as n becomes infinite. The following statement will enable one to determine from the previous theorems the necessary and sufficient conditions that this shall be true.

THEOREM 4: *If n is as in the preceding theorems and I_n is a number depending on it, then a necessary and sufficient condition that two numbers, M and n_M, exist so that I_n shall be numerically less than M when $n > n_M$ is that, for every positive divergent sequence $\{q_n\}$, $\lim_{n=\infty} |I_n| / q_n = 0$.*

APPLICATIONS.

6. **Fourier's Coefficients.**—In this section I discuss the rates at which the coefficients of Fourier approach zero.† Let f be defined and absolutely L-integrable in the interval $(c, c + 2\pi)$. As is customary, we write

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(t) \sin nt\, dt, \qquad b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(t) \cos nt\, dt.$$

Let $\{q_n\}$ be a positive sequence which diverges at an arbitrarily small, positive rate. From the preceding theorems the following may be easily established:

(*a*) *The coefficients a_n and b_n approach zero.*‡

(*b*) *There exist continuous, odd* § *functions for which $q_n a_n$ does not approach zero, and continuous, even functions for which $q_n b_n$ does not approach zero.*

(*c*) *If f has limited variation, na_n and nb_n remain finite.*||

(*d*) *If f is continuous and has limited variation, but not necessarily if f has limited variation and is not continuous, na_n approches zero when c is an odd multiple of $\pi/2$ and n is odd, and nb_n approaches zero when c is an even multiple of $\pi/2$.*

(*e*)¶ *Let f belong to $G_1^{(i)}$, let $f^{(0)} = f$, $f^{(p)}$ be the pth derivative of f, and let $p = 0$,*

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* Cf., e. g., § 6, c, f, and § 7, c, f.

† Cf. also a footnote to an article by Bôcher, Annals of Mathematics, ser. 2, vol. 7 (1906), p. 109.

‡ This follows from Theorem 1 for F_1, and is well known.

§ This shows that (*a*) is the most that may be said even of families of functions as restricted as those in (*b*). The function $f(t)$ is odd in $(c, c + 2\pi)$, if $f(c + \pi - t) = -f(c + \pi + t)$; even, if $f(c + \pi - t) = f(c + \pi + t)$.

|| This is known.

¶ Cf. Young, Proceedings of the London Mathematical Society, ser. 2, vol. 10 (1911), pp. 256–7.

$1, \cdots, i-1$. *Then $n^i a_n$ approaches zero if $c = 0$ or a multiple of π, and $f^{(p)}(c+2\pi) = f^{(p)}(c)$ when p is even; $n^i b_n$ approaches zero if $c = 0$ or a multiple of π, and $f^{(p)}(c+2\pi) = f^{(p)}(c)$ when p is odd; both approach zero if $f^{(p)}(c+2\pi) = f^{(p)}(c)$ for all the values of p.*

(*f*) *If f belongs to $G_5^{(i)}$, then with the same provisions, respectively, as in* (*e*), *$n^{i+1} a_n$ and $n^{i+1} b_n$ remain finite.*

Proof of (*b*).—Consider condition 1° of Theorem 1 for F_{4a}.

$$\int_c^{c+\pi} q_n \left| \begin{matrix} \sin nt \\ \cos nt \end{matrix} \right| dt \gtreqqless q_n \int_c^{c+\pi} \begin{matrix} \sin^2 nt \\ \cos^2 nt \end{matrix} dt = \frac{\pi}{2} q_n,$$

which is not limited.

Proof of (*c*).—Consider Theorem 1 for F_5.

$$\frac{n}{q_n} \left| \int_c^{c+t} \begin{matrix} \sin nt \\ \cos nt \end{matrix} dt \right| = \frac{1}{q_n} \left| \begin{matrix} \cos nc - \cos n(c+t) \\ \sin n(c+t) - \sin nc \end{matrix} \right| \eqqless \frac{2}{q_n},$$

which approaches zero.

Proof of (*d*).—Consider Theorem 1 for F_{5a}.

$$n \left| \int_c^{c+t} \begin{matrix} \sin nt \\ \cos nt \end{matrix} dt \right| \eqqless 2.$$

$$n \left| \int_a^{c+2\pi} (t-a) \sin nt \, dt \right| = \frac{1}{n} \left| \sin nc - \sin na + n(a-c-2\pi) \cos nc \right|,$$

which approaches zero if nc is an odd multiple of $\pi/2$. Similarly,

$$n \left| \int_a^{c+2\pi} (t-a) \cos nt \, dt \right| = \frac{1}{n} \left| \cos nc - \cos na + n(c+2\pi-a) \sin nc \right|,$$

which approaches zero if c is an even multiple of $\pi/2$. The conditions of the theorems for F_{5a} are thus satisfied, but condition (2°) for F_5 is not satisfied. For example, if $c = 0$ and $t = \pi/2$,

$$n \left| \int_0^{\frac{\pi}{2}} \cos nt \, dt \right| = \left| \sin \frac{n\pi}{2} \right|,$$

which does not approach zero.

Proof of (*e*) and (*f*).—I use here a well-known method.* Integration by parts is allowable† because f is an indefinite integral, and so we have

$$\int_c^{c+2\pi} f \sin nt \, dt = -\frac{\cos nc}{n} [f(c+2\pi) - f(c)] + \frac{1}{n} \int_c^{c+2\pi} f' \cos nt \, dt.$$

* E. g., Hobson, *Theory of Functions of a Real Variable* (1907), p. 718.

† Lebesgue, loc. cit., p. 46.

Therefore, if (1) either $\cos nc$ or $f(c+2\pi)-f(c)$ equals zero,

$$a_n = \frac{b_n'}{n}, \qquad \text{where} \qquad b_n' = \frac{1}{\pi}\int_c^{c+2\pi} f' \cos nt\, dt.$$

Likewise

$$\int_c^{c+2\pi} f \cos nt\, dt = \frac{\sin nc}{n}\,[f(c+2\pi)-f(c)] - \frac{1}{n}\int_c^{c+2\pi} f' \sin nt\, dt,$$

and, if (2) either $\sin nc$ or $f(c+2\pi)-f(c)$ equals zero,

$$b_n = \frac{a_n'}{n}, \qquad \text{where} \qquad a_n' = \frac{1}{\pi}\int_c^{c+2\pi} f' \sin nt\, dt.$$

We can obviously satisfy (2) by choosing $c=0$ or a multiple of π. We can satisfy both by making $f(c+2\pi)=f(c)$.

Repeating these processes and applying (a) we have (e); applying (c) we have (f).

From the foregoing propositions follow many corollaries concerning the convergence of Fourier's series. I will state two of them:

COROLLARY 1: *If f has limited variation, the series whose general term is $|a_n^p \sin^p nx| + |b_n^p \cos^p nx|$ converges uniformly when $p>1$.* For, by (c), this may be compared with the series whose general term is $1/n^p$.

COROLLARY 2: *If f is the integral of a function of limited variation and if $f(c)=f(c+2\pi)$, the series of Corollary 1 converges uniformly when $p>1/2$.* For, by (f) for G_5', it may be compared with the series whose general term is $1/n^{2p}$.

7. **Fourier's Development.**—In this section n and q_n are used as in the preceding section, and R_n represents the difference between $f(x)$ and the sum of the first terms of its Fourier's series up to and including the terms whose argument is nx. The function $f(t)$ is supposed defined everywhere, to be periodic of period 2π, and absolutely L-integrable in the interval $(-\pi, \pi)$. At x the limits $f(x+0)$ and $f(x-0)$ are supposed to exist, and $f(x)$ to be equal to one half their sum. As is well known,*

$$(1) \qquad \pi R_n = \int_{x-\pi}^{x+\pi} [f(t)-f(x)] \frac{\sin \frac{2n+1}{2}(t-x)}{2\sin\frac{t-x}{2}}\, dt$$

$$(2) \qquad = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [f(x+2v)-f(x)] \frac{\sin \nu v}{\sin v}\, dv \qquad (\nu = 2n+1).$$

Set

$$\psi(v) = \frac{f(x+2v)-f(x)}{\sin v} \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

* E. g., Pierpont, *Theory of Functions of Real Variables*, vol. 2, pp. 421, 2.

Reasoning analogous to that of the preceding article gives us the following propositions:

(*a*) *For some function* ψ *which is absolutely L-integrable,* $q_n R_n$ *does not approach zero, though it is known that* R_n *does approach zero.*

(*b*) *For some function* $f(t)$ *which is continuous in* $(-\pi, \pi)$ *and has a first derivative at* x, *the same is true.*

(*c*) *If* ψ *has limited variation,* nR_n *remains finite.*

(*d*) *The same is true if* $1/v[f(x+2v)-f(x)]$ *has limited variation in* $(-\pi/2, \pi/2)$.

(*e*) *If* ψ *is continuous and has limited variation,* nR_n *approaches zero, but this is not true of all functions having limited variation.*

(*f*) *If* ψ *is the integral of a function having limited variation (i. e., if* ψ *belongs to* G_5'), $n^2 R_n$ *remains finite.*

(*g*) *There exists a function* ψ *belonging to all the other G families, for which* $n^2 R_n$ *does not approach zero.*

Proof of (*a*).—We use (2) and condition 1° of Theorem 1 for F_1, putting $\phi = q_n \sin \nu v$. This function is not uniformly limited in $(-\pi/2, \pi/2)$. Therefore there exists an absolutely L-integrable function $g(v)$, such that the integral from $-\pi/2$ to $\pi/2$ of $g \sin \nu v$ does not approach zero. It can now be shown that a function $f(t)$ may be defined so that $g(v)$ has the form $\psi(v)$ above.

Proof of (*b*).—We use (1). The function $f(t)-f(x)$ satisfies the conditions on f in Corollary 1 of Theorem 2 for F_{6a}. We have, then, corresponding to ϕ,

$$\frac{q_n \sin \frac{\nu}{2}(t-x)}{2 \sin \frac{1}{2}(t-x)}.$$

Since, in the interval $(x-\pi, x+\pi)$, except at most at the three points, $x-\pi$, x, $x+\pi$,

$$\left| \frac{t-x}{2 \sin \frac{1}{2}(t-x)} \right| \geqq 1,$$

$$\int_{x-\pi}^{x+\pi} |(t-x)\phi| \, dt \geqq q_n \int_{x-\pi}^{x+\pi} \left| \sin \frac{\nu}{2}(t-x) \right| dt$$

$$\geqq q_n \int_{x-\pi}^{x+\pi} \sin^2 \frac{\nu}{2}(t-x) \, dt = q_n \pi,$$

and is not uniformly limited.

Proof of (*c*).—Using (2) and Theorem 1 for F_5, we have

$$\left| \frac{n}{q_n} \int_{-\frac{\pi}{2}}^{t} \sin \nu v \, dv \right| = \frac{n}{q_n \nu} \left| - \cos \nu t \right| \overset{=}{<} \frac{n}{q_n \nu},$$

and this approaches zero.

Proof of (*d*).—We use the same theorem, but now corresponding to ϕ we have $v \sin \nu v / \sin v$. By an inequality,* due to Lebesgue, for monotone increasing functions, and since in $(0, \pi/2)$ $v/\sin v$ is monotone increasing from 1 to $\pi/2$, we have,† if $0 \overset{=}{<} t \overset{=}{<} \pi/2$,

$$\left| \int_0^t \frac{v \sin \nu v}{\sin v} dv \right| < \frac{3\pi}{2} \max_{\lambda \text{ in } (0,\, t)} \left| \int_0^\lambda \sin \nu v \, dv \right| \overset{=}{<} \frac{3\pi}{2} \frac{2}{\nu}.$$

The same is true if $-\pi/2 \overset{=}{<} t \overset{=}{<} 0$, since the integrand is an odd function. Therefore

$$\frac{n}{q_n} \left| \int_{-\frac{\pi}{2}}^{t} \frac{v \sin \nu v}{\sin v} dv \right| < \frac{n}{q_n} \frac{6\pi}{\nu},$$

which approaches zero as n becomes infinite.

Proof of (*e*).—Using (2) and Theorem 1 for F_{5a}, we have

$$n \left| \int_{-\frac{\pi}{2}}^{t} \sin \nu v \, dv \right| \overset{=}{<} \frac{2n}{\nu} \overset{=}{<} 1,$$

$$n \left| \int_a^{\frac{\pi}{2}} (v - a) \sin \nu v \, dv \right| = \frac{n}{\nu^2} \left| \sin \frac{\nu \pi}{2} - \sin \nu a \right| \overset{=}{<} \frac{2n}{\nu^2},$$

$$n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \nu v \, dv = 0.$$

These show that the conditions in question are satisfied, and establish the first part of (*e*). As for the second part, we note that

$$n \left| \int_{-\frac{\pi}{2}}^{t} \sin \nu v \, dv \right| = \frac{n}{\nu} \left| \cos \nu t \right|.$$

This equals n/ν when $t = 0$, and therefore does not approach zero for each t. We may now proceed as in the proof of (*a*).

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* Lebesgue, loc. cit., p. 36, B. See also my paper (second footnote), p. 54.

† "$\underset{\lambda}{\text{Max}}$" means the maximum of the function with respect to the parameter λ. I use the words "maximum" and "minimum" in the sense of upper and lower limits, not of greatest and least values.

Proof of (f).—Using (2) and Theorem 3 for G_5', we have

$$\frac{n^2}{q_n}\left|\int_{-\frac{\pi}{2}}^{t} dt \int_{-\frac{\pi}{2}}^{t} \sin \nu v\, dv\right| = \frac{n^2}{q_n \nu^2}\,|\sin \nu t \pm 1| \lesseqgtr \frac{2n^2}{q_n \nu^2},$$

$$\frac{n^2}{q_n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \nu v\, dv = 0.$$

Proof of (g).—It would be practicable to show that for no G are the conditions of Theorem 3 satisfied when $\phi = n^2 \sin \nu v$, but it is easier to see directly that for the function $\psi(v) = v$, which belongs to all the families, $n^2 |R_n|$ remains greater than $2/9$ if $n > 1$. $\psi = v$, if $f(t) = (t/2)\sin(t/2)$ in $(-\pi, \pi)$ and $x = 0$.

$$n^2 |R_n| = n^2 \left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v \sin \nu v\, dv\right| = \frac{2n^2}{\nu^2} > \frac{2}{9}, \quad \text{if } n > 1.$$

8. **Legendre's Development.**—In this section n and q_n are used as hitherto, $f(t)$ is supposed defined and absolutely L-integrable in the interval $(-1, 1)$, P_n is Legendre's polynomial of order n, and A_n is the coefficient of P_n in Legendre's development of f:

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f P_n\, dt.$$

In the following lemmas, M is some fixed number independent of t and t_i.

LEMMA 1:

(a) $$\left|\int_{-1}^{t} P_n\, dt\right| < \frac{M}{n^{\frac{3}{2}}(1-t^2)^{\frac{1}{2}}} \qquad (t^2 \neq 1,\ n \geqq 2),$$

(b) $$\left|\int_{-1}^{t} P_n dt\right| < \frac{M}{n^{1.1}} \qquad (n \geqq 2).$$

LEMMA 2: *If $P_n(t)$ be integrated i times, with respect to the variables $t_0, t_1, \cdots t_{i-1}$,*

(a) $$\left|\int_{-1}^{t_i} \cdots \int_{-1}^{t_1} P_n\, dt_0\right| < \frac{M}{n^{i+\frac{1}{2}}(1-t_i^2)^{\frac{1}{2}}} \qquad (t_i^2 \neq 1,\ n \geqq 2i),$$

(b) $$\left|\int_{-1}^{t_i} \cdots \int_{-1}^{t_1} P_n\, dt_0\right| < \frac{M}{n^{i+0.1}} \qquad (n \geqq 2i).$$

LEMMA* 3:

(a) $$\int_{-1}^{t} P_n^2\, dt < \frac{1}{n} \qquad (n \geqq 1),$$

(b) $$\int_{-1}^{t_i}\left(\int_{-1}^{t_{i-1}} \cdots \int_{-1}^{t_1} P_n\, dt_0\right)^2 dt_{i-1} < \frac{M}{n^{2i-1}} \qquad (n \geqq 2i-2).$$

* Proposition (a) is known.

LEMMA* 4

(a) $$\int_{-1}^{t} |P_n|\, dt < \frac{M}{\sqrt{n}} \qquad (n \geqq 1),$$

(b) $$\int_{-1}^{t_i} \left| \int_{-1}^{t_{i-1}} \cdots \int_{-1}^{t_1} P_n\, dt_0 \right| dt_{i-1} < \frac{M}{n^{i-\frac{1}{2}}} \qquad (n \geqq 2i-1).$$

Proof of Lemma 1.—From the known equation

$$(2n+1)\, P_n = \frac{d}{dt}(P_{n+1}) - \frac{d}{dt}(P_{n-1}),$$

we derive

(1) $$\left| \int_{-1}^{t} P_n\, dt \right| = \left| \int_{-1}^{1} \right| + \left| \int_{1}^{t} \right| = \frac{1}{2n+1} \left| P_{n+1}(t) - P_{n-1}(t) \right| \quad (n > 0).$$

Hobson† has proved the inequality

(2) $$(1-t^2)^{\frac{1}{4}} \left| P_n(t) \right| < \frac{M}{\sqrt{n}} \qquad (n > 0).$$

This with (1) proves that

$$\left| \int_{-1}^{t} P_n\, dt \right| < \frac{M}{2n+1} \left(\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n-1}} \right) \frac{1}{(1-t^2)^{\frac{1}{4}}} < \frac{3M}{n^{\frac{3}{2}} (1-t^2)^{\frac{1}{4}}}$$

$$(t^2 \neq 1,\ n \geqq 2),$$

which is (a). To establish (b) we note first that‡

(3) $$\left| \int_{-1}^{-1+x} \frac{dt}{(1-t^2)^{\frac{1}{4}}} \right|, \quad \left| \int_{1-x}^{1} \frac{dt}{(1-t^2)^{\frac{1}{4}}} \right| \leqq \frac{4}{3} x^{\frac{3}{4}} \qquad (0 \leqq x \leqq 1).$$

Case 1: $-1 + 1/n^{0.8} \leqq t \leqq 1 - 1/n^{0.8}$.

(4) $$\left| \int_{-1}^{t} P_n\, dt \right| \leqq \int_{-1}^{-1+1/n^{0.8}} |P_n|\, dt + \left| \int_{-1+1/n^{0.8}}^{t} P_n\, dt \right| = \mathrm{I} + \mathrm{II}.$$

By (2) and (3),

(5) $$\mathrm{I} < \frac{4}{3} \left(\frac{1}{n^{0.8}} \right)^{\frac{3}{4}} \frac{M}{\sqrt{n}} = \frac{4M}{3n^{1.1}} \qquad (n > 0).$$

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* I do not need this, but I put it down as a simple corollary of the others; (a) is known.

† Hobson, Proceedings of the London Mathematical Society, ser. 2, vol. 7 (1909), p. 25.

‡ For example, the first of these equals, if we put $t = z - 1$,

$$\left| \int_0^x \frac{dz}{(2z - z^2)^{\frac{1}{4}}} \right| \leqq \left| \int_0^x \frac{dz}{z^{\frac{1}{4}}} \right| = \frac{4}{3} x^{\frac{3}{4}}, \text{ if } 0 \leqq x \leqq 1.$$

By (2), since $1 - t^2 \overline{\overline{>}} 1/n^{1.6}$, and

$$\int_a^b P_n\,dt = \frac{1}{2n+1}\,[P_{n+1}(t) - P_{n-1}(t)]_a^b,$$

$$\text{(6)} \qquad \text{II} < \frac{1}{2n+1}\cdot\frac{4M}{\sqrt{n-1}}\cdot n^{0.4} < \frac{4M}{2n}\cdot\sqrt{\frac{2}{n}}\cdot n^{0.4} < \frac{4M}{n^{1.1}} \qquad (n>1).$$

The inequalities (5) and (6) establish Case 1.

Case 2: $-1 \overline{\overline{<}} t \overline{\overline{<}} -1 + 1/n^{0.8}$.

$$\left|\int_{-1}^{t} P_n dt\right| \overline{\overline{<}} \int_{-1}^{-1+1/n^{0.8}} |P_n|\,dt,$$

and by (5) this is less than $M/n^{1.1}$ if $n > 0$. Likewise, if

Case 3: $1 - 1/n^{0.8} \overline{\overline{<}} t \overline{\overline{<}} 1$,

$$\left|\int_{-1}^{t} P_n dt\right| \overline{\overline{<}} \left|\int_{-1}^{1-1/n^{0.8}} P_n dt\right| + \int_{1-1/n^{0.8}}^{t} |P_n|\,dt < \frac{2M}{n^{1.1}} \qquad (n>0),$$

by Case 1, (2), and (3).

Proof of Lemma 2.—Lemma 2 is proved by means of Lemma 1 and equalities like the following:

$$\int_{-1}^{t_2} dt_1 \int_{-1}^{t_1} P_n\,dt_0 = \frac{1}{2n+1}\left[\frac{1}{2n+3}P[_{n+2}(t_2) - P_n(t_2)] - \frac{1}{2n-1}[P_n(t_2) - P_{n-2}(t_2)]\right],$$

$$\left|\int_{-1}^{t_2} dt_1 \int_{-1}^{t_1} P_n\,dt_0\right| = \left|\int_{-1}^{t_2} \frac{1}{2n+1}[P_{n+1}(t_1) - P_{n-1}(t_1)]\,dt_1\right| \overline{\overline{<}} \frac{1}{2n+1}\left(\left|\int_{-1}^{t_2} P_{n+1}\,dt_1\right| + \left|\int_{-1}^{t_2} P_{n-1}\,dt_1\right|\right).$$

Proof of Lemma 3(*b*).—By (*a*) of Lemma 2,

$$\int_{-1}^{t_i} (\qquad)^2 \cdot dt_{i-1} < \int_{-1}^{1} \frac{M^2}{n^{2i-1}\sqrt{(1-t_{i-1}^2)}}\,dt_{i-1} = \frac{\pi M^2}{n^{2i-1}} \quad (n \overline{\overline{>}} 2i-2).$$

Proof of Lemma 4(*b*).—By (*a*) of Lemma 2, and (3),

$$\int_{-1}^{t_i} |\qquad|\,dt_{i-1} < \int_{-1}^{1} \frac{M}{n^{i-\frac{1}{2}}(1-t_{i-1}^2)^{\frac{1}{2}}}\,dt_{i-1} = \int_{-1}^{0} + \int_{0}^{1} < \frac{2M}{n^{i-\frac{1}{2}}}\,\frac{4}{3} < \frac{4M}{n^{i-\frac{1}{2}}}$$

$$(n \overline{\overline{>}} 2i-2).$$

The following statements are now easily established.

(*a*) *For every absolutely L-integrable function, A_n/n approaches zero, but there exists such a function for which $q_n A_n/n$ does not approach zero.*

(*b*) *For every function whose square is L-integrable,* $A_n / \sqrt{n}$ *approaches zero, but there exists such a function for which* $q_n A_n / \sqrt{n}$ *does not approach zero.*

(*c*) *There exists a continuous* function for which* $q_n A_n$ *does not approach zero.*

(*d*) *For every function of limited variation,* $n^{0.1} A_n$ *approaches zero.*

(*e*) *For every function of* $G_2^{(i)}$, $n^{i-\frac{1}{2}} A_n$ *approaches zero.*

(*f*) *For every function of* $G_5^{(i)}$, $n^{i+0.1} A_n$ *approaches zero.*

Proof of (*a*).—The first part of (*a*) follows from Theorem 1 for F_1, ϕ being† P_n, for P_n is limited, and by Lemma 1 (*b*) its integral from -1 to t approaches zero. But if ϕ be $q_n P_n$, condition (1°) of that theorem is not satisfied.

Proof of (*b*).—Using Theorem 1 for F_2, $\phi = \sqrt{n} P_n$, and Lemma 1 (*b*), we have

$$\int_{-1}^{1} \frac{n^2 P_n^2}{n} dt = \frac{2n}{2n+1} < 1 \qquad (n \geqq 1),$$

$$\int_{-1}^{t} \frac{n P_n}{\sqrt{n}} dt < \frac{M}{n^{0.6}}.$$

Proof of (*c*).—Condition (1°) of Theorem 1 for F_{4a} is not satisfied by $\phi = q_n n P_n$, for

$$q_n \int_{-1}^{1} n \,|\, P_n \,|\, dt \geqq q_n n \int_{-1}^{1} P_n^2 \, dt = \frac{2n\, q_n}{2n+1}.$$

Proof of (*d*).—Putting $\phi = n^{1.1} P_n$ in Theorem 1 for F_5, we have, by Lemma 1 (*b*),

$$\left| \int_{-1}^{t} n^{1.1} P_n \, dt \right| < M \qquad (n \geqq 2),$$

and, by Lemma 1 (*a*),

$$\left| \int_{-1}^{t} n^{1.1} P_n \, dt \right| < \frac{n^{1.1} M}{n^{1.5} (1 - t^2)^{\frac{1}{2}}} \quad \text{if } t^2 \neq 1 \qquad (n \geqq 2),$$

$$= 0 \qquad \text{if } t^2 = 1 \qquad (n \geqq 1).$$

Proof of (*e*).—Consider G_2'. (The reasoning is analogous for the other G_2's.) Applying the first condition of Theorem 3 for G_2', and using $\phi = n^{\frac{3}{2}} P_n$, we have, by Lemma 3 (*b*), and Lemma 2 (*b*),

$$\int_{-1}^{1} \left(\int_{-1}^{t_1} n^{\frac{3}{2}} P_n \, dt_0 \right)^2 dt_1 < \frac{n^3 M}{n^3} = M \qquad (n \geqq 2),$$

$$\int_{-1}^{t_2} \left(\int_{-1}^{t_1} n^{\frac{3}{2}} P_n \, dt_0 \right) dt_1 < \frac{n^{\frac{3}{2}} M}{n^{2.1}} \qquad (n \geqq 4),$$

$$\int_{-1}^{1} n^{\frac{3}{2}} P_n \, dt_0 = 0 \qquad (n \geqq 1).$$

* But I cannot show that for every continuous function A_n remains limited.

† Exactly, $\phi = [(2n+1)/2n] P_n$, but this approaches the same limit as P_n. Similar remarks hold for the other proofs.

Proof of (f).—Consider for example G_5'. Using Theorem 3 for G_5' and $\phi = n^{2.1} P_n$, and Lemma 2 (b) and (a), we have

$$\left| \int_{-1}^{t_2} \left(\int_{-1}^{t_1} n^{2.1} P_n \, dt_0 \right) dt_1 \right| < \frac{n^{2.1} M}{n^{2.1}} = M \qquad (n \gtreqless 4),$$

$$< \frac{n^{2.1} M}{n^{2.5} (1 - t_2^2)^{\frac{1}{4}}} \quad \text{if } t_2^2 \neq 1 \quad (n \gtreqless 4),$$

$$= 0 \qquad \text{if } t_2^2 = 1 \quad (n \gtreqless 2),$$

the last result being obtained by use of the proof of Lemma 2. Similarly,

$$\int_{-1}^{1} n^{2.1} P_n \, dt_0 = 0 \qquad (n \gtreqless 1).$$

COROLLARY 1: *If f is the integral of a function which has limited variation in the interval $(-1, 1)$, its Legendre development converges absolutely uniformly in the same interval.*

For, by (f), the absolute value series may be compared with the series whose general term is $1/n^{1.1}$.

COROLLARY 2: *If f has limited variation in $(-1, 1)$, the series whose general term* is $|A_n| P_n^2$ converges uniformly in any interval within $(-1, 1)$.*

For, by (2), in any such interval nP^2 remains uniformly limited, and therefore, by (d), the series in question may be compared with the series whose general term is $1/n^{1.1}$.

We may now apply to the developments of Legendre the same reasoning as was used in the case of Fourier's developments, and obtain the following results concerning their rates of convergence:†

COROLLARY 3: *Let R_n be the remainder of the formal development of f at the point x, and let $\psi(t)$ be the fraction $[f(t) - f(x)]/(t - x)$.*

(a) If ψ is absolutely L-integrable, R_n/n approaches zero.

(b) If ψ^2 is L-integrable, $R_n/\sqrt{n}$ approaches zero.

(c) If ψ has limited variation, $n^{0.1} R_n$ approaches zero.

(d) If ψ belongs to $G_5^{(i)}$, $n^{i+0.1} R_n$ approaches zero.

9. **Definite Integrals.**—Much of this section, in a somewhat different form, has been established by W. H. Young‡ independently of Lebesgue's theorems, but by means of reasoning closely analogous to that of Lebesgue. The present paper shows that the results are really only corollaries of Lebesgue's

* À fortiori this is true of the series whose general term is $A_n^2 P_n^2$.

† Cf. D. Jackson, *On the Degree of Convergence*, etc., these Transactions, vol. 13 (1912), pp. 305–318; and *On Approximation by Trigonometric Sums*, etc., ibid., pp. 491–515.

‡ Young, Proceedings of the London Mathematical Society, ser. 2, vol. 9 (1910), pp. 463–485.

fundamental theorem (§ 2), and possesses the advantage that the conditions stated here are thus shown to be necessary as well as sufficient. Young has nothing to do with the family F_6, and in Theorem 6 I consider several families where Young considers but one. His account includes, however, two topics which mine omits: non-absolutely convergent L-integrals, and integrals over infinite intervals.

Let us consider the evaluation of the definite integral

$$\int_k^l f(t)\, g(t)\, dt, \tag{1}$$

by means of the common method of expanding $g(t)$ in a series,

$$g(t) = u_1(t) + u_2(t) + \cdots \tag{2}$$

(the sum of the first n terms of which shall be called $s_n(t)$), multiplying it through by $f(t)$, and integrating termwise from k to l, thus:

$$\int_k^l fg dt = \int_k^l fu_1\, dt + \int_k^l fu_2\, dt + \cdots. \tag{3}$$

Concerning the validity of this process we have the following

THEOREM 5: *Let the various integrals of* (3) *exist as absolutely convergent L-integrals. A necessary and sufficient condition that* (3) *shall be valid for all functions f belonging to one of the families F of Theorems* 1 *and* 2, *is that* $g - s_n$ *shall satisfy the conditions imposed on* ϕ *in that part of those theorems that applies to the F in question.*

In the cases of* F_1, F_2, *and* F_3, *the condition stated remains sufficient if* (2°) *of Theorem* 1 *be replaced by the condition that* (2) *be valid in* (k, l), *except perhaps at a null set.*

For it is necessary and sufficient that

$$\int_k^l (g - s_n) f\, dt$$

approach zero, and therefore Theorem 1 (with corollary) and Theorem 2 are applicable.

Continuing the same problem and notation, let us now suppose that it is not convenient to express g or its 1st, 2d, . . . , or $(i-1)$th integral $(i \gtreqless 1)$ by means of a series, but that its ith integral may be so expressed, thus:

$$J(t_i) = \int_k^{t_i} \cdots \int_k^{t_i} g\, dt_0 = U_1(t_i) + U_2(t_i) + \cdots. \tag{1}$$

Writing

$$S_n = U_1 + \cdots + U_n, \qquad R_n = J - S_n, \tag{2}$$

* Stated in another paper (see second footnote), pp. 63, 64.

and denoting differentiation by primes, we have, except at most at a null set,

$$J^{(i)} = g = s_n + r_n. \tag{3}$$

We now give the conditions that

$$\int_k^l fg\,dt = \lim_{n=\infty} \int_k^l s_n\,dt\,; \tag{4}$$

that is, that (1) may be differentiated termwise i times, then multiplied through by f and integrated termwise from k to l, giving the integral of fg.

THEOREM 6: *Let* ($a°$) *the U's have i derivatives, and let all the integrals involved exist as absolutely convergent L-integrals. Let* ($b°$) $\lim_n R_n^{(p)}(k) = \lim_n R_n^{(p)}(l) = 0$, $p = 0, 1, \cdots, i-1$. *Then a necessary and sufficient condition that* (4) *be valid, for all functions belonging to the family $G_j^{(i)}$, is that R_n satisfy the conditions on ϕ in Theorem* 1, *or* 2, *for $F_j^{(i)}$.*

In the cases of $G_1^{(i)}$, $G_2^{(i)}$, and $G_3^{(i)}$, the conditions remain sufficient if it be required that (1) *be valid in* (k, l), *except perhaps at a null set, instead of that R_n satisfy* (2°) *of Theorem* 1.

Proof.—From ($a°$) and successive applications of (d), § 4, it follows that

$$\begin{aligned} \int_k^{t_{i-p}} \cdots \int_k^{t_1} r_n\,dt_0 &= R_n^{(p)}(t_{i-p}) + [\,P_0 R_n^{(p)}(k) + \cdots + P_{i-p-1} R_n^{i-1}(k)\,] \\ &= R_n^{(p)} + \eta \qquad (p = 0, \cdots, i-1), \end{aligned} \tag{5}$$

where P_s is a polynomial of degree s in t_{i-p}. By ($b°$), η approaches zero uniformly with respect to t_{i-p}. It is necessary and sufficient that

$$\lim_{n=\infty} \int_k^l fr_n\,dt = 0, \tag{6}$$

and for this the conditions of Theorem 3 relative to $G_j^{(i)}$ are necessary and sufficient. To see that the theorem is true it is now only necessary to write out the conditions involved in that theorem and in our present hypothesis in full. This will be done for the case of $G_1^{(i)}$ only. Theorem 3 says that the first of the following integrals shall remain uniformly limited if n is large enough (except perhaps at a null set), and that the others shall approach zero.

$$\int_k^{t_i} \cdots \int_k^{t_1} r_n\,dt_0, \tag{7}$$

$$\int_k^{t_{i+1}} \cdots \int_k^{t_1} r_n\,dt_0, \tag{8}$$

$$\int_k^l \int_k^{t_{i-1}} \cdots \int_k^{t_1} r_n\,dt_0, \quad \int_k \int_k^{t_{i-2}} \cdots \int_k^{t_1} r_n\,dt_0, \quad \cdots, \quad \int_k^l r_n\,dt_0. \tag{9}$$

Sufficiency.—By ($b°$) $\lim_{n=\infty} R_n^{(p)}(l) = 0$, $p = 0, \cdots, i-1$, and this with (5) shows that (9) approach zero. By hypothesis and (1°) of Theorem 1, $|R_n(t_i)| < M$, $n > n_M$, except perhaps at a null set, and this with (5) shows that the same is true of (7), and again, by (2°) of Theorem 1,

$$\lim_{n=\infty} \int_k^{t_{i+1}} R_n \, dt_i = 0,$$

and therefore, since η approaches zero uniformly, (8) approaches zero.

Necessity.—If either (1°) or (2°) of Theorem 1 is not satisfied by R_n, it follows from (5) that either (7) or (8), both necessary conditions, is not satisfied.

The corollary of Theorem 1 applies to the second part of the theorem.

10. **Termwise Integration.**—We might now consider the following similar problem. Let g be differentiable i times, and let its ith derivative be represented by a series. Under what conditions may this series be integrated termwise i times, then multiplied through by f and again integrated termwise, giving the integral of fg? But this method of evaluating a definite integral would not ordinarily be of advantage, because it is integration and not differentiation that usually leads to simpler functions, though it does frequently happen that it is no *harder* to evaluate an integral in this way than in one of the others, for what is lost in making the expansion more difficult is gained from the extra generality of the function f by which its integral may be multiplied. However, I wish to present the matter in another form. As already noted, a necessary and sufficient condition that termwise integration of a convergent sequence $\{\phi(n)\}$ be permissible is (1°) of Theorem 3. Let us suppose that ϕ satisfies not only this condition but also the severer condition (1°) of Theorem 1. Is anything more than termwise integration permissible? An answer to this question is found in Theorem 7, for which I introduce the following definition and lemmas.

Definition: Let us denote by $E_j^{(i)}(x)$ the family of functions f such that, for a certain preassigned x in (k, l), $f(t)(t-x)^i$ belongs to F_j, i being a positive integer or zero.*

Lemma 1: *In order that the integral of Theorem* 1 *shall exist and approach zero for all functions f belonging to the family $E_j^{(i)}$, it is necessary and sufficient that the conditions of Theorem* 1 *for F_j be satisfied when ϕ is replaced by $\phi/(t-x)^i$.*

This is a simple corollary of Theorem 1.

Lemma 2: *If $g(t)$ is continuous in (k, l), the ith iterated integral $J(t_i)$ may be expressed thus:*

$$J(t_i) = \int_x^{t_i} \cdots \int_x^{t_1} g \, dt_0 = \frac{(t_i - x)^i}{i!} g(\tau), \tag{1}$$

where τ is a point of the interval (x, t_i).

* E. g., $t^{-\frac{3}{2}}$ belongs to $E_1'(0)$ in $(0, 1)$. As far as Lemma 1 is concerned i might be any real number.

For, by successive applications of (*a*), § 4, we may learn that, for each p $(p = 0, 1, \cdots, i-1)$ and at each point of (k, l),

$$\frac{d^p J}{dt^p} = \int_x^{t_{i-p}} \cdots \int_x^{t_1} g\, dt_0, \qquad \frac{d^i J}{dt^i} = g. \tag{2}$$

By a well-known theorem we may therefore write

$$J(t_i) = J(x) + \left.\frac{dJ}{dt}\right|_x \frac{t_i - x}{1!} + \cdots + \frac{(t_i - x)^i}{i!} g(\tau),$$

and, evidently, by (1) and (2), all the terms on the right vanish except the last.

THEOREM 7: *If, in the interval* (k, l), (1°) *of Theorem* 1 *is satisfied, and* $\{\phi(n)\}$ *is a convergent sequence having continuous terms and a continuous limit, it is permissible* (*a*) *to integrate* $\{\phi(n)\}$ *termwise* i *times from* x *to* t, *and* (*b*) *to multiply both sides of the resulting equation by any function* f *belonging to* $E_1^{(i)}(x)$, *and again to integrate termwise from* a *to* b, a, b, *and* x *being any points of the interval, thus:*

$$\lim_{n=\infty} \int_x^{t_i} \cdots \int_x^{t_1} \phi(n)\, dt_0 = \int_x^{t_i} \cdots \int_x \lim_{n=\infty} \phi(n)\, dt_0, \tag{a}$$

$$\lim_{n=\infty} \int_a^b f(t_i)\, dt_i \int_x^{t_i} \cdots \int_x^{t_1} \phi(n)\, dt_0 = \int_a^b f(t_i)\, dt_i \int_x^{t_i} \cdots \int_x^{t_1} \lim_{n=\infty} \phi(n)\, dt_0. \tag{b}$$

Proof.—Without loss of generality we may suppose that $\lim \phi(n) = 0$, for otherwise the limit of the sequence $\{\phi(n) - \lim \phi(n)\}$ is zero, and this latter sequence possesses the properties ascribed to $\{\phi\}$ above.

Case 1: $(i = 0)$. We have to show merely that, for f in $(E_1^0 = F_1)$,

$$\lim_{n=\infty} \int_a^b f\phi(n)\, dt = 0; \tag{b}$$

and this has been established already in the corollary to Theorem 1.

Case 2: $(i > 0)$. Let us suppose that (*b*) is true when i is replaced by $r - 1$; if we choose $f = 1$, the result is (*a*) for $i = r$. Hence (*a*) may be obtained by induction from (*b*) for Case 1. To establish (*b*) for Case 2 it is, by Lemmas 1 and 2, necessary and sufficient that

$$\lim_{n=\infty} \int_a^b f(t_i)\, \phi(n, \tau) \frac{(t_i - x)^i}{i!} dt_i = 0.$$

From the corollary to Theorem 1 and Lemma 1, it is clear that for this it is sufficient that $\phi(n, \tau)$ be uniformly limited in (a, b), and that for each τ its limit be zero. Both these conditions are satisfied by hypothesis.

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CONGRUENCES AND COMPLEXES OF CIRCLES*

BY

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The differential geometry of circle systems has received a large amount of attention of late. The subject has been approached from two quite distinct sides. On the one hand we have Koenigs, Cosserat, Moore, Bompiani and others who fix their attention on what we may call the descriptive differential properties of circles. The totality of circles in three-dimensional space may be represented by an S_6^5, lying in an S_9, and this variety may be studied by the now familiar methods of projective differential geometry. The theorems so reached are invariant under the twenty-four parameter group of sphere transformations. The other class of writers, wherein we may include Bianchi, Tzitzeica, Guichard and Eisenhart, have confined themselves largely to congruences (two-parameter systems) of circles, frequently to normal congruences. The methods employed have been the general ones of differential geometry and the center of interest has been rather more in the axes of the circles than the circles themselves.

It has seemed for some time to the present writer that the last word on these subjects had not by any means been written, and that by a different method of approach not only might we obtain simpler proofs of known theorems but discover a number of new theorems as well. The most interesting properties of circles are those which are invariant under inversion, or under the ten-parameter group of conformal transformations of space; the best approach to this group is through the use of pentaspherical coördinates. It is true that some of the writers mentioned above have made use of these coördinates, and still more has been done by Darboux in his *Théorie des Surfaces*, yet the possibilities of these coördinates have been by no means exhausted, and in the present paper they are more systematically applied to problems in differential circle geometry than has been the case in the past.

A circle may be regarded in two different aspects, either as a locus of points, or an envelop of spheres. The two points of view are, of course, closely related, but the change of emphasis leads naturally to rather different sets of theorems. The first section of the present article is devoted to preliminary formulæ for points and spheres in pentaspherical coördinates, and certain

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fundamental theorems concerning cospherical and orthogonal circles. In section 2 a parametric representation is given of the points of a circle, and it is shown how this leads to direct and elegant proofs of the usual theorems about circle congruences, in so far as they are invariant under the conformal group. In the third section the circle is considered in its other aspect, and the classic methods of the Kummer line-geometry are applied to circle congruences, thus giving a goodly number of new theorems. The fourth section is devoted to three-parameter families of circles.

§ 1. Preliminary Formulæ.

In the present work we shall confine ourselves to points in the finite region of space, locating each by a system of homogeneous pentaspherical coördinates, whose relation to the homogeneous rectangular Cartesian system is exhibited by the following equations:

$$
(1)\qquad
\begin{aligned}
\rho x_0 &= i\,(x^2 + y^2 + z^2 + t^2), & \sigma x &= x_2,\\
\rho x_1 &= x^2 + y^2 + z^2 - t^2, & \sigma y &= x_3,\\
\rho x_2 &= 2xt, & \sigma z &= x_4,\\
\rho x_3 &= 2yt, & \sigma t &= -\,(ix_0 + x_1),\\
\rho x_4 &= 2zt,
\end{aligned}
$$

$$
(2)\qquad \sum_{i=0}^{i=4} x_i^2 \equiv (xx) = 0.
$$

A sphere will be given by an equation of the form

$$
(3)\qquad \sum_{i=0}^{i=4} y_i\, x_i \equiv (yx) = 0.
$$

The sphere will be null if (y) satisfy the identity (2) in which case y_j will be the coördinates of the center in our present system. The angle of two not null spheres (y) and (z) will be

$$
(4)\qquad \theta = \cos^{-1} \frac{(yz)}{\sqrt{(yy)}\,\sqrt{(zz)}}.
$$

It is convenient when a sphere is not null to assume that its coördinates (coefficients) satisfy the identity

$$
(5)\qquad (yy) = 1.
$$

If (x) and (x') be two points, then every sphere (y), where

$$
y_i = \lambda x_i + \mu x_i',
$$

will be a sphere of inversion to interchange the two. We see, in fact, that every sphere orthogonal to (x) and (x') is orthogonal to (y) also. By our identity,

$$(xx) = (x'x') = 0.$$

If then

$$(xx') = 0$$

the sphere (y) will be null. The circle common to the null spheres whose centers are (x) and (x') will contain none but null spheres, i. e., (x) and (x') lie on an isotropic line. Suppose that we have two infinitely near points (x) and $(x + dx)$,

$$(xdx) = -\tfrac{1}{2}(dxdx).$$

Since this is a differential expression of the second order, we may take (dx) as the coördinates of the sphere through (x) orthogonal to the direction of advance to $(x) + (dx)$. The condition that two directions of advance should be mutually perpendicular is thus

$$(dx\delta x) = 0. \tag{6}$$

If a direction of advance be isotropic we have

$$(dxdx) = 0. \tag{7}$$

Through each circle of finite radius will pass two null spheres whose centers are called the *foci* of the circle. The necessary and sufficient condition that a circle should be null is that the foci should fall together. If, thus, the circle be given by the spheres (y) and (z) it will be null if we have equal roots in the quadratic

$$\lambda^2(yy) + 2\lambda\mu(yz) + \mu^2(zz) = 0,$$

that is,

$$(yy)(zz) - (yz)^2 = 0.$$

The following well-known theorems are given for reference:

A. A necessary and sufficient condition that two circles of finite radius should be cospherical is that their foci should be concyclic.

B. A necessary and sufficient condition that two circles of finite radius should touch one another is that their foci should lie on two intersecting isotropic lines.

C. If two circles be cospherical they will be cospherical and orthogonal to an infinite number of circles; if they be not cospherical and their foci do not lie in pairs on two isotropics, there will be two circles cospherical and orthogonal to both. The last two circles are in bi-involution, i. e., every sphere through the one is orthogonal to every sphere through the other.

There remains the case where the foci of the two circles lie in pairs on two skew isotropics. These two will be generators of one system of a sphere.

There is a one-parameter group of conformal transformations leaving invariant these two generators as well as all generators of the other system. For instance, if the fixed sphere were

$$x_0 = 0$$

we might consider transformations of the quaternion type

$$(x_1' + ix_2' + jx_3' + kx_4') = (a + bi + cj + dk)(x_1 + ix_2 + jx_3 + kx_4),$$
$$i^2 = j^2 = k^2 = ijk = -1.$$

It appears thus that circles cospherical and orthogonal to our given circles will be continuously transformed by this group, so that there are an infinite number of such circles. Our original circles shall be said in this case to be paratactic.*

D. A necessary and sufficient condition that two non-cospherical circles should be cospherical and orthogonal to more than two circles is that they should be paratactic.

§ 2. The Parametric Method. The Classical Theorems of Ribaucour and Darboux.

Let a circle be determined by a sphere (β) to which it is orthogonal, and the two points of intersection therewith (α) and (γ). The coördinates of every point thereon may be exhibited in the parametric form

$$x_i = \rho^2 \alpha_i + \rho\beta_i + \gamma_i,$$
$$(\alpha\alpha) = (\alpha\beta) = (\beta\gamma) = (\gamma\gamma) = 0, \tag{8}$$
$$(\beta\beta) = -2(\alpha\gamma) = 1.$$

This form is especially suitable to the study of congruences of circles. We therefore assume that $(\alpha), (\beta), (\gamma)$ are functions of two independent variables u and v, the various ratios being naturally supposed not to be all constant or functions of a single variable. We have the additional equations

$$\left(\alpha\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\alpha}{\partial v}\right) = \left(\beta\frac{\partial\beta}{\partial u}\right) = \left(\beta\frac{\partial\beta}{\partial v}\right) = \left(\gamma\frac{\partial\gamma}{\partial u}\right) = \left(\gamma\frac{\partial\gamma}{\partial v}\right) = 0,$$
$$\left(\alpha\frac{\partial\beta}{\partial u}\right) + \left(\beta\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\beta}{\partial v}\right) + \left(\beta\frac{\partial\alpha}{\partial v}\right) = \left(\beta\frac{\partial\gamma}{\partial u}\right) + \left(\gamma\frac{\partial\beta}{\partial u}\right) = \tag{9}$$
$$\left(\beta\frac{\partial\gamma}{\partial v}\right) + \left(\gamma\frac{\partial\beta}{\partial v}\right) = \left(\alpha\frac{\partial\gamma}{\partial u}\right) + \left(\gamma\frac{\partial\alpha}{\partial u}\right) = \left(\alpha\frac{\partial\gamma}{\partial v}\right) + \left(\gamma\frac{\partial\alpha}{\partial v}\right) = 0.$$

Our groups of equations (8) and (9) lead to easy proofs of all the classical

* The concept of paratactic circles seems first to have been introduced in an article by the Author, *A Study of the Circle Cross*, These Transactions, vol. 14 (1913), pp. 149–174. Proofs of our theorems *A–D* will there be found.

theorems about circle congruences, in so far, that is, as these theorems are invariant for inversion. We state a number of these theorems which are particularly important. They will be found for the most part in the second volume of Darboux's *Theorie generale des surfaces,* but mostly trace their origin to Ribaucour.*

THEOREM 1. *The circles of a congruence are, in general, tangent to four surfaces.*

In practice it is more convenient to say that a circle in general position intersects four adjacent circles. When this intersection arises from the fact that it is cospherical with two adjacent circles, i. e., strictly speaking, when each circle is a generator of two annular surfaces of the congruence, the latter is said to be *focal.* The corresponding generating spheres through the circle are called the *focal spheres.* The points where the circle touches the four surfaces are the *focal points,* and must not be confused with the *foci* mentioned above. If the circles of a congruence be orthogonal to an infinite number of surfaces, the congruence is said to be *normal.* In counting the number of orthogonal trajectories attached to a given congruence, we count the number of points on each circle where it meets some surface orthogonally. Thus, the circles orthogonal to a given sphere are said to have (at least) *two* orthogonal trajectories.

THEOREM 2. *If the circles of a congruence be orthogonal to more than two surfaces, the congruence is normal.*

THEOREM 3. *Any four orthogonal trajectories of the circles of a normal congruence will meet those circles in sets of points having a fixed cross ratio.*

THEOREM 4. *The foci of the circles of a focal congruence generate the two nappes of the envelope of a congruence of spheres.*

THEOREM 5. *If the foci of the circles of a congruence be the pairs of points of contact of the spheres of a non-parabolic congruence with their envelope, the congruence of circles is focal.*

THEOREM 6. *A necessary and sufficient condition that a focal congruence should be normal is that the focal spheres through a circle in general position should be mutually orthogonal, or that it should consist in circles through two points.*

THEOREM 7. *Every normal congruence is focal.*

THEOREM 8. *In a normal congruence the lines of curvature correspond to one another in all orthogonal trajectories, and give the annular surfaces of the congruence.*

THEOREM 9. *If a congruence of spheres have an envelope of two nappes whereon the lines of curvature correspond to one another, the circles orthogonal to the various spheres at their points of contact will generate a normal congruence.*

* See his two notes *Sur les systemès cyclicques* and *Sur les faisceaux de cercles,* Paris, Comptes Rendus, vol. 76 (1873); also an earlier note *Sur la déformation des surfaces,* ibid., vol. 70 (1870).

THEOREM 10. *If the circles of a congruence have two orthogonal trajectories the congruence will be focal when, and only when it is normal.*

THEOREM 11. *If the circles of a congruence be normal to two surfaces which do not consist in a sphere counted twice, and can be assembled into a one-parameter family of annular surfaces, the congruence is normal.*

As examples of the method of applying our equations (8), (9) to the proof of these theorems, we select theorems 2, 3, 6, and 8.

2. We wish to make t in (8) such a function of u and v that if dx_i be the differential along this surface, and δx_i the differential along the circle, then

$$(dx\,\delta x) = 0.$$

This involves

$$-\frac{\partial t}{\partial u} = \left(\alpha\frac{\partial\beta}{\partial u}\right)t^2 + 2\left(\alpha\frac{\partial\gamma}{\partial u}\right)t + \left(\beta\frac{\partial\gamma}{\partial u}\right),$$

$$-\frac{\partial t}{\partial v} = \left(\alpha\frac{\partial\beta}{\partial v}\right)t^2 + 2\left(\alpha\frac{\partial\gamma}{\partial v}\right)t + \left(\beta\frac{\partial\gamma}{\partial v}\right).$$

The condition for compatibility becomes, in the light of (9),

$$\begin{aligned}
&\left\{\left(\frac{\partial\alpha}{\partial v}\frac{\partial\beta}{\partial u}\right) - \left(\frac{\partial\alpha}{\partial u}\frac{\partial\beta}{\partial v}\right) - 2\left[\left(\alpha\frac{\partial\beta}{\partial u}\right)\left(\alpha\frac{\partial\gamma}{\partial v}\right) - \left(\alpha\frac{\partial\beta}{\partial v}\right)\left(\alpha\frac{\partial\gamma}{\partial u}\right)\right]\right\}t^2 \\
&+ 2\left\{\left(\alpha\frac{\partial\beta}{\partial v}\right)\left(\beta\frac{\partial\gamma}{\partial u}\right) - \left(\alpha\frac{\partial\beta}{\partial u}\right)\left(\beta\frac{\partial\gamma}{\partial v}\right) + \left(\frac{\partial\alpha}{\partial v}\frac{\partial\gamma}{\partial u}\right) - \left(\frac{\partial\alpha}{\partial u}\frac{\partial\gamma}{\partial v}\right)\right\}t \\
&+ \left\{\left(\frac{\partial\beta}{\partial v}\frac{\partial\gamma}{\partial u}\right) - \left(\frac{\partial\beta}{\partial u}\frac{\partial\gamma}{\partial v}\right) + 2\left[\left(\alpha\frac{\partial\gamma}{\partial v}\right)\left(\beta\frac{\partial\gamma}{\partial u}\right) - \left(\alpha\frac{\partial\gamma}{\partial u}\right)\left(\beta\frac{\partial\gamma}{\partial v}\right)\right]\right\} = 0.
\end{aligned} \tag{10}$$

Since this equation is quadratic, if it have more than two solutions it is an identity.

3. Let (α) and (γ) be so chosen as to trace two orthogonal trajectories

$$(\alpha d\beta) = (\beta d\alpha) = (\gamma d\beta) = (\beta d\gamma) = 0. \tag{11}$$

For another orthogonal trajectory we must have

$$-\frac{\partial t}{\partial u} = 2\left(\alpha\frac{\partial\gamma}{\partial u}\right)t, \qquad -\frac{\partial t}{\partial v} = 2\left(\alpha\frac{\partial\gamma}{\partial v}\right)t.$$

The condition of compatibility is

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\gamma}{\partial v}\right) = \left(\frac{\partial\alpha}{\partial v}\frac{\partial\gamma}{\partial u}\right). \tag{12}$$

This equation will be of great importance later. If t be one solution of our system of differential equations above, rt is another, if r be constant. But

this constant is the cross ratio of the four points corresponding to the parameter values ∞, 0, t, rt.

6. Our congruence being focal, we may take u and v as focal parameters. We take for (β) the sphere whose points of contact with its envelop will, by theorem 4, be the foci of our circles; these foci shall be (s) and (t)

$$(ss) = (s\alpha) = (s\beta) = (s\gamma) = (sd\beta) = 0,$$

$$(tt) = (t\alpha) = (t\beta) = (t\gamma) = (td\beta) = 0.$$

From the first equations we get

$$A\alpha_i + B\beta_i + C\gamma_i + D\frac{\partial\beta_i}{\partial u} + Es_i = 0,$$

$$A'\alpha_i + B'\beta_i + C'\gamma_i + D'\frac{\partial\beta_i}{\partial v} + E's_i = 0.$$

Multiplying through by t_i and summing, then by β_i and summing, we have

$$E = B = 0, \qquad D \neq 0,$$

$$\frac{\partial\beta_i}{\partial u} = a\alpha_i + c\gamma_i, \qquad \frac{\partial\beta_i}{\partial v} = a'\alpha_i + c'\gamma_i.$$

Since all points of the circle $(\alpha) + (\partial\alpha/\partial u)\,du$, $(\gamma) + (\partial\gamma/\partial u)\,du$, $(\beta) + (\partial\beta/\partial u)\,du$ lie on a sphere through our circle, a focal sphere,

$$\frac{\partial\gamma_i}{\partial u} = l\alpha_i + m\beta_i + n\gamma_i + r\frac{\partial\alpha_i}{\partial u},$$

$$\frac{\partial\gamma_i}{\partial v} = \lambda\alpha_i + \mu\beta_i + \nu\gamma_i + \rho\frac{\partial\alpha_i}{\partial v}.$$

If the congruence be normal, (10) must be satisfied identically. On substituting, the coefficient of t^2 and the constant vanish. The condition is thus

$$\left[cc' - 4\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right)\right](r - \rho) = 0.$$

If $r = \rho$, every solution of

$$(x\alpha) = (x\beta) = (x\gamma) = (xd\alpha) = 0$$

is also a solution of

$$(xd\beta) = (xd\gamma) = 0.$$

Each two adjacent circles are co-spherical, and all of our circles pass through two points or lie on one sphere. The latter would not however yield a normal congruence.

If $r \neq \rho$ we may take as focal spheres

$$\rho y_i = \left| \alpha_j \, \beta_k \, \gamma_l \, \frac{\partial \alpha_m}{\partial u} \right|, \qquad \sigma z_i = \left| \alpha_j \, \beta_k \, \gamma_l \, \frac{\partial \alpha_m}{\partial v} \right|.$$

The condition that these should be mutually orthogonal is

$$cc' - 4 \left(\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v} \right) = 0 .$$

8. We have a normal congruence, so that we have the use of equations (11). We may in two ways (usually distinct) solve the following equations for s_i, du, dv

$$(s\alpha) = (s\beta) = (s\gamma) = (sd\alpha) = (sd\gamma) .$$

By a change of parameters we may write

$$\frac{\partial \gamma_i}{\partial u} = l\alpha_i + b\beta_i + m\gamma_i + n \frac{\partial \alpha_i}{\partial u},$$

$$\frac{\partial \gamma_i}{\partial v} = \lambda\alpha_i + b' \beta_i + \mu\gamma_i + \rho \frac{\partial \alpha_i}{\partial u}.$$

Multiplying through by β_i and summing, we get

$$b = b' = 0 .$$

The condition for a normal congruence (12) gives

$$n \left(\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v} \right) = \rho \left(\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v} \right).$$

If $n = \rho$, then

$$d\gamma_i = P\alpha_i + Q\gamma_i + Rd\alpha_i .$$

The points (α), (γ), $(\alpha) + (d\alpha)$, $(\gamma) + (d\gamma)$ are always concyclic. Hence the circles with these as foci pass through two points or lie on a sphere, since any two adjacent ones are cospherical. The first case is inadmissible as (α) and (γ) would trace a circle. In the other case they trace surfaces mutually inverse in this sphere and the lines of curvature correspond.

If $n \neq \rho$, then

$$\left(\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v} \right) = - \left(\alpha \frac{\partial^2 \alpha}{\partial u \, \partial v} \right) = 0,$$

$$\left(\frac{\partial \gamma}{\partial u} \frac{\partial \gamma}{\partial v} \right) = - \left(\gamma \frac{\partial^2 \gamma}{\partial u \, \partial v} \right) = 0 .$$

The condition of compatibility of the two partial differential equations for (γ) is

$$(R - P)\frac{\partial^2 \alpha_i}{\partial u\, \partial v} + \beta \frac{\partial \alpha_i}{\partial u} + C\frac{\partial \alpha_i}{\partial v} + D\alpha_i + E\gamma_i = 0.$$

Multiplying through by α_i and summing, we find $E = 0$ and

$$\frac{\partial^2 \alpha_i}{\partial u\, \partial v} = a\alpha_i + b\frac{\partial \alpha_i}{\partial u} + c\frac{\partial \alpha_i}{\partial v},$$

$$\frac{\partial^2 \gamma_i}{\partial u\, \partial v} = a'\alpha_i + b'\frac{\partial \alpha_i}{\partial u} + c'\frac{\partial \alpha_i}{\partial v}.$$

These equations are characteristic for the lines of curvature, for they show that the Gauss coefficient D' must vanish for both surfaces, while we saw above

$$\left(\frac{\partial \alpha}{\partial u}\frac{\partial \alpha}{\partial v}\right) = \left(\frac{\partial \gamma}{\partial u}\frac{\partial \gamma}{\partial v}\right) = 0,$$

showing that the u and v curves are mutually orthogonal.*

§ 3. The Kummer Method.

Since the totality of spheres can be put into one-to-one correspondence with that of all points of a four-dimensional space (if planes be counted as spheres) so there is a one-to-one correspondence between the circles of our space and the lines of a four-dimensional space. Moreover our formula (4) for the angle of two spheres is identical with the non-euclidean distance formula, so that the angles of spheres will correspond to the non-euclidean distances of points. We are thus led to the idea of approaching our circle systems by the classical methods of Kummer for line geometry, with the modifications necessary for a four-dimensional universe, and non-euclidean measurement.† We shall fix a circle by two mutually orthogonal spheres (y) and (z) whose coördinates are supposed to be functions of u and v

$$y_i = y_i(u, v), \qquad z_i = z_i(u, v), \qquad (yy) = (zz) = 1, \qquad (yz) = 0.$$

We have the three fundamental equations

$$\begin{aligned} (dy\,dy) - (zdy)^2 &= Edu^2 + 2Fdu\,dv + Gdv^2, \\ (dz\,dz) - (ydz)^2 &= E'du^2 + 2F'du\,dv + G'dv^2, \\ (dy\,dz) &= e\,du^2 + (f + f')\,du\,dv + ydv^2. \end{aligned} \tag{13}$$

More specifically

* Conf. Darboux, *Surfaces*, vol. I, p. 221.

† The Kummer formulæ for non-euclidean space were first worked out by Fibbi, *I sistemi doppiamente infiniti di raggi negli spazii di curvatura costante*, Annali della R. Scuola Normale Superiore, Pisa, 1891. See also the Author's *Elements of Non-Euclidean Geometry*, Oxford, 1909, Chap. 16.

$$\left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial u}\right)-\left(z\frac{\partial y}{\partial u}\right)^2=E,\quad \left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\right)-\left(z\frac{\partial y}{\partial u}\right)\left(z\frac{\partial y}{\partial v}\right)=F,$$

$$\left(\frac{\partial y}{\partial v}\frac{\partial y}{\partial v}\right)-\left(z\frac{\partial y}{\partial v}\right)^2=G,$$

$$(14)\quad \left(\frac{\partial z}{\partial u}\frac{\partial z}{\partial u}\right)-\left(y\frac{\partial z}{\partial u}\right)^2=E',\quad \left(\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)-\left(y\frac{\partial z}{\partial u}\right)\left(y\frac{\partial z}{\partial v}\right)=F',$$

$$\left(\frac{\partial z}{\partial v}\frac{\partial z}{\partial v}\right)-\left(y\frac{\partial z}{\partial v}\right)^2=G',$$

$$\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial u}\right)=e,\quad \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)=f,\quad \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right)=f',\quad \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial v}\right)=g.$$

These various coefficients are connected by the following syzygy:

$$(15)\quad \begin{vmatrix} y_0 & \cdots & y_4 & 0 \\ z_0 & \cdots & z_4 & 0 \\ \frac{\partial y_0}{\partial u} & \cdots & \frac{\partial y_4}{\partial u} & 0 \\ \frac{\partial y_0}{\partial v} & \cdots & \frac{\partial y_4}{\partial v} & 0 \\ \frac{\partial z_0}{\partial u} & \cdots & \frac{\partial z_4}{\partial u} & 0 \\ \frac{\partial z_0}{\partial v} & \cdots & \frac{\partial z_4}{\partial v} & 0 \end{vmatrix}^2 = \begin{vmatrix} E & F & e & f' \\ F & G & f & g \\ e & f & E' & F' \\ f' & g & F' & G' \end{vmatrix} = 0.$$

We saw in theorem D that two circles in general position will be cospherical and orthogonal with two other circles, the latter being in bi-involution, i. e., all spheres through the one are orthogonal to all through the other. We start with two adjacent circles of our congruence; the circles orthogonal to them will be determined by two mutually orthogonal spheres through the first, each orthogonal to one of two mutually orthogonal spheres through the second. Let the spheres through the first circle be $\cos\varphi\,(y)+\sin\varphi\,(z)$, $-\sin\varphi\,(y)+\cos\varphi\,(z)$, while those through the second are $\lambda\,(y+dy)+u\,(z+dz)$, $-\mu\,(y+dy)+\lambda\,(z+dz)$; writing the conditions for criss-cross orthogonality we have two linear homogeneous equations in $\lambda:\mu$. Equating the discriminant to zero, and remembering that

$$(y\,dy)=-\tfrac{1}{2}(dy\,dy),\quad (z\,dz)=-\tfrac{1}{2}(dz\,dz),\quad (y\,dz)+(z\,dy)=-(dy\,dz),$$

we have

$$\begin{vmatrix} [1-\frac{1}{2}(dy\,dy)]\sin\varphi-(z\,dy)\cos\varphi & (y\,dz)\sin\varphi-[1-\frac{1}{2}(dz\,dz)]\cos\varphi \\ -[1-\frac{1}{2}(dz\,dz)]\sin\varphi-(y\,dz)\cos\varphi & (z\,dy)\sin\varphi+[1-\frac{1}{2}(dy\,dy)]\cos\varphi \end{vmatrix} = 0.$$

Expanding, and casting aside infinitesimals of higher order, we have

$$\begin{aligned}(16)\qquad & [e\,du^2+(f+f')\,du\,dv+g\,dv^2](\cos^2\varphi-\sin^2\varphi)+[(E-E')\,du^2 \\ & \qquad +2(F-F')\,du\,dv+(G-G')\,dv^2]\sin\varphi\cos\varphi=0.\end{aligned}$$

In order to discuss this equation, we write the two

$$e\,du^2+(f+f')\,du\,dv+g\,dv^2=0,$$

$$(E-E')\,du^2+2(F-F')\,du\,dv+(G-G')\,dv^2=0.$$

They are equivalent to

$$(dy\,dz)=(dy\,dy)-(dz\,dz)=0.$$

Suppose then that the foci of our circle are (α), (γ). We may write

$$(17)\qquad \alpha_i=\frac{-\sqrt{-1}\,y_i+z_i}{2},\qquad \gamma_i=-\frac{\sqrt{-1}\,y_i+z_i}{2};$$

$$y_i=\sqrt{-1}\,(\alpha_i+\gamma_i),\qquad z_1=\alpha_i-\gamma_i.$$

Our pair of differential equations may thus be replaced by

$$(d\alpha\,d\alpha)=(d\gamma\,d\gamma)=0.$$

We have the following possibilities with regard to these two equations:

A. They have, in general, no common root. The isotropic curves do not correspond on the surfaces by the foci. We shall call these *non-conformal congruences*.

B. The equations have one common root, in general. Then on the two surfaces of foci (which must not be confused with the focal surfaces to which our circles are tangent) one system of isotropic curves will be in correspondence. These congruences shall be called *semi-conformal*.

C. The two equations are identical. The two surfaces of foci when they exist are conformally related, and the congruence is said to be *conformal*.

Let us begin with the non-conformal congruence. Here if $du:dv$ be given we usually get a unique value of $\tan 2\varphi$, that is two mutually orthogonal spheres, and so the two circles in bi-involution required. On the other hand if φ be given we have a quadratic in $du:dv$ so that on each sphere through a circle

in general position of a non-conformal congruence will lie two circles orthogonal to this circle, cospherical and orthogonal with two adjacent circles of the congruence. These circles will fall together if the equation in $du : dv$ have equal roots, i. e., if

$$
\begin{aligned}
&[eg - \tfrac{1}{2}(f+f')^2](\tan^2\varphi - 1)^2 \\
(18)\qquad &+ [e(G-G') - (F-F')(f+f') + g(E-E')](\tan^2\varphi - 1)\tan\varphi \\
&\qquad + [(E-E')(G-G') - (F-F')^2]\tan^2\varphi = 0.
\end{aligned}
$$

This equation is unaltered if we replace φ by $\pi/2 + \varphi$.

THEOREM 12. *Through each circle in general position of a non-conformal congruence will pass two pairs of mutually orthogonal spheres on each of which there is but one circle orthogonal to the given circle, cospherical and orthogonal to an adjacent circle of the congruence.*

These spheres shall be called the *limiting spheres.* They correspond to maximum and minimum values for $\tan\varphi$. We see, in fact, that if we equate to zero the partial derivatives of (16) with regard to du and dv, we get

$$\left[edu + \frac{f+f'}{2}dv\right](\tan^2\varphi - 1) - [(E-E')du + (F-F')dv]\tan\varphi = 0,$$

$$\left[\frac{f+f'}{2}du + gdv\right](\tan^2\varphi - 1) + [(F-F')du + (G-G')dv]\tan\varphi = 0.$$

Eliminating $du : dv$ we fall back on (18). In the case of a real congruence, the spheres containing real circles orthogonal and cospherical to the adjacent circle will lie in determinate angular openings between these limiting spheres. We next make the further assumption that our congruence is focal. The following equations are characteristic of a focal sphere

$$\cos\varphi\, y_i + \sin\varphi\, z_i \equiv \cos(\varphi + d\varphi)(y_i + dy_i) + \sin(\varphi + d\varphi)(z_i + dz_i),$$

$$dy_i \cos\varphi + dz_i \sin\varphi - (y_i \sin\varphi - z_i \cos\varphi)\, d\varphi = 0.$$

Multiplying through by y_i and summing, we have

$$d\varphi = (ydz) = -(zdy).$$

Substituting this value of $d\varphi$ in the last equation, we get

$$
\begin{aligned}
&\left[\left(\frac{\partial y_i}{du}\cos\varphi + \frac{\partial z_i}{\partial u}\sin\varphi\right) - (y_i \sin\varphi - z_i\cos\varphi)\left(y\frac{\partial z}{\partial u}\right)\right]du \\
&\qquad + \left[\left(\frac{\partial y_i}{\partial v}\cos\varphi + \frac{\partial z_i}{\partial v}\sin\varphi\right) - (y_i\sin\varphi - z_i\cos\varphi)\left(y\frac{\partial z}{\partial v}\right)\right]dv = 0.
\end{aligned}
$$

Multiplying through by $\partial z_i/\partial u$ and summing, then doing the same for $\partial z_i/\partial v$,

$$[edu + fdv]\cos\varphi + [E'du + F'dv]\sin\varphi = 0,$$

$$[f'du + gdv]\cos\varphi + [F'du + G'dv]\sin\varphi = 0.$$

Similarly

$$[edu + f'dv]\sin\varphi + [Edu + Fdv]\cos\varphi = 0,$$

$$[fdu + gdv]\sin\varphi + [Fdu + Gdv]\cos\varphi = 0.$$

Eliminating φ,

$$(19)\qquad \begin{aligned} &(E'f' - F'e)\,du^2 + [E'g - F'(f - f') - G'e]\,du\,dv \\ &\qquad\qquad + (F'g - G'f)\,dv^2 = 0, \\ &(Ef - Fe)\,du^2 + [Eg - F(f' - f) - Ge]\,du\,dv + (Fg - Gf')\,dv^2 = 0. \end{aligned}$$

Eliminating $du : dv$,

$$(20)\qquad \begin{aligned} &(E'G' - F'^2)\tan^2\varphi + [E'g - F'(f + f') + G'e]\tan\varphi + (eg - ff') = 0, \\ &(eg - ff')\tan^2\varphi + [Eg - F(f + f') + Ge]\tan\varphi + (EG - F^2) = 0, \end{aligned}$$

$$(21)\qquad \begin{aligned} &[(eg - ff') - (E'G' - F'^2)]\tan^2\varphi + [(E - E')g - (F - F') \\ &\times (f + f') + (G - G')e]\tan\varphi + [(EG - F^2) - (eg - ff')] = 0. \end{aligned}$$

It is necessary, in order that a congruence should be focal, that the two equations (19) and the two equations (20) should be equivalent. It should be noted further that the middle coefficient is the same in (18) and (21). This coefficient will vanish when the corresponding pairs of spheres make equal angles with (y). Now if two spheres be coaxal with a third, and make equal angles therewith, they are mutually inverse therein and conversely, and we have

THEOREM 13. *In a focal congruence, the pairs of limiting spheres are mutually inverse in the same two spheres as are the two focal spheres.*

The necessary and sufficient conditions that a congruence should be focal are best reached in another form. Retaining (α) and (γ) for our foci there must, by (4) and (5), be one sphere through all the points (α), (γ), $(\alpha)+(d\alpha)$, $(\gamma) + (d\gamma)$. This gives the equations

$$\begin{vmatrix} \dfrac{\partial\gamma}{\partial u} & \alpha & \gamma & \dfrac{\partial\alpha}{\partial u} & \dfrac{\partial\alpha}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial\gamma}{\partial v} & \alpha & \gamma & \dfrac{\partial\alpha}{\partial u} & \dfrac{\partial\alpha}{\partial v} \end{vmatrix} = 0,$$

which are equivalent to

$$\begin{vmatrix} \dfrac{\partial z}{\partial u} & y & z & \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial z}{\partial v} & y & z & \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = 0.$$

Squaring we have

$$(22)\qquad \begin{vmatrix} E & F & e \\ F & G & f \\ e & f & E' \end{vmatrix} = \begin{vmatrix} E & F & f' \\ F & G & g \\ f' & g & G' \end{vmatrix} = 0.$$

The condition that two adjacent circles intersect is that their common orthogonal sphere should be null. This will give the equation

$$(Edu^2 + 2Fdu\,dv + Gdc^2)\,(E'\,du^2 + 2F'\,du\,dv + G'\,dv^2)$$
$$- (edu^2 + (f + f')\,du\,dv + g'\,dv^2)^2 = 0.$$

The fact that this equation is of the fourth degree proves theorem 1. For a focal congruence, it must be the square of a quadratic expression. When we know that our congruence is focal, and take u and v as the focal parameters, we have the simple equations

$$\frac{\partial z_i}{\partial u} = ay_i + ac\,z_i + c\frac{\partial y_i}{\partial u}, \qquad \frac{\partial z_i}{\partial v} = a'\,y_i + a'\,c'\,z_i + c'\frac{\partial y_i}{\partial v},$$

$$(23)\qquad E' = ce, \qquad F' = cf' = c'f, \qquad G' = c'g,$$

$$E = \frac{1}{c}e, \qquad F = \frac{1}{c}f = \frac{1}{c'}f', \qquad G = \frac{1}{c'}g.$$

The focal spheres will be

$$c\,(y) - (z), \qquad c'\,(y) - (z).$$

The condition for a normal congruence is thus, by theorem 6,

$$cc' + 1 = 0.$$

This may be written in the invariant form

$$(24)\qquad (EG - F^2) = (E'\,G' - F'^2) = -\,(eg - ff').$$

These equations and (20) will give necessary and sufficient conditions for a normal congruence.

The normal congruence of circles seems superficially the most natural analog of the normal line congruence. From our present point of view, however, the natural analog is that circle congruence which corresponds to a normal congruence of lines in four-dimensional space. A little reflection shows that such a congruence has the following interpretation in circle geometry. Through each circle (perhaps only in a limited region) will pass one sphere whose points of contact with the two nappes of its envelop are mutually inverse in that circle. (The extension of inversion to the geometry on a sphere is immediate. Two points of a sphere are inverse in a circle thereof, if all

circles of the sphere through them are orthogonal to that circle.) Suppose that such a sphere have the coördinates

$$y'_i = \cos\varphi\, y_i + \sin\varphi\, z_i .$$

Then every sphere through the points of contact of (y') with its envelop cuts (y') in a circle orthogonal to the given circle, i. e., every such sphere is orthogonal to the sphere through the given circle orthogonal to (y'). In algebraic form, every sphere

$$\left[(y') + \left(\frac{\partial y'}{\partial u}\right)\right] du + \left(\frac{\partial y'}{\partial v}\right) dv$$

is orthogonal to the sphere

$$-\sin\varphi\,(y) + \cos\varphi\,(z).$$

This yields the equations

$$-\left(y\frac{\partial z}{\partial u}\right)\sin^2\varphi + \left(z\frac{\partial y}{\partial u}\right)\cos^2\varphi + \frac{\partial\varphi}{\partial u}$$

$$= -\left(y\frac{\partial z}{\partial v}\right)\sin^2\varphi + \left(z\frac{\partial y}{\partial v}\right)\cos^2\varphi + \frac{\partial\varphi}{\partial v} = 0, \tag{25}$$

$$\frac{\partial\varphi}{\partial u} = \left(y\frac{\partial z}{\partial u}\right), \qquad \frac{\partial\varphi}{\partial v} = \left(y\frac{\partial z}{\partial v}\right).$$

The condition of compatibility will be

$$f = f'. \tag{26}$$

The differential equations (25) are equally well solved by φ and $\varphi + k$ where k is any constant.

THEOREM 14. *If through each circle of a congruence it be possible to pass one sphere whose points of contact with the two nappes of the envelop are mutually inverse in that circle, then an infinite number of such spheres may be passed through each circle. The spheres generate a one-parameter family of congruences, the corresponding spheres of any two congruences will intersect at a constant angle.*

We shall say in this case that our congruence of circles is *pseudo-normal.* Suppose that we have a congruence which is of this sort, and also is focal. We find from (21)

$$(c - c')f = 0.$$

If

$$c = c',$$

$$dz_i = py_i + qz_i + rdy_i,$$

any two adjacent circles of the congruence will be cospherical, i. e., all the circles lie on a sphere, or pass through two points, two cases which we may rule out. Hence

$$f = f' = 0.$$

Solving (20) in view of (21) we find both roots are solutions of (18), hence

THEOREM 15. *The necessary and sufficient condition that a focal congruence should be pseudo-normal is that the focal spheres should coincide with a pair of limiting spheres.*

Let us see what relations subsist between the foci of the circles of a pseudo-normal congruence. We see from (17) that if we have

$$\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right) = \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right),$$

we also have

$$\left(\frac{\partial \alpha}{\partial u}\frac{\partial \gamma}{\partial v}\right) = \left(\frac{\partial \alpha}{\partial v}\frac{\partial \gamma}{\partial u}\right).$$

If then our congruence be focal as well as pseudo-normal, and (β) be the sphere whose points of contact with its envelop are (α) and (γ), by theorem 4 we see that the circle orthogonal to (β) at the points (α) and (γ) (and so in bi-involution with the given circle) will, by equation (12), generate a normal congruence. We next notice that if u and v be the focal parameters for the pseudo-normal congruence, our equations (23) lead to the equations used in proving theorem 8, so that they are focal parameters for the normal congruence also and give the lines of curvature of the (α) and (γ) surfaces.

THEOREM 16. *The necessary and sufficient condition that a focal congruence of circles should be pseudo-normal is that the lines of curvature should correspond in the surfaces traced by the foci of the circles.*

We may find still another necessary and sufficient condition for a pseudo-normal congruence. Revert to our equations (23); the focal spheres were

$$c\,(y) - (z), \qquad c'\,(y) - (z).$$

The spheres orthogonal to our circle through the pairs of focal points are

$$\left(\frac{\partial y}{\partial u} z\right) z_i - \frac{\partial y_i}{\partial u}, \qquad \left(\frac{\partial y}{\partial v} z\right) z_i - \frac{\partial y_i}{\partial v}.$$

These will be mutually orthogonal if

$$F = f = f' = 0.$$

When, however, two spheres are orthogonal to a circle and to one another, their pairs of intersections with the circle separate one another harmonically, and vice versa.

THEOREM 17. *The necessary and sufficient condition that a focal congruence of circles should be pseudo-normal is that the pairs of focal points should separate one another harmonically.*

We get at once from theorem 16

THEOREM 18. *If a normal congruence be given, the congruence of circles whose foci are the pairs of intersections of the circles of the normal congruence with any two orthogonal trajectories is focal and pseudo-normal.*

THEOREM 19. *If a congruence be focal and pseudo-normal, the foci are the pairs of intersections of the circles of a normal congruence with two orthogonal trajectories.*

When a normal and pseudo-normal congruence are related to one another in this way, we shall call them *associated*. The normal congruence has the parametric form

$$x_i = t^2 \alpha_i + t\beta_i + \gamma_i, \qquad \rho\beta_i = \left| y_j \; z_k \; \frac{\partial y_l}{\partial u} \; \frac{\partial y_m}{\partial v} \right|.$$

Here u and v are supposed to be the common focal parameters of both congruences. Let (y') be one of those spheres whose points of contact with the envelop are mutually inverse in the (y) (z) circle. We write

$$y'_i = \cos\varphi \, y_i + \sin\varphi \, z_i,$$

$$\frac{\partial y'_i}{\partial u} = \cos\varphi \frac{\partial y_i}{\partial u} + \sin\varphi \frac{\partial z_i}{\partial u} + \left(y \frac{\partial z}{\partial u} \right) (- y_i \sin\varphi + z_i \cos\varphi),$$

$$\frac{\partial y'_i}{\partial v} = \cos\varphi \frac{\partial y_i}{\partial v} + \sin\varphi \frac{\partial z_i}{\partial v} + \left(y \frac{\partial z}{\partial v} \right) (- y_i \sin\varphi + z_i \cos\varphi),$$

as we see with the aid of (25). We also see from (17) and the above value of (β) that all points of the circle of the normal congruence lie on the spheres $(\partial y'/\partial u)$, $(\partial y'/\partial v)$, so that, in particular, the points of contact of (β) with its envelop lie on this circle. The circle is orthogonal to (y'), since $(\partial y'/\partial u)$ and $(\partial y'/\partial v)$ are orthogonal thereto.

THEOREM 20. *If a focal and pseudo-normal congruence be given, the spheres whose pairs of points of contact with their envelops are mutually inverse in the circles of the congruence will generate the orthogonal trajectories of the circles of the associated normal congruence.*

Since our two congruences have the same focal parameters

THEOREM 21. *If a normal and pseudo-normal congruence be associated, the annular surfaces will correspond in the two.*

THEOREM 22. *If a normal congruence be given, not consisting in the circles through two points, the pairs of intersections with any two orthogonal trajectories may be taken as foci of the circles of an associated pseudo-normal congruence.*

The other orthogonal trajectories will then be paired in such a way that the intersections of each circle with a pair of trajectories are mutually inverse in the associated circle.

The pseudo-normal circle congruence enjoys a sort of indestructibility, akin to that of the normal line congruence. Let our congruence be determined by the spheres (y) and (z). We may determine (z') in such a way that

$$(z'z') = 1, \qquad (z'y) = 0, \qquad (z'z) = \cos\theta, \qquad \left(\frac{\partial z'}{\partial u}\frac{\partial y}{\partial v}\right) = \left(\frac{\partial z'}{\partial v}\frac{\partial y}{\partial u}\right).$$

Here θ is supposed to have a fixed value, the spheres (y) and (z') will determine a second pseudo-normal congruence of such sort that each of its circles is cospherical with one circle of the original pseudo-normal congruence, and makes therewith a fixed angle. The sphere through the circle common to (z) and (z'), making an angle φ with the former is (z''), where

$$z_i'' = \frac{\sin(\theta - \varphi)}{\sin\theta} z_i + \frac{\sin\varphi}{\sin\theta} z_i'.$$

This cuts (y) in a circle coaxal with the two circles already determined thereon, and making an angle φ with the first of them. If, then, φ be constant

$$\left(\frac{\partial z''}{\partial u}\frac{\partial y}{\partial v}\right) = \left(\frac{\partial z''}{\partial v}\frac{\partial y}{\partial u}\right).$$

THEOREM 23. *If two pseudo-normal congruences be so related that corresponding circles are cospherical and make a fixed angle, then the congruence of circles coaxal with them, and making fixed angles with both, is also pseudo-normal.*

Another theorem of the same sort is obtained as follows. Suppose that we have an analytical complex (three-parameter family) of spheres. A sphere of the complex and its next neighbors are all orthogonal to one sphere which we may speak of as *correlative* to the first. The complex generated by these spheres shall be said to be *correlative* to the first complex. Analytically, if our complex be given parametrically in the form

$$x_i = x_i(u, v, w),$$

the correlative sphere is (y), where

$$\rho y_i = \left| x_j \quad \frac{\partial x_k}{\partial u} \quad \frac{\partial x_l}{\partial v} \quad \frac{\partial x_m}{\partial w} \right|,$$

Since

$$(xy) = \left(y\frac{\partial x}{\partial u}\right) = \left(y\frac{\partial x}{\partial v}\right) = \left(y\frac{\partial x}{\partial w}\right) = \left(x\frac{\partial y}{\partial u}\right) = \left(x\frac{\partial y}{\partial v}\right) = \left(x\frac{\partial y}{\partial w}\right) = 0,$$

we see that (x) and (y) bear a reciprocal relation, or each is the correlative

of the other. This is true in general; there are special cases where (y) may depend on less than three parameters, but these we explicitly exclude by saying that our complex is *non-developable.* Suppose, then, that we have a pseudo-normal congruence. Through each circle will pass at least one sphere of any chosen non-developable complex. We take this for our sphere (y), the correlative sphere is (t), where

$$\rho t_i = \left| y_j \frac{\partial y_k}{\partial u} \frac{\partial y_l}{\partial v} s_m \right|,$$

where s_m is a function of u, v, w. We further put

$$z_i' = \frac{\sin(\varphi - \theta)}{\sin\theta} t_i + \frac{\sin\varphi}{\sin\theta} z_i,$$

so that θ is the angle of (z) and (t), while φ is the angle of (z') and (t), and assume finally that the ratio of $\sin\varphi$ to $\sin\theta$ has the constant value k. We have

$$(yt) = \left(y\frac{\partial t}{\partial u}\right) = \left(y\frac{\partial t}{\partial v}\right) = 0,$$

$$\left(\frac{\partial z'}{\partial u}\frac{\partial y}{\partial v}\right) - \left(\frac{\partial z'}{\partial v}\frac{\partial y}{\partial u}\right) = k\left[\left(\frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) - \left(\frac{\partial z}{\partial v}\frac{\partial y}{\partial u}\right)\right],$$

and this last equation gives the four-dimensional equivalent of the Malus-Dupin theorem.

THEOREM 24. *Through each circle c of a pseudo-normal congruence a sphere is passed belonging to a given non-developable complex in such a way as to generate an analytic congruence of that complex. This sphere meets the correlative sphere in c', and a circle c'' is found coaxal with c and c' so that the sines of the angles of c and c'' with c' bear to one another a constant ratio. The circles c'' will also generate a pseudo-normal congruence.*

We now pass from the non-conformal congruence to the other types noted above. The semi-conformal type has the property that each circle in general position is tangent or paratactic to one adjacent circle. The members of such a congruence are, however, essentially imaginary, and we pass on to the more interesting conformal congruences. Here we have

$$(E - E') : (F - F') : (G - G') = e : \frac{f + f'}{2} : g. \tag{27}$$

Equations (16) become

$$[\, edu^2 + (f + f')\, du\, dv + gdv^2]\, [\cos 2\varphi + k \sin 2\varphi] = 0.$$

The roots of the second factor give two mutually orthogonal spheres, which we may take for our spheres (y) and (z). A circle cospherical and orthogonal

to a circle in general position and one of its next neighbors must therefore, in general, lie either on (y) or (z). The sphere (y) is thus orthogonal to (z) and to $(z)+(dz)$ so that

$$(28)\qquad \begin{gathered}(ydz) = (zdy) = -\tfrac{1}{2}(dy\,dz) = 0,\\ e = f + f' = g = 0.\end{gathered}$$

Reverting to (17) we see

$$\left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial u}\right) = \left(\frac{\partial\gamma}{\partial u}\frac{\partial\gamma}{\partial u}\right), \quad \left(\frac{\partial\alpha}{\partial u}\frac{\partial\alpha}{\partial v}\right) = \left(\frac{\partial\gamma}{\partial u}\frac{\partial\gamma}{\partial v}\right), \quad \left(\frac{\partial\alpha}{\partial v}\frac{\partial\alpha}{\partial v}\right) = \left(\frac{\partial\gamma}{\partial v}\frac{\partial\gamma}{\partial v}\right).$$

This shows that if (α) and (γ) really depend on two parameters, their surfaces are conformally related. There is the additional possibility that all of these terms should vanish identically; (α) and (γ) would then trace two minimal curves. Conversely, let (α) and (γ) trace two conformally related surfaces. We have

$$(d\alpha\,d\alpha) = \sigma\,(d\gamma\,d\gamma).$$

Replacing α_i by $\sqrt[4]{\sigma}\,\alpha_i$ and γ_i by $1/\sqrt[4]{\sigma}\,\gamma_i$

$$(d\alpha\,d\alpha) = (d\gamma\,d\gamma), \qquad (dy\,dz) = 0,$$

we have surely a conformal congruence. We next write

$$y_i' = y_i\cos\varphi + z_i\sin\varphi, \qquad y_i'' = y_i\cos\varphi - z_i\sin\varphi,$$

where φ is constant

$$(dy'\,dy') = (dy''\,dy'').$$

Conversely, if this equation holds, and if φ be a fixed angle

$$(dy\,dz) = 0.$$

We shall say that two congruences of spheres are *conformally related* if the infinitesimal angle of two adjacent spheres of the one is equal to the corresponding infinitesimal angle of the other. We thus get

THEOREM 26. *Through each circle of a conformal congruence we may, in an infinite number of ways, pass two spheres, making a constant angle, and generating conformally related congruences as the circle describes the given congruence.*

THEOREM 25. *If two sphere congruences be conformally related, and corresponding spheres in the two meet at a fixed angle, then the circles of intersection of such spheres will generate a conformal congruence.*

We now make the additional assumption that our circle congruence is focal. The isotropic curves will correspond on the two surfaces generated by the foci. If we take these as our parameter curves u and v we have two conceivable cases

$$(A) \qquad \frac{\partial \gamma_i}{\partial u} = p\alpha_i + q\gamma_i + r\frac{\partial \alpha_i}{\partial v}, \qquad \frac{\partial \gamma_i}{\partial v} = p'\,\alpha_i + q'\,\gamma_i + r'\frac{\partial \alpha_i}{\partial u},$$

$$(B) \qquad \frac{\partial \gamma_i}{\partial u} = p\alpha_i + q\gamma_i + r\frac{\partial \alpha_i}{\partial u}, \qquad \frac{\partial \gamma_i}{\partial v} = p'\,\alpha_i + q'\,\gamma_i + r'\frac{\partial \alpha_i}{\partial v}.$$

Leaving aside for the moment the question of whether both of these cases actually exist, let us take them up in turn. In case A we have

$$\left(\frac{\partial \alpha}{\partial u}\frac{\partial \gamma}{\partial v}\right) = \left(\frac{\partial \alpha}{\partial v}\frac{\partial \gamma}{\partial u}\right).$$

Our congruence is pseudo-normal, the focal parameters u' and v' will give also the focal directions for the associated normal congruence (by 21), so that they give mutually orthogonal directions of advance for (α) and (γ). We have the partial differential equations, analogous to those previously found for a normal congruence,

$$\frac{\partial \gamma_i}{\partial u'} = bc\alpha_i + b\gamma_i + c\frac{\partial \alpha_i}{\partial u'}, \qquad \frac{\partial \gamma_i}{\partial v'} = b'\,c'\,\alpha_i + b'\,\gamma_i + c'\frac{\partial \alpha_i}{\partial v'}.$$

The relation among the coefficients of α_i, γ_i comes from the equations

$$\left(\alpha\frac{\partial \gamma}{\partial u'}\right) + \left(\gamma\frac{\partial \alpha}{\partial u'}\right) = \left(\alpha\frac{\partial \gamma}{\partial v'}\right) + \left(\gamma\frac{\partial \alpha}{\partial v'}\right) = 0.$$

The condition that our surfaces (α) and (γ) should be conformally related is

$$\left(\frac{\partial \alpha}{\partial u'}\frac{\partial \alpha}{\partial u'}\right)du'^2 + \left(\frac{\partial \alpha}{\partial v'}\frac{\partial \alpha}{\partial v'}\right)dv'^2 = \left(\frac{\partial \gamma}{\partial u'}\frac{\partial \gamma}{\partial u'}\right)du'^2 + \left(\frac{\partial \gamma}{\partial v'}\frac{\partial \gamma}{\partial v'}\right)dv'^2.$$

This gives $c^2 = c'^2$. Now if $c = c'$ we have at once

$$d\gamma_i = P\alpha_i + Q\gamma_i + Rd\alpha_i,$$

each two adjacent circles are cospherical, and we have the circles through two points or on one sphere, which cases we may exclude. Hence we must have

$$c + c' = 0.$$

This shows that the focal spheres, whose coördinates are $c(\alpha) - (\gamma)$, $c'(\alpha) - (\gamma)$ are mutually orthogonal, and the congruence is, by theorem 6, a normal one. Our processes here are entirely reversible, and we have the theorems

THEOREM 27. *If a congruence be both conformal and pseudo-normal, it is normal.*

THEOREM 28. *If a congruence be both normal and conformal it is pseudo-normal, or else consists in the circles touching a given circle at a given point.*

With regard to the existence of such congruences, we construct an example as follows. Let a cylinder be given whose elements are all parallel to a given plane, and of such cross section that a sphere whose center moves perpendicularly to the direction of the elements, while it remains tangent to the cylinder and the plane, will trace paths of equal length on the two. The points of contact of the spheres with plane and cylinder will be the required foci. As a matter of fact, however, these congruences are well enough known.* In type B we see that the isotropic parameters are the focal ones. We thus see by theorem B that each circle is tangent to two adjacent circles, or the circles of the congruence are the osculating circles of two different one-parameter families of curves. This much is true, if the congruence exist; unfortunately the present writer has signally failed in all his attempts to answer this interesting question.

§ 4. Complexes of Circles.

Our leading idea in approaching our circle geometry by the Kummer method was to interpret in three dimensions those theorems of the line geometry of four dimensions which are easily reached by an extension of the well-known methods of line geometry. The idea lies close at hand that in changing our line geometry from three to four dimensions, we might have done well to allow ourselves an extra parameter, in other words, to busy ourselves with three-parameter circle systems, or complexes. To such systems we now give our attention. We write

$$(29)\qquad \begin{gathered} y_i = y_i(u_1 u_2 u_3), \qquad z_i = z_i(u_1 u_2 u_3), \\ (yy) = (zz) = 1, \qquad (yz) = 0. \end{gathered}$$

$$(30)\qquad \begin{gathered} (dy\,dy) - (zdy)^2 \equiv \Sigma a_{ij}\,du_i\,du_j, \qquad a_{ij} = a_{ji}, \\ (dz\,dz) - (ydz)^2 \equiv \Sigma b_{ij}\,du_i\,du_j, \qquad b_{ij} = b_{ji}, \\ (dydz) = \Sigma c_{ij}\,du_i\,du_j. \end{gathered}$$

$$(31)\qquad \begin{gathered} \left(\frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_i}\right) - \left(z\frac{\partial y}{\partial u_i}\right)^2 = a_{ii}, \\ \left(\frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_j}\right) - \left(z\frac{\partial y}{\partial u_i}\right)\left(z\frac{\partial y}{\partial u_j}\right) = a_{ij} + a_{ji} = 2a_{ij} = 2a_{ji} \quad (i \neq j). \end{gathered}$$

* The problem of finding a conformal focal congruence is sometimes called the problem of Ribaucour, for it amounts to finding a congruence of spheres which establish a conformal relation between the two nappes of their envelop. For an interesting discussion of the congruences of type A see Darboux, *Sur les surfaces isothermiques*, Annales de l'Ecole Normale, series 3, vol. 16 (1899), pp. 498 ff. Darboux there proves that this is the only type of conformal focal congruence, but his proof is erroneous, as he has acknowledged in a letter to the present writer. He doubted, however, whether any congruences of type B really existed. The theorem that congruences of type A are normal was casually mentioned by Cosserat, *Sur le problème di Ribaucour*, Bulletin de l'Académie des Sciences de Toulouse, vol. 3 (1900), pp. 267 ff.

$$(32)\qquad \begin{aligned} &\left(\frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_i}\right) - \left(y\frac{\partial z}{\partial u_i}\right)^2 = b_{ii}, \\ &\left(\frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_j}\right) - \left(y\frac{\partial z}{\partial u_i}\right)\left(y\frac{\partial z}{\partial u_j}\right) = b_{ij} + b_{ji} = 2b_{ij} = 2b_{ji} \quad (i \neq j). \end{aligned}$$

$$(33)\qquad \left(\frac{\partial y}{\partial u_i}\frac{\partial z}{\partial u_j}\right) = c_{ij}.$$

The coefficients are connected by various relations

$$(34)\qquad \sum_{ijk} \begin{vmatrix} \frac{\partial y_i}{\partial u_1} & \frac{\partial y_j}{\partial u_1} & \frac{\partial y_k}{\partial u_1} \\ \frac{\partial y_i}{\partial u_2} & \frac{\partial y_j}{\partial u_2} & \frac{\partial y_k}{\partial u_2} \\ \frac{\partial y_i}{\partial u_3} & \frac{\partial y_j}{\partial u_3} & \frac{\partial y_k}{\partial u_3} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial z_i}{\partial u_1} & \frac{\partial z_j}{\partial u_1} & \frac{\partial z_k}{\partial u_1} \\ \frac{\partial z_i}{\partial u_2} & \frac{\partial z_j}{\partial u_2} & \frac{\partial z_k}{\partial u_2} \\ \frac{\partial z_i}{\partial u_3} & \frac{\partial z_j}{\partial u_3} & \frac{\partial z_k}{\partial u_3} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix},$$

$$(35)\qquad \begin{aligned} &\left|z\,y\,\frac{\partial y}{\partial u_1}\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_3}\right|^2 = |a_{ij}|, \qquad \left|y\,z\,\frac{\partial z}{\partial u_1}\frac{\partial z}{\partial u_2}\frac{\partial z}{\partial u_3}\right|^2 = |\,b_{ij}\,|, \\ &z_i = \frac{\left|y_j\,\frac{\partial z_k}{\partial u_1}\frac{\partial z_l}{\partial u_2}\frac{\partial z_m}{\partial u_3}\right|}{\sqrt{|b_{ij}|}}, \qquad y_i = \frac{\left|z_j\,\frac{\partial y_k}{\partial u_1}\frac{\partial y_l}{\partial u_2}\frac{\partial y_m}{\partial u_3}\right|}{\sqrt{|\,a_{ij}\,|}}. \end{aligned}$$

If, then, we write

$$B_{kl} \equiv \frac{\partial\,|\,b_{ij}\,|}{\partial b_{kl}}, \qquad A_{kl} \equiv \frac{\partial\,|\,a_{ij}\,|}{\partial a_{kl}},$$

we shall have

$$(36)\qquad a_{ij} = -\frac{\sum_{kl} c_{ik}\,c_{jl}\,B_{kl}}{|\,b_{ij}\,|}, \qquad b_{ij} = -\frac{\sum_{kl} c_{ik}\,c_{jl}\,A_{kl}}{|\,a_{ij}\,|}.$$

$$(37)\qquad |\,a_{ij}\,| \cdot |\,b_{ij}\,| = |\,c_{ij}\,|^2.$$

We next look for the circles cospherical and orthogonal to a circle in general position, and to one of its next neighbors. Following the method which previously led up to (16) we now have

$$(38)\qquad \sum_{ij} c_{ij}\,du_i\,du_j\,(\cos^2\varphi - \sin^2\varphi) + \sum_{ij}(a_{ij} - b_{ij})\,du_i\,du_j \sin\varphi\cos\varphi = 0.$$

As in the case of the congruence, we must here consider the equations

$$(39)\qquad (d\alpha\,d\alpha) = 0, \qquad (d\gamma\,d\gamma) = 0,$$

where (α) and (γ) are the foci of our circle. When these equations are identical, the complex of circles effect a conformal transformation of space, and the complex is said to be *conformal;* if they have four or fewer common

solutions, the complex is called *non-conformal.* As the equations are irreducible, there is no type of complex corresponding to the semi-conformal congruence. We begin with the non-conformal complex. The circles cospherical and orthogonal to two given circles are indeterminate when, and only when, the given circles are cospherical or are paratactic. The four common solutions of the equations (39) will give four circles adjacent to a given circle and either tangent or paratactic thereto. To determine which of these cases arises, let us see how many circles adjacent to a circle in general position of a complex are cospherical therewith. For cospherical adjacent circles we must have

$$dy_i \cos\varphi + dz_i \sin\varphi \sin\varphi - (y_i \sin\varphi - z_i \cos\varphi)\, d\varphi = 0,$$

$$(ydz) = -(zdy) = d\varphi,$$

$$(40) \qquad \sum_{n=1}^{n=3} \left[\frac{\partial y_i}{\partial u_n} \cos\varphi + \frac{\partial z_i}{\partial u_n} \sin\varphi - (y_i \sin\varphi - z_i \cos\varphi) \left(y \frac{\partial z}{\partial u_n} \right) \right] du_n = 0.$$

Multiplying through by $\partial z_i / \partial u_1$, $\partial z_i / \partial u_2$, $\partial z_i / \partial u_3$ and summing, we find

$$(41) \qquad \sum_{j=1}^{j=3} c_{ij}\, du_j + \left[\sum_{j=1}^{j=3} b_{ij}\, du_j \right] \tan\varphi = 0 \qquad (i = 1, 2, 3),$$

$$(42) \qquad \begin{vmatrix} (c_{11} + b_{11} \tan\varphi) & \cdots \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ \cdots & (c_{33} + b_{33} \tan\varphi) \end{vmatrix} = 0.$$

These equations show that a circle in general position will not, usually, be cospherical with more than three adjacent circles. Conversely, if we have the equations (41), (42) and replace the right-hand side of (40) by c_i while $(ydz) = d\varphi$ we find

$$(yt) = (zt) = \left(\frac{\partial y}{\partial u_1} t \right) = \left(\frac{\partial y}{\partial u_2} t \right) = \left(\frac{\partial y}{\partial u_3} t \right) = 0,$$

hence

$$t_i \equiv 0 \qquad (i = 0 \cdots 4).$$

Lastly, we see that the determination of these three circles depends upon the solution of a (usually) irreducible cubic equation, and so must give different results from the solution of the irreducible quartic arising from (39).

THEOREM 29. *A circle in general position in a non-conformal complex lies usually on three annular surfaces thereof.**

THEOREM 30. *A circle in general position in a non-conformal complex is usually paratactic with four adjacent circles of the complex.*

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* This theorem is due to COSSERAT, *Sur le cercle consideré comme élément générateur de l'espace,* Annales de la Faculté des Sciences de Toulouse, vol. 3 (1889) p. E 33.

The qualification *pseudo-normal* may be applied to complexes as well as to congruences of circles. A complex of circles shall be said to be *pseudo-normal* if it be composed of the circles of intersection of corresponding spheres of two correlative complexes. To find the condition that a given complex should be of this type, we put

$$y_i' = y_i \cos\varphi + z_i \sin\varphi, \qquad z_i' = -y_i \sin\varphi + z_i \cos\varphi.$$

The complex will be pseudo-normal if

$$(y'\,dz') = (z'\,dy') = 0.$$

Proceeding exactly as before, we find

$$(ydz) = d\varphi, \tag{43}$$

$$c_{ij} = c_{ji}. \tag{44}$$

We note that if the differential equation for φ have one solution, it will have an infinite number differing by an additive constant.

THEOREM 31. *The circles of a pseudo-normal complex are the intersections of corresponding spheres in a singly infinite system of pairs of correlative complexes. The spheres of any two of these complexes through the same circle make with one another a constant angle.*

We get the necessary and sufficient geometrical conditions for a pseudo-normal complex as follows. Let the three sets of solutions of equations (41), (42) be

$$d^{(1)}\,u_1\,d^{(1)}\,u_2\,d^{(1)}\,u_3\,\varphi_1, \qquad d^{(2)}\,u_1\,d^{(2)}\,u_2\,d^{(2)}\,u_3\,\varphi_2, \qquad d^{(3)}\,u_1\,d^{(3)}\,u_2\,d^{(3)}\,u_3\,\varphi_3.$$

Multiply the three equations (41) through by the corresponding factors $d^{(m)}\,u_1\,d^{(m)}\,u_2\,d^{(m)}\,u_3$ and add; we get equations of the form

$$\sum_{ij} c_{ij}\,d^{(m)}\,u_i\,d^{(n)}\,u_j + \tan\varphi_m \sum_{ij} b_{ij}\,d^{(m)}\,u_i\,d^{(n)}\,u_j = 0 \qquad (m \neq n),$$

$$\sum_{ij} c_{ij}\,d^{(n)}\,u_i\,d^{(m)}\,u_j + \tan\varphi_n \sum_{ij} b_{ij}\,d^{(n)}\,u_i\,d^{(m)}\,u_j = 0 \qquad (m \neq n).$$

If now our complex be pseudo-normal, so that (44) are satisfied, we get, by subtraction,

$$\sum_{j} b_{ij}\,d^{(m)}\,u_i\,d^{(n)}\,u_j = 0 \qquad (m \neq n). \tag{45}$$

Conversely, when these equations hold, the complex is pseudo-normal. On the other hand if we mean by the *focal points* of a circle the pairs of points where it meets an adjacent cospherical circle, that is, its pairs of points of contact with the envelop on an annular surface of the complex, we find the

spheres through these pairs of focal points, orthogonal to the given circle exactly as we did for the congruence. We wish to find a linear combination of (y) and (z), which is also a linear combination of $(y)+(dy)$ and $(z)+(dz)$, the differentials being determined by a set of solutions to our equations (41). The three spheres sought will thus be

$$\sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(1)},\qquad \sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(2)},$$

$$\sum_{k=1}^{k=3}\left[\left(\frac{\partial z}{\partial u_k}y\right)z_i-\frac{\partial z_i}{\partial u_k}\right]du_k^{(3)}.$$

The equations above give us the necessary and sufficient conditions that these should be mutually orthogonal in pairs.

THEOREM 32. *A necessary and sufficient condition that a complex should be pseudo-normal is that the pairs of focal points on a circle in general position should separate one another harmonically.*

On an arbitrary sphere there will lie a finite number of circles belonging to a given complex, unless all are orthogonal to one sphere. Let two pseudo-normal complexes be determined by the three-parameter sphere systems $(y)\,(z)$ and $(y)\,(z')$ where (z) and (z') make with one another a fixed angle. Then if

$$z_i''=\frac{\sin(\varphi-\theta)}{\sin\theta}z_i+\frac{\sin\varphi}{\sin\theta}z_i',$$

where φ and θ have constant values, we see that (y) and (z'') determine a pseudo-normal complex.

THEOREM 33. *If two pseudo-normal complexes be so related that corresponding circles are cospherical and make a fixed angle with one another, then the complex of circles coaxial respectively with these corresponding pairs, and making fixed angles with them, is also pseudo-normal.*

Suppose next, that (s) is the sphere correlative to the sphere (y); we may write

$$\rho s_i=\left|y_j\ \frac{\partial y_k}{\partial u_1}\ \frac{\partial y_l}{\partial u_2}\ \frac{\partial y_m}{\partial u_3}\right|.$$

If, then, we put

$$z_i'=\frac{\sin(\varphi-\theta)}{\sin\theta}s_i+\frac{\sin\varphi}{\sin\theta}z_i,$$

and require the ratio of $\sin\varphi$ to $\sin\theta$ to be constant, we see that (y) and (z') determine a pseudo-normal complex.

THEOREM 34. *Through each circle of a pseudo-normal complex a sphere is passed belonging to a determinate non-developable complex and the original circle*

is replaced by such a circle coaxial with it and the circle cut by the correlative sphere, that the sines of the angles formed therewith by the original and the replacing circle have a fixed ratio; then will the replacing circles also generate a pseudo-normal complex.

The consideration of the circles adjacent to and cospherical with a circle in general position possesses special features when the complex is conformal, we therefore turn our attention to complexes of that type. Suppose that a pair of foci, which are corresponding points in a conformal transformation of space, are (α) and (γ), and that these are concyclic with the adjacent foci $(\alpha)+(d\alpha)$, $(\gamma)+(d\gamma)$, the circles being then cospherical. If we follow our transformation with an inversion which interchanges (α) and (γ), we have a conformal transformation leaving invariant (α) and the circle through (α), (γ), $(\alpha+d\alpha)$. We are thus led to the consideration of the invariant lineal elements in a conformal transformation of space with a fixed point. We have four possibilities:

1. One proper and two isotropic fixed lineal elements.
2. One proper fixed lineal element, and all elements orthogonal thereto also fixed.
3. All lineal elements of an isotropic plane fixed.
4. All lineal elements fixed.

These four cases for the transformation where (α) is fixed give the possible cases for our given conformal transformation, and a fixed isotropic lineal element through (α) corresponds to the case where (α), (γ), $(\alpha)+(d\alpha)$, $(\gamma)+(d\gamma)$ lie on a null circle, i. e., where adjacent circles are tangent to one another. We thus have

THEOREM 35. *If a circle in general position in a conformal complex be cospherical with but three adjacent circles, it will touch two of them.*

THEOREM 36. *If a circle in general position in a conformal complex be cospherical with three adjacent circles but not tangent to any one of them, it is cospherical with a one-parameter family of adjacent circles, including two of the given ones at least.*

THEOREM 37. *If a circle in general position in a conformal complex be tangent to but one adjacent circle, it will be cospherical with a one-parameter family of adjacent circles including this one, and with no others.*

THEOREM 38. *If a circle in general position in a conformal complex be tangent to three adjacent circles it is cospherical with every adjacent circle, and the complex consists in the totality of circles on a sphere.*

Of course these last four theorems could be restated is more precise form; when we say that adjacent circles are cospherical we mean that our circles can be assembled into the generators of annular surfaces, when we say that they are tangent we mean that we can assemble them into the osculating circles of curves.

Two sphere complexes are said to be *conformally related* when the angle of two adjacent spheres of one is equal to the corresponding infinitesimal angle in the other. We prove exactly as in the case of the conformal congruence:

THEOREM 39. *If a conformal complex be given we may, in an infinite number of ways, pass two spheres through each circle which shall generate conformally related complexes, as the circle generates the given complex.*

THEOREM 40. *If two sphere complexes be conformally related, and corresponding spheres make a fixed angle, their circles of intersection generate a conformal complex.*

HARVARD UNIVERSITY, CAMBRIDGE, MASS.,
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SUR LA NOTION DE DIFFÉRENTIELLE D'UNE FONCTION DE LIGNE*

PAR

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Introduction

1. Avant d'étudier le cas particulier des fonctions de lignes, il m'a paru utile de montrer que la définition de la différentielle que je propose s'inspire de considérations générales applicables à une fonctionnelle quelconque. Je commence donc aux §§ 2, 3 par rappeler les points de vue de M. Volterra et de M. Hadamard. Ces auteurs se sont occupés uniquement de la dérivée des fonctionnelles telle qu'on l'entend dans le Calcul des Variations. Je pense qu'il y a lieu de définir d'abord la *différentielle* en généralisant la définition donnée par Stolz dans le cas d'une fonction ordinaire de plusieurs variables. J'arrive ainsi aux §§ 4, 5, 6, 7 à une définition générale de la différentielle des fonctionnelles qu'il suffit de préciser dans chaque cas particulier.

Je m'arrête ensuite au cas des fonctions de lignes et je déduis (§§ 8, 9) l'expression de la " variation " de celle de la différentielle. Cette différentielle est une fonctionnelle linéaire des accroissements des coordonnées des points

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de la courbe. Mais cette fonctionnelle linéaire n'est pas arbitraire et satisfait à une condition que j'obtiens aux §§ 10, 11.

Je mets ensuite la fonctionnelle linéaire sous la forme de somme d'intégrales de Stieltjes,—forme qui se déduit immédiatement d'un théorème de F. Riesz. On peut ainsi écrire explicitement (§§ 11, 12) la condition à laquelle j'ai fait allusion. Elle permet de mettre (§§ 13, 14, 15, 16) l'expression de la différentielle sous une forme qui n'est plus assujettie à aucune condition d'égalité —dans le cas où la courbe est rectifiable et à tangente continue—

$$\int_a^b (y_0' \Delta z - z_0' \Delta y)\, dp + \int_a^b (z_0' \Delta x - x_0' \Delta z)\, dq + \int_a^b (x_0' \Delta y - y_0' \Delta x)\, dr$$
$$+ A\Delta x(a) + B\Delta y(a) + C\Delta z(a) + A'\Delta x(b) + B'\Delta y(b) + C'\Delta z(b),$$

où p, q, r sont des fonctions à variation bornée quelconques, où A, B, C, A', B', C' sont des constantes quelconques et où les intégrales sont prises au sens de Stieltjes.

J'applique ensuite le mode de décomposition—que j'ai présenté ailleurs—d'une fonctionnelle linéaire, pour mettre en évidence (§§ 17, 18, 19, 20, 21, 22) des termes réguliers et irréguliers dans la différentielle d'une fonction de ligne.

Enfin, j'indique comment en suivant une méthode indiquée par M. Lebesgue on peut, en compliquant l'élément d'intégration, se passer des intégrales de Stieltjes (§§ 23–25). Toute cette théorie est établie en supposant que les seuls voisinages imposés soient d'ordre zéro au sens adopté en Calcul des Variations. Les résultats seraient un peu plus compliqués si l'on imposait aux voisinages d'être du premier ordre.

Aperçu historique

2. Le premier essai pour appliquer aux fonctionnelles les procédés du Calcul Différentiel semble être dû à M. Volterra. Le point de départ de sa méthode est la définition de la dérivée d'une fonctionnelle. Cette définition est l'extension de celle qui s'était imposée dans le Calcul des Variations. Dans le cas où l'argument de la fonctionnelle U_L est une ligne continue L plane par exemple, M. Volterra suppose que si l'on déforme un peu L au voisinage d'un point M de L, la variation de U_L ainsi produite est un infiniment petit d'ordre au moins égal à celui de l'aire balayée dans la déformation et que la limite du quotient ne dépend que de L et de M. Cette supposition se trouve être exacte dans les cas classiques du Calcul des Variations, au moins entre les extrémités de L; et elle conduit à la même formule pour la variation de U_L, soit une formule telle que

$$\delta U_L = \int_L U_{L,x}\, \delta y dx \tag{1}$$

où $U_{L,x}$ dépend de la ligne L et du point de L d'abscisse x, mais non de la variation δy de l'ordonnée en ce point.

3. M. Volterra ne manqua pas de remarquer qu'une telle définition n'était pas entièrement satisfaisante, puisqu'elle laisse de côté une grande partie des expressions qui interviennent dans le Calcul des Variations, à savoir celles des variations des intégrales définies où les limites ne sont pas fixes. De telles variations comportent en effet, outre une intégrale définie de la forme (1), des termes finis aux limites. Il convint donc d'ajouter au second membre de (1) des termes qui dépendent, *d'une manière spéciale*, selon son expression, de certains points exceptionnels. En adoptant la définition de M. Volterra, on s'inspire des premierès applications qui se sont présentées et que M. Volterra a traitées avec un succés qui justifie pratiquement sa définition. Mais il était souhaitable au point de vue logique et pour assurer le développement futur de la théorie de déduire la définition d'un principe unique et général. M. Hadamard proposa donc de " considérer comme fonctionnelles auxquelles on peut étendre les méthodes du Calcul Infinitésimal, toutes les fonctionnelles U_y dont la variation est une fonctionnelle linéaire de la variation de y." (La définition d'une fonctionnelle linéaire sera donnée plus tard (§ 6)).

4. Je me propose de poursuivre encore plus loin la critique de M. Hadamard. Notons d'abord en passant qu'une fonctionnelle linéaire peut comporter encore un terme d'une nature différente des deux termes mentionnés plus haut (intégrale et terme fini); nous retrouverons cette remarque au § 7. Mais surtout observons que la première chose à faire pour appliquer au Calcul Fonctionnel les méthodes du Calcul Infinitésimal, c'est de définir la *différentielle* et d'en tirer seulement comme conséquence l'expression de la *variation*. On comprendra notre pensée en l'appliquant au cas des fonctions ordinaires de plusieurs variables numériques. C'est en effet seulement après avoir défini la différentielle totale de telles fonctions qu'il y a lieu de donner la règle de dérivation des fonctions composées.

Nous serons donc conduit à essayer de définir d'abord la notion de différentielle, et pour cela à nous poser deux questions: Quelle doit être la forme de la différentielle? De quelle manière doit elle se déduire de la fonctionnelle donnée? La réponse à la première question nous est fournie par la remarque de M. Hadamard que le résultat fondamental du Calcul Différentiel est le suivant: " La différentielle d'une fonction est une fonction linéaire des différentielles des variables." Nous supposerons donc que la différentielle d'une fonctionnelle est une fonctionnelle linéaire—nous préciserons plus loin (§ 6) le sens de cette expression—de l'accroissement de l'argument. La remarque de M. Hadamard suffirait si nous désirions seulement arriver à une définition de la " variation." Car la réponse à la seconde question est contenue dans la définition de la variation: c'est la dérivée par rapport à α de la fonction de α qui est la valeur

de la fonctionnelle quand l'argument dépend aussi de α. Mais la réponse est moins immédiate quand nous voulons arriver à définir une différentielle totale. Ici les dérivées partielles n'ont plus d'analogues. J'ai donc été amené à essayer de reprendre l'ancienne définition aujourd'hui généralement abandonnée: la différentielle est la partie principale de l'accroissement de la fonction quand l'accroissement de la variable est considéré comme infiniment petit.

Cas des fonctions ordinaires de plusieurs variables

5. Il faut bien remarquer en effet que si cette ancienne définition—qui est la première en date, la plus naturelle et la plus commode—a prêté à des objections très fondées, c'est qu'elle a été mal formulée. Tout au contraire, c'est à elle que sont revenus des auteurs désireux de plus de rigueur dans la théorie des fonctions de plusieurs variables. MM. Stolz,* Pierpont,† W. H. Young,‡ formulent ainsi la définition de la différentielle d'une fonction de plusieurs variables:

Une fonction $f(x, y)$ admet une différentielle totale au point x_0, y_0 si (1) $f(x, y)$ admet en ce point des dérivées partielles en x et en y, *et si* (2) *on a, quels que soient les accroissements* Δx, Δy,

$$(2) \quad f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \Delta x \frac{\partial f}{\partial x_0} + \Delta y \frac{\partial f}{\partial y_0} + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

ϵ_1 et ϵ_2 tendant vers zéro quand Δx, Δy tendent simultanément vers zéro. Cette différentielle est alors

$$\Delta x \frac{\partial f}{\partial x_0} + \Delta y \frac{\partial f}{\partial y_0}.$$

Sans connaître cette nouvelle définition, j'ai été amené en appliquant les idées exposées plus haut à la définition suivante qui lui est entièrement équivalente mais dont la forme se rapproche plus de la définition historique:

Une fonction $f(x, y)$ admet au point x_0, y_0 une différentielle totale s'il existe une fonction linéaire des accroissements des variables, soit $A\Delta x + B\Delta y$, qui ne diffère de l'accroissement de la fonction que par un infiniment petit relativement à la distance $\sqrt{\Delta x^2 + \Delta y^2}$ du point x_0, y_0 au point voisin $x_0 + \Delta x$, $y_0 + \Delta y$. Au lieu de $\sqrt{\Delta x^2 + \Delta y^2}$, il est équivalent de prendre $|\Delta x| + |\Delta y|$, de sorte que d'après ma définition

$$(3) \qquad \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - [A\Delta x + B\Delta y]}{|\Delta x| + |\Delta y|} = \epsilon,$$

ϵ tendant vers zéro avec $|\Delta x| + |\Delta y|$. Il est facile de voir que dans ce

* *Grundzüge der Differential- und Integralrechnung*, t. 1, p. 133.

† *The Theory of Functions of Real Variables*, t. 1, p. 268.

‡ *The Fundamental Theorems of Differential Calculus.*

cas, on a $A = \partial f / \partial x_0$, $B = \partial f / \partial y_0$ et que la formule (3) est équivalente à la formule (2) de Stolz.

M. W. H. Young a, le premier, nettement montré les simplifications que procurent la définition nouvelle.* J'ai moi-même repris et complété cette exposition.† De sorte que je pourrai admettre ici comme établie la supériorité de la définition de Stolz sur la définition ordinaire (où l'on omet généralement la condition (2) de Stolz).

Différentielle d'une fonctionelle

6. Il nous faut maintenant revenir à la définition de la différentielle d'une fonctionnelle. J'ai d'abord considéré dans les Comptes Rendus du Congrès des Sociétés Savantes (Paris, 1912, p. 44) le cas des fonctionnelles dont l'argument est une fonction continue. Je rappelle brièvement les définitions et théorèmes concernant ce cas et qui nous serviront dans la suite.

Une fonctionnelle linéaire est une fonctionnelle distributive et continue. Dans le cas actuel:

Une fonctionnelle U_f définie dans le champ C des fonctions $f(x)$ continues dans (a, b) est continue dans ce champ si U_{f_n} tend vers U_f lorsque $f_n(x)$ converge *uniformément* vers $f(x)$.

Elle est distributive si l'on a identiquement $U_{f_1+f_2} = U_{f_1} + U_{f_2}$. Nous dirons qu'une fonctionnelle U_f définie dans la champ C a une différentielle pour l'argument f_0 s'il existe une fonctionnelle linéaire $T_{\Delta f}$ de l'accroissement Δf de l'argument, qui ne diffère de l'accroissement de la fonctionnelle U_f que par une quantité infiniment petite par rapport au maximum de la valeur absolue de l'accroissement $\Delta f(x)$ dans (a, b). En appelant $m\Delta f$ ce maximum, on aura donc

$$U_{f+\Delta f} = U_f + T_{\Delta f} + \epsilon m \Delta f, \tag{4}$$

ϵ tendant vers zéro avec $m\Delta f$.

Diverses formes des fonctionnelles linéaires

7. Quant à la forme des fonctionnelles linéaires, on peut facilement prévoir l'existence de deux types différents parmi ces fonctionnelles, l'un tel que

$$\int_a^b f(x)\, \alpha(x)\, dx \tag{5}$$

où $\alpha(x)$ est une fonction sommable‡ quelconque, indépendante de l'argument

* Loc. cit.

† Nouvelles Annales de Mathématiques, 4e série, t. 12 (1912), pp. 385, 433.

‡ Dans tout le cours de cet article, les intégrales ordinaires seront prises au sens de M. Lebesgue.

$f(x)$; l'autre pouvant s'écrire

$$\sum_n A_n f(c_n) \tag{6}$$

où les A et c sont des constantes indépendantes de l'argument $f(x)$ de la fonctionnelle, où les c sont entre a et b, et la série ΣA_n est absolument convergente. Mais il existe des fonctionnelles linéaires qui n'appartiennent à aucun de ces deux types.

La représentation générale d'une fonctionnelle linéaire *quelconque* U_f peut s'effectuer sous les trois formes équivalentes suivantes. D'après M. Hadamard, on peut écrire

$$U_f = \lim_{n=\infty} \int_a^b f(x)\, \alpha_n(x)\, dx, \tag{7}$$

où les $\alpha_n(x)$ sont des fonctions indépendantes de $f(x)$, qui peuvent être continues et qu'on peut même astreindre à être des polynomes. D'après M. Lebesgue, on a

$$U_f = \int_a^b f[\varphi(x)]\, \beta(x)\, dx, \tag{8}$$

où $\varphi(x)$ et $\beta(x)$ sont des fonctions indépendantes de l'argument f, $\varphi(x)$ étant une fonction non décroissante et $\beta(x)$ une fonction sommable qui prend seulement les valeurs ± 1. Enfin la représentation qui paraît la plus commode, puisqu'elle détermine la fonctionnelle au moyen d'une seule fonction ordinaire $u(x)$, est celle de M. F. Riesz,

$$U_f = \int_a^b f(x)\, d[u(x)], \tag{9}$$

où $u(x)$ est une fonction à variation bornée et où l'intégrale est prise au sens de Stieltjes. C'est à dire que l'on a

$$U_f = \lim \sum_i f(\xi_i)\, [u(x_i) - u(x_{i-1})], \tag{10}$$

où l'on a pris arbitrairement les x et ξ tels que $a = x_0 \leqq \xi_1 \leqq x_1 \leqq \xi_2 \leqq \cdots \leqq \xi_n \leqq x_n = b$ et où l'on fait tendre vers zéro la longueur maximum des intervalles (x_{i-1}, x_i).

Il faut bien remarquer cependant que si l'on veut se rendre compte de la structure d'une fonctionnelle linéaire, il n'est pas nécessaire d'employer l'intégrale de Stieltjes sous sa forme la plus générale, c'est à dire en prenant pour $u(x)$ une fonction quelconque à variation bornée. J'ai montré en effet* que toute fonctionnelle linéaire peut se mettre *et d'une seule* manière sous la forme d'une somme de trois fonctionnelles linéaires de types essentiellement

* M. Fréchet, *Sur la Notion de Différentielle dans le Calcul Fonctionnel*, Comptes Rendus du Congrès des Sociétés Savantes (Grenoble, 1913).

différents et représentables sous les formes respectives (5), (6) et (9), où la fonction $u(x)$ qui figure dans (9) doit non seulement être à variation bornée mais encore posséder une dérivée nulle presque partout (c'est à dire sauf en un ensemble de points de mesure nulle). Un exemple de ce troisième type nous est fourni par la fonctionnelle

$$U_f = \lim_{\text{mesure } S_E = 0} \left[\frac{\int_{S_E} f(x)\, dx}{\int_{S_E} dx} \right]$$

où S_E est un ensemble dénombrable d'intervalles non empiétant et couvrant l'ensemble parfait de mesure nulle E.

Différentielles des fonctions de lignes continues

8. J'appellerai *ligne continue* toute suite ordonnée de points équivalente au point de vue de l'Analysis Situs à un segment de droite et par conséquent représentable sous la forme

$$(L) \qquad x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad a \leqq t \leqq b,$$

où f, g, h sont trois fonctions arbitraires uniformément continues dans (a, b) et qui ne sont à la fois constantes dans aucun intervalle de valeurs de t.

Si à toute ligne continue L correspond un nombre bien déterminé U_L, on dit que U_L est une fonction de ligne.

9. On voit que la donnée d'une ligne équivaut à celle d'un ensemble de trois fonctions continues de sorte que les définitions de la continuité et de la différentiation d'une fonction de ligne se déduisent immédiatement des définitions correspondantes pour les fonctionnelles dont l'argument est une seule fonction continue dans (a, b). Il est bon cependant de les énoncer explicitement pour éviter tout malentendu.

Une fonctionnelle $U_{f,\, g,\, h}$ dépendant de trois fonctions $f(t)$, $g(t)$, $h(t)$ uniformément continues dans (a, b) est continue en (f_0, g_0, h_0) si $U_{f,\, g,\, h}$ tend vers $U_{f_0,\, g_0,\, h_0}$ de quelque manière que f, g, h convergent uniformément et *simultanément* vers f_0, g_0, h_0. Elle est distributive si l'on a identiquement

$$U_{f+f_1,\, g+g_1,\, h+h_1} \equiv U_{f,\, g,\, h} + U_{f_1,\, g_1,\, h_1}.$$

Enfin la fonctionnelle $U_{f,\, g,\, h}$ admet une différentielle en (f_0, g_0, h_0) s'il existe une fonctionnelle $T_{\Delta f,\, \Delta g,\, \Delta h}$ linéaire par rapport à l'ensemble des accroissements Δf, Δg, Δh des arguments, qui ne diffère de l'accroissement de la fonctionnelle que par une quantité infiniment petite par rapport à l'ensemble des accroissements Δf, Δg, Δh. Pour préciser, nous supposons

$$(11) \qquad \frac{U_{f_0+\Delta f,\, g_0+\Delta g,\, h_0+\Delta h} - U_{f_0,\, g_0,\, h_0} - T_{\Delta f,\, \Delta g,\, \Delta h}}{m\Delta f + m\Delta g + m\Delta h} = \epsilon,$$

$m\Delta f$, $m\Delta g$, $m\Delta h$ désignant les maxima dans (a, b) des valeurs absolues $|\Delta f|$, $|\Delta g|$, $|\Delta h|$, et ϵ tendant vers zéro quand la somme de ces maxima tend vers zéro. On peut aussi remplacer le dénominateur par $m\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$; la définition n'en est pas altérée.

Nous pouvons maintenant remarquer que, si $V_{f, g, h}$ est une fonctionnelle linéaire par rapport à l'ensemble (f, g, h), c'est la somme de trois fonctionnelles linéaires dépendant chacune d'une seule des fonctions f, g, h. Car on a

$$V_{f, g, h} = V_{f, 0, 0} + V_{0, g, 0} + V_{0, 0, h}.$$

Si maintenant on décompose ainsi la différentielle $T_{\Delta f, \Delta g, \Delta h}$ d'une fonctionnelle $U_{f, g, h}$ en trois fonctionnelles linéaires

$$T_{\Delta f, \Delta g, \Delta h} = \mathcal{T}_{\Delta f}^{(f)} + \mathcal{T}_{\Delta g}^{(g)} + \mathcal{T}_{\Delta h}^{(h)}, \tag{12}$$

la formule (11) montre immédiatement que la fonctionnelle $U_{f, g, h}$ admet respectivement ces trois fonctionnelles comme différentielles par rapport à f seul, g seul, ou h seul. Mais la réciproque n'est pas vraie.

Expression de la variation

10. Nous pouvons passer maintenant au calcul de la "variation" de la fonctionnelle $U_{f, g, h}$. Supposons que f, g, h soient fonctions non seulement de la variable t, mais d'un paramètre α. Alors U sera lui-même une fonction de α

$$F(\alpha) = U_{f(t, \alpha), g(t, \alpha), h(t, \alpha)}.$$

Si pour $\alpha = \alpha_0$, $f(t, \alpha)$, $g(t, \alpha)$, $h(t, \alpha)$ se réduisent à $f_0(t)$, $g_0(t)$, $h_0(t)$, la variation de U pour f_0, g_0, h_0 sera la différentielle de F par rapport à α pour $\alpha = \alpha_0$. Pour pouvoir calculer cette variation, il va falloir faire certaines hypothèses sur $f(t, \alpha)$, $g(t, \alpha)$, $h(t, \alpha)$. En effet dans le cas classique et plus simple où $g \equiv h \equiv 0$ et où

$$U_{f(t, \alpha)} = \int_a^b f(t, \alpha)\, dt,$$

on a déjà à introduire ces hypothèses. On suppose généralement que $f(t, \alpha)$ est dérivable par rapport à α pour $\alpha = \alpha_0$ et que $[f(t, \alpha) - f(t, \alpha_0)]/(\alpha - \alpha_0)$ converge uniformément vers $\partial f/\partial \alpha_0$ quand α tend vers α_0. *Nous ferons donc cette hypothèse pour $f(t, \alpha)$ ainsi que pour $g(t, \alpha)$ et $h(t, \alpha)$. Elle nous suffira en supposant bien entendu que $U_{f, g, h}$ admet une différentielle pour* f_0, g_0, h_0. On aura en effet d'après (11) et (12)

$$\frac{F(\alpha)-F(\alpha_0)}{\alpha-\alpha_0} = \mathcal{T}^{(f_0)}_{\frac{f(t,\alpha)-f(t,\alpha_0)}{\alpha-\alpha_0}} + \mathcal{T}^{(g_0)}_{\frac{g(t,\alpha)-g(t,\alpha_0)}{\alpha-\alpha_0}} + \mathcal{T}^{(h_0)}_{\frac{h(t,\alpha)-h(t,\alpha_0)}{\alpha-\alpha_0}}$$
$$+ \epsilon\left\{m\left[\frac{f(t,\alpha)-f(t,\alpha_0)}{\alpha-\alpha_0}\right] + m\left[\frac{g(t,\alpha)-g(t,\alpha_0)}{\alpha-\alpha_0}\right]\right.$$
$$\left. + m\left[\frac{h(t,\alpha)-h(t,\alpha_0)}{\alpha-\alpha_0}\right]\right\}.$$

D'après les hypothèses faites $\partial f/\partial\alpha_0$, $\partial g/\partial\alpha_0$, $\partial h/\partial\alpha_0$ sont des fonctions continues, et le second membre de l'équation précédente tendra vers

$$\mathcal{T}^{(f_0)}_{\frac{\delta f}{\delta\alpha_0}} + \mathcal{T}^{(g_0)}_{\frac{\delta g}{\delta\alpha_0}} + \mathcal{T}^{(h_0)}_{\frac{\delta h}{\delta\alpha_0}}.$$

Donc $F(\alpha)$ a une dérivée en α_0 et cette dérivée est $T_{\delta f/\delta\alpha_0,\, \delta g/\delta\alpha_0,\, \delta h/\delta\alpha_0}$.

Avec les notations ordinaires du Calcul des Variations:

$$\delta U_{f_0,\, g_0,\, h_0} = T_{\delta f_0,\, \delta g_0,\, \delta h_0}$$

ou encore

$$\delta U_L = T_{\delta x,\, \delta y,\, \delta z}. \tag{13}$$

Autrement dit, *on obtient, sous les conditions indiquées plus haut, la "variation" d'une fonction de ligne en remplaçant dans l'expression de sa différentielle les accroissements des coordonnées par leurs variations.*

La différentielle d'une fonction de ligne n'est pas une fonctionnelle linéaire arbitraire des accroissements des coordonnées

11. Si tout système de fonctions f, g, h continues et non à la fois constantes définit bien une courbe L, la réciproque n'est pas vraie; tous les systèmes obtenus en faisant dans f, g, h une substitution de la forme $t = \theta(t')$, où $\theta(t')$ est une fonction continue de t' qui croît de la valeur a à la valeur b quand t croît de a à b définit la même courbe L. On peut donc s'attendre à ce que, pour une ligne L_0 déterminée, la différentielle de U ne soit pas une fonctionnelle linéaire quelconque des Δf, Δg, Δh. C'est ce que nous allons montrer. Bornons-nous pour simplifier au cas très général où la courbe L_0 est rectifiable et a partout une tangente variant continuement, c'est à dire, où $f_0(t)$, $g_0(t)$, $h_0(t)$ ont des dérivées uniformément continues dans (a, b) et telles que $f_0'^2 + g_0'^2 + h_0'^2$ reste différent de zéro; et appliquons la formule (13) lorsqu'on prend pour ligne variée la ligne L_0 elle-même. Considérons dans ce but une fonction $\varphi(t)$ continue entre a et b, nulle en a et b, et admettant une dérivée continue dans (a, b). Pour λ assez petit, la fonction $\theta(t) = t + \lambda\varphi(t)$ est continue et croît de a à b quand t varie de a à b. Dès lors la courbe

$$x = f_0[\theta(t)], \qquad y = g_0[\theta(t)], \qquad z = h_0[\theta(t)]$$

représente la même courbe L_0, et les seconds membres convergent uniformément vers $f_0(t)$, $g_0(t)$, $h_0(t)$ quand λ tend vers zéro. Mais de plus $\{f_0[\theta(t)] - f_0(t)\}/\lambda$, $\{g_0[\theta(t)] - g_0(t)\}/\lambda$, $\{h_0[\theta(t)] - h_0(t)\}/\lambda$ convergeront uniformément vers $\varphi(t) f_0'(t)$, $\varphi(t) g_0'(t)$, $\varphi(t) h_0'(t)$; nous aurons donc le droit d'appliquer la formule (13) où δU sera évidemment nul et nous aurons donc

$$T_{\phi f_0', \phi g_0', \phi h_0'} = 0. \tag{14}$$

Ainsi le fait que la fonctionnelle U est une fonction de ligne se traduit par l'égalité (14) qui a lieu quelle que soit la fonction $\varphi(t)$ nulle aux extrémités et à dérivée continue.

12. Pour mettre cette condition (14) sous une forme indépendante de la fonction arbitraire $\varphi(t)$, nous utiliserons la représentation (9) des fonctionnelles linéaires due à M. F. Riesz. La fonctionnelle T est la somme de trois fonctionnelles linéaires et pourra donc s'écrire

$$T_{\Delta f, \Delta g, \Delta h} = \int_a^b \Delta f(t)\, d[u(t)] + \int_a^b \Delta g(t)\, d[v(t)] + \int_a^b \Delta h(t) d[w(t)], \tag{15}$$

u, v, w désignant trois fonctions à variation bornées qui ne dépendent que de f_0, g_0, h_0, U. On aura donc

$$0 = \int_a^b \varphi(t) f_0'(t) d[u(t)] + \int_a^b \varphi(t) g_0'(t) d[v(t)] + \int_a^b \varphi(t) h_0'(t) d[w(t)]. \tag{14'}$$

Posons

$$F(t) = \int_a^t f_0'(t)\, d[u(t)].$$

Pour étudier ses discontinuités, introduisons la variation totale $U(t)$ de $u(t)$ dans (a, b) et le maximum $\Omega(\epsilon)$ de l'oscillation de la fonction continue $f_0'(t)$ dans tout intervalle de longueur $\leqq \epsilon$. On aura évidemment

$$\begin{aligned} \int_{t_{i-1}}^{t_i} f_0'(t)\, d[u(t)] &= \lim \Sigma f_0'(\xi_j')[u(t_j') - u(t_{j-1}')] \\ &= \lim \{\Sigma [f_0'(\xi_j') - f_0'(\xi)][u(t_j') - u(t_{j-1}')] \\ &\qquad + f_0'(\xi) \Sigma [u(t_j') - u(t_{j-1}')]\} \end{aligned} \tag{16}$$

si $t_{i-1} = t_0' \leqq \xi_1' \leqq t_1' \leqq \cdots \leqq t_n' = t_i$ et $t_{i-1} \leqq \xi \leqq t_i$. D'où

$$\left| \int_{t_{i-1}}^{t_i} f_0'(t)\, d[u(t)] - f_0'(\xi)[u(t_i) - u(t_{i-1})] \right| \leqq \Omega(\epsilon) |U(t_i) - U(t_{i-1})|, \tag{17}$$

ϵ étant supérieur ou égal à la longueur de l'intervalle $(t_i - t_{i-1})$. Si maintenant on fait tendre t_{i-1} vers $t' = t_i$ par valeurs plus petites ou t_i vers $t' = t_{i-1}$ par valeurs plus grandes, on voit qu'on aura

$$F(t' \pm 0) - F(t') = f_0'(t')[u(t' \pm 0) - u(t)]. \tag{18}$$

D'autre part, l'égalité (16) donne évidemment

$$\Sigma \mid F(t_i) - F(t_{i-1}) \mid \leqq M \cdot U(b),$$

M étant le maximum de $f_0'(t)$ dans (a, b). Donc $F(t)$ est à variation bornée dans (a, b). Par suite, on peut écrire

$$\int_a^b \varphi(t) f_0'(t)\, d[u(t)] = \int_a^b \varphi(t)\, d[F(t)], \tag{19}$$

car le second membre a un sens et le premier est, d'après (17), la limite de

$$\begin{aligned} \Sigma\varphi(\xi_i) f_0'(\xi_i)[u(t_i) - u(t_{i-1})] &= \Sigma\varphi(\xi_i)[F(t_i) - F(t_{i-1})] \\ &\quad + \theta q \Omega(\epsilon_0)\, U(b), \end{aligned} \tag{20}$$

avec $|\theta| < 1$, ϵ_0 représentant la longueur du plus grand des intervalles $(t_i - t_{i-1})$, et q le maximum de $\varphi(t)$. Or le second membre de (20) tend vers le second membre de (19) quand ϵ_0 tend vers zero.

Ceci étant, posons

$$K(t) = \int_a^t x_0'(t)\, d[u(t)] + \int_a^t y_0'(t)\, d[v(t)] + \int_a^t z_0'(t)\, d[w(t)] \tag{21}$$

en ecrivant x_0, y_0, z_0 pour f_0, g_0, h_0. On verra comme pour $F(t)$ que c'est une fonction à variation bornée et que l'égalité (14′) peut s'écrire

$$\int_a^b \varphi(t)\, d[K(t)] = 0.$$

Or on voit facilement que

$$\int_a^b \varphi(t)\, d[K(t)] = [\varphi(t) K(t)]_a^b - \int_a \varphi'(t) K(t)\, dt,$$

et puisque $\varphi(b) = \varphi(a) = 0$, l'égalité (14′) est finalement équivalente à

$$\int_a^b \varphi'(t) K(t)\, dt = 0.$$

Cette condition, vérifiée quelle que soit la fonction $\varphi(t)$ nulle aux extrémités et à dérivée continue, ne peut être satisfaite si $K(t)$ est arbitraire. Remarquons d'abord qu'au moyen d'un changement de variable simple, on peut supposer $a = 0$, $b = 2\pi$ et qu'alors on aura le droit de prendre pour $\varphi(\theta)$, θ étant la nouvelle variable, l'une des fonctions

$$\sin n\theta, \qquad \cos n\theta - 1$$

et par suite pour $\varphi'(t)$ l'une des fonctions

$$n \cos n\theta, \qquad - n \sin n\theta,$$

n étant un entier quelconque. Si donc $n \neq 0$, on aura

$$\int_0^{2\pi} K_1(\theta) \cos n\theta \, d\theta = \int_0^{2\pi} K_1(\theta) \sin n\theta \, d\theta = 0 \quad (n = 1, 2, 3 \cdots),$$

où $K_1(\theta) = K(t)$. La série de Fourier de $K_1(\theta)$ se réduit donc à une constante, et d'autre part, on sait que la série de Fourier d'une fonction à variation bornée $K_1(\theta)$ converge partout vers $K_1(\theta)$ sauf peut-être aux points de discontinuité de $K_1(\theta)$ qui forment un ensemble dénombrable.

En revenant à la variable primitive t, on voit donc que $K(t)$ est égal en tout point de (a, b) à une constante déterminée k, sauf peut-être aux points d'un certain ensemble dénombrable N. Il est d'ailleurs facile de déterminer N. En raisonnant pour $K(t)$ comme pour $F(t)$, on obtient en effet l'égalité analogue à (18),

$$\begin{aligned} K(t \pm 0) - K(t) = x_0'(t)[u(t \pm 0) - u(t)] + \cdots \\ + z_0'(t)[w(t \pm 0) - w(t)]. \end{aligned} \tag{22}$$

Par suite $K(t)$ ne peut être discontinue qu'aux points où l'une des fonctions u, v, w est discontinue.

Ainsi lorsque la fonction de ligne U_L admet une différentielle pour une ligne $L_0[x_0(t), y_0(t), z_0(t)]$ à tangente continue, si l'on met cette différentielle sous la forme

$$\int_a^b \Delta x(t) \, d[u(t)] + \cdots + \int_a^b \Delta z(t) \, d[w(t)], \tag{23}$$

les fonctions à variation bornée u, v, w ne peuvent être arbitraires; elles vérifient nécessairement l'égalité

$$\int^t x_0'(t) \, d[u(t)] + \cdots + \int_a^t z_0'(t) \, d[w(t)] = \textit{constante} \tag{24}$$

en tout point t de (a, b)—sauf peut-être en un ensemble dénombrable de points qui ne peuvent être que des points de discontinuité de u, v ou w.

Forme explicite de la différentielle

13. La forme (23) de la différentielle est très utile en ce sens qu'elle détermine la différentielle seulement au moyen de trois fonctions u, v, w, et que réciproquement ces trois fonctions sont déterminées par la différentielle—du moins, d'après F. Riesz, à un changement près de leur valeur en un ensemble dénombrable de points autres que a et b. Mais elle a l'inconvénient que ces trois fonctions u, v, w soient assujetties à la condition (24). Nous allons donner une autre expression de la différentielle où les fonctions qui déterminent la différentielle sont des fonctions à variation bornée choisies arbitrairement.

Nous avons supposé que x_0', y_0', z_0' sont des fonctions continues de t et que $x_0'^2 + y_0'^2 + z_0'^2$ a un minimum positif h. Par suite les expressions

$$
\begin{aligned}
p(t) &= \int_a^t \frac{y_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[w(t)] - \int_a^t \frac{z_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[v(t)]; \\
(25) \quad q(t) &= \int_a^t \frac{z_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[u(t)] - \int_a^t \frac{x_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[w(t)]; \\
r(t) &= \int_a^t \frac{x_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[v(t)] - \int_a^t \frac{y_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[u(t)]
\end{aligned}
$$

sont calculables comme intégrales de Stieltjes. De plus, en raisonnant comme au § 12 pour la fonction $F(t)$, on voit que p, q, r sont des fonctions à variation bornée dans (a, b). Ceci étant, calculons la différence

$$
\begin{aligned}
\delta = \int_a^b \Delta x(t)\, d[u(t)] + \int_a^b \Delta y(t)\, d[v(t)] + \int_a^b \Delta z(t)\, d[w(t)] \\
- \int_a^b [y_0'(t)\Delta z(t) - z_0'(t)\Delta y(t)]\, d[p(t)] \\
- \int_a^b [z_0'(t)\Delta x(t) - x_0'(t)\Delta z(t)]\, d[q(t)] \\
- \int_a^b [x_0'(t)\Delta y(t) - y_0'(t)\Delta x(t)]\, d[r(t)].
\end{aligned}
$$

C'est une somme algébrique d'intégrales de Stieltjes et c'est par suite la limite de trois termes tels que

$$
(26) \quad \begin{aligned}
\sum_i \Delta x(\xi_i)\{[u(t_i) - u(t_{i-1})] + y_0'(\xi_i)[r(t_i) - r(t_{i-1})] \\
- z_0'(\xi_i)[q(t_i) - q(t_{i-1})]\}.
\end{aligned}
$$

L'accolade peut s'écrire

$$
(27) \quad \begin{aligned}
u(t_i) - u(t_{i-1}) - \int_{t_{i-1}}^{t_i} \frac{x_0'(t)x_0'(\xi_i) + y_0'(t)y_0'(\xi_i) + z_0'(t)z_0'(\xi_i)}{x_0'^2(t) + y_0'^2(t) + z_0'^2(t)} d[u(t)] \\
+ \int_{t_{i-1}}^{t_i} \frac{x_0'(t)}{x_0'^2(t) + y_0'^2(t) + z_0'^2(t)} \{x_0'(\xi_i)\, d[u(t)] + y_0'(\xi_i)\, d[v(t)] \\
+ z_0'(\xi_i)\, d[w(t)]\}.
\end{aligned}
$$

D'autre part, remarquons que si l'on prend pour t_{i-1} et t_i des points où u, v, w sont continues, on aura d'après (24)

$$
\int_{t_{i-1}}^{t_i} x_0'(t)\, d[u(t)] + \int_{t_{i-1}}^{t_i} y_0'(t)\, d[v(t)] + \int_{t_{i-1}}^{t_i} z_0'(t)\, d[w(t)] = 0.
$$

Retranchons de (27) cette expression multipliée par

$$x_0'(\xi_i)/[x_0'^2(\xi_i)+y_0'^2(\xi_i)+z_0'^2(\xi_i)].$$

L'expression (27) peut donc s'écrire

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i}\left\{\frac{x_0'(t)[x_0'(t)-x_0'(\xi_i)]+y_0'(t)[y_0'(t)-y_0'(\xi_i)]+z_0'(t)[z_0'(t)-z_0'(\xi_i)]}{x_0'^2(t)+y_0'^2(t)+z_0'^2(t)}\right\}d[u(t)]\\
&\quad+\int_{t_{i-1}}^{t_i}\left[\frac{x_0'(t)\,x_0'(\xi_i)}{x_0'^2(t)+y_0'^2(t)+z_0'^2(t)}-\frac{x_0'(t)\,x_0'(\xi_i)}{x_0'^2(\xi_i)+y_0'^2(\xi_i)+z_0'^2(\xi_i)}\right]du\\
(28)\qquad&\quad+\int_{t_{i-1}}^{t_i}\left[\frac{x_0'(t)\,y_0'(\xi_i)}{x_0'^2(t)+y_0'^2(t)+z_0'^2(t)}-\frac{y_0'(t)\,x_0'(\xi_i)}{x_0'^2(\xi_i)+y_0'^2(\xi_i)+z_0'^2(\xi_i)}\right]dv\\
&\quad+\int_{t_{i-1}}^{t_i}\left[\frac{x_0'(t)\,z_0'(\xi_i)}{x_0'^2(t)+y_0'^2(t)+z_0'^2(t)}-\frac{z_0'(t)\,x_0'(\xi_i)}{x_0'^2(\xi_i)+y_0'^2(\xi_i)+z_0'^2(\xi_i)}\right]dw.
\end{aligned}$$

Soit maintenant M le plus grand maximum de $|x_0'(t)|$, $|y_0'(t)|$, $|z_0'(t)|$ dans (a, b); $\omega(\epsilon)$ la plus grande oscillation de $x_0'(t)$, $y_0'(t)$, $z_0'(t)$ dans un intervalle de longueur au plus égale à la plus grande longueur ϵ des intervalles (t_{i-1}, t_i); $U(t)$, $V(t)$, $W(t)$ les variations totales de u, v, w dans (a, t). D'après la définition des intégrales de Stieltjes, on a en général

$$\left|\int_{t_{i-1}}^{t_i} x(t)\,d[u(t)]\right| \leqq [\text{maximum de } |x(t)| \text{ dans } (t_{i-1}, t_i)] \times [U(t_i)-U(t_{i-1})].$$

En appliquant une formule analogue à chacune des intégrales que nous venons d'écrire, on voit que la valeur absolue de l'expression (27) est au plus égale à

$$\begin{aligned}
&\frac{3M\omega(\epsilon)}{h}[U(t_i)-U(t_{i-1})]+\frac{6M^3\omega(\epsilon)}{h^2}[U(t_i)-U(t_{i-1})]\\
&\quad+\frac{11M^3\omega(\epsilon)}{h^2}[V(t_i)-V(t_{i-1})]+\frac{11M^3\omega(\epsilon)}{h^2}[W(t_i)-W(t_{i-1})].
\end{aligned}$$

Ceci ne s'applique que si t_i et t_{i-1} ne sont pas de points de discontinuité de u, v ou w. Si l'on suppose d'abord que a et b ne sont pas de tels points de discontinuité, on pourra s'arranger pour que tous les t_i soient ainsi choisis et la valeur absolue de la somme des trois termes analogues à (26) sera au plus égale à

$$N\left[\frac{3M\omega(\epsilon)}{h}+\frac{6M^3\omega(\epsilon)}{h^2}+\frac{22M^3\omega(\epsilon)}{h^2}\right][U(b)+V(b)+W(b)]=\omega(\epsilon)\times K$$

où N désigne le plus grand des maxima de $|\Delta x(t)|$, $|\Delta y(t)|$, $|\Delta z(t)|$

dans (a, b), et K un nombre fixe. Quand ϵ tend vers zéro, $\omega(\epsilon) \times K$ tend vers zéro. Donc δ serait nulle. Mais il peut arriver que u, v ou w soit discontinu en a ou b. Alors nous aurons seulement prouvé que l'on a

$$\delta = \lim_{\epsilon=0} \sum_i \left[\frac{\Delta x(\xi_i) x_0'(\xi_i) + \Delta y(\xi_i) y_0'(\xi_i) + \Delta z(\xi_i) z_0'(\xi_i)}{x_0'^2(\xi_i) + y_0'^2(\xi_i) + z_0'^2(\xi_i)} \right] \times [K(t_i) - K(t_{i-1})].$$

En prenant comme précédemment les t_i distincts de a et de b, il restera seulement les termes extrêmes et on aura à la limite

$$\delta = \frac{\Delta x(b) x_0'(b) + \Delta y(b) y_0'(b) + \Delta z(b) z_0'(b)}{x_0'^2(b) + y_0'^2(b) + z_0'^2(b)} [K(b) - K(b-0)]$$
$$+ \frac{\Delta x(a) x_0'(a) + \Delta y(a) y_0'(a) + \Delta z(a) z_0'(a)}{x_0'^2(a) + y_0'^2(a) + z_0'^2(a)} [K(a+0) - K(a)].$$

En définitive *nous obtenons maintenant l'expression suivante de la différentielle d'une fonction de ligne:*

$$\begin{aligned} \int_a^b [y_0' \Delta z - z_0' \Delta y]\, dp + \int_a^b [z_0' \Delta x - x_0' \Delta z]\, dq + \int_a^b [x_0' \Delta y - y_0' \Delta x]\, dr \\ + R[x_0'(a) \Delta x(a) + y_0'(a) \Delta y(a) + z_0'(a) \Delta z(a)] \\ + S[x_0'(b) \Delta x(b) + y_0'(b) \Delta y(b) + z_0'(b) \Delta z(b)], \end{aligned} \tag{29}$$

où $p(t)$, $q(t)$, $r(t)$ sont trois fonctions à variation bornée et où R, S sont deux constantes et où les intégrales sont prises au sens de Stieltjes.

Forme géométrique de la différentielle

14. Cette formule est susceptible d'une interprétation géométrique simple. Appelons λ, μ, ν les cosinus directeurs de la normale au plan T qui passe par la tangente à L_0 en un point P et par le déplacement Δx, Δy, Δz de ce point. Soient Δ_t, Δ_n les composantes de ce déplacement sur la tangente et suivant la normale à L_0 dans T et soient $\Delta\sigma_{yz}$, $\Delta\sigma_{zx}$, $\Delta\sigma_{xy}$ les projections sur les trois plans de coordonnées de l'aire du parallélogramme ayant pour cotés le déplacement $(\Delta x, \Delta y, \Delta z)$ de P et un segment unitaire sur la tangente à L_0 en P. On peut écrire la formule (29) en supposant que la variable initiale soit l'arc $s = AP$ de L_0. Seulement p, q, r, R, S ne seront pas nécessairement les mêmes fonctions, ni les mêmes constantes. On voit alors que *l'expression de la différentielle peut aussi s'écrire*

$$(30)\quad \int_0^l \Delta\sigma_{yz}(s)\,d[p_0(s)] + \int_0^l \Delta\sigma_{zx}(s)\,d[q_0(s)] + \int_0^l \Delta\sigma_{xy}(s)\,d[r_0(s)] + R_0\Delta_t(0) + S_0\Delta_t(l)$$

ou encore

$$(30\ bis)\quad \int_0^l \lambda(s)\Delta_n(s)\,d[p_0(s)] + \int_0^l \mu(s)\Delta_n(s)\,d[q_0(s)] + \int_0^l \nu(s)\Delta_n(s)\,d[r_0(s)] + R_0\Delta_t(0) + S_0\Delta_t(l).$$

Cette seconde expression montre que la valeur de la différentielle dépend des valeurs du déplacement normal Δ_n, mais qu'elle ne dépend du déplacement tangentiel Δ_t que par les valeurs de celui ci aux extrémités de la courbe. Ce résultat est d'autant plus intéressant que l'expression de notre différentielle s'applique à un déplacement infiniment petit *ou fini* et non pas seulement à la variation.

On remarquera l'analogie de l'expression (30) avec celle adoptée par M. Volterra pour la dérivée fonctionnelle. Elle s'en distingue d'abord par le fait que les Δ sont relatifs à des déplacements finis et non à des variations et en outre par l'introduction d'intégrales de Stieltjes, qui lui donne une plus grande généralité.

Remarque

15. Il y a lieu d'observer que les fonctions $p(t)$, $q(t)$, $r(t)$ ne sont assujetties à aucune condition analogue à (24). D'une manière précise, nous allons démontrer que *si R, S sont des constantes quelconques et si $p(t)$, $q(t)$, $r(t)$ sont des fonctions quelconques à variation bornée, on peut trouver des fonctions à variation bornée $u(t)$, $v(t)$, $w(t)$ satisfaisant à la condition* (24) *et telles que l'expression* (29) *soit de la forme* (23).* En effet, posons d'abord

$$(31)\quad \begin{aligned} u_3(t) &= \int_a^t z_0'(t)\,d[q(t)] - \int_a^t y_0'(t)\,d[r(t)], \\ v_3(t) &= \int_a^t x_0'(t)\,d[r(t)] - \int_a^t z_0'(t)\,d[p(t)], \\ w_3(t) &= \int_a^t y_0'(t)\,d[p(t)] - \int_a^t x_0'(t)\,d[q(t)]. \end{aligned}$$

Ces intégrales de Stieltjes ont un sens et on démontre—comme au § 12 pour la fonction $F(t)$—qu'elles définissent des fonctions $u_3(t)$, $v_3(t)$, $w_3(t)$ à variation bornée. Posons, pour $a < t < b$, $u(t) = u_3(t)$, $v(t) = v_3(t)$,

* Il faut remarquer que le résultat actuel serait tout aussi exact, si les coefficients de $\Delta x(a)$, $\Delta y(a)$, $\Delta z(a)$, $\Delta x(b)$, $\Delta y(b)$, $\Delta z(b)$ étaient quelconques et non respectivement proportionnels aux dérivées de x, y, z en a et b.

$w(t) = w_3(t)$. Alors non seulement $u(t)$, $v(t)$, $w(t)$ sont définies par ces formules pour $a < t < b$, mais encore aussi $u(a+0)$, $u(b-0)$, $\cdots$, $w(a+0)$, $w(b-0)$. Pour connaître partout u, v, w, il suffit donc de poser encore

$$(32) \qquad u(a+0) - u(a) = Rx_0'(a) + u_3(a+0) - u_3(a),$$
$$u(b) - u(b-0) = Sx_0'(b) + u_3(b) - u_3(b-0)$$

et de même pour v, w. Nous remarquons alors que

$$\int_a^b \Delta x(t)\{d[u(t)] - d[u_3(t)]\} + \int_a^b \Delta y(t)\{d[v(t)] - d[v_3(t)]\}$$
$$+ \int_a^b \Delta z(t)\{d[w(t)] - d[w_3(t)]\}$$
$$= R[x_0'(a)\Delta x(a) + y_0'(a)\Delta y(a) + z_0'(a)\Delta z(a)]$$
$$+ S[x_0'(b)\Delta x(b) + y_0'(b)\Delta y(b) + z_0'(b)\Delta z(b)].$$

Pour prouver que l'expression (29) peut se mettre sous la forme (23), il nous suffit donc de démontrer que trois expressions de la forme

$$(33) \quad \int_a^b \Delta x(t)\, d[u_3(t)] - \int_a^b \Delta x(t) z_0'(t)\, d[q(t)] + \int_a^b \Delta x(t) y_0'(t)\, d[r(t)]$$

sont nulles. Or cette dernière est la limite d'une somme telle que

$$(34) \quad \sum_i \Delta x(\xi_i)\left\{[u_3(t_i) - u_3(t_{i-1})] - z_0'(\xi_i)\int_{t_{i-1}}^{t_i} d[q(t)] + y_0(\xi_i)\int_{t_{i-1}}^{t_i} d[r(t)]\right\}$$

qui est égale d'après (31) à

$$\sum_i \Delta x(\xi_i)\left\{\int_{t_{i-1}}^{t_i} [z_0'(t) - z_0'(\xi_i)]\, d[q(t)] - \int_{t_{i-1}}^{t_i} [y_0'(t) - y_0'(\xi_i)]\, d[r(t)]\right\}$$

lequel est en valeur absolue inférieur à

$$N\omega(\epsilon)[Q(b) + R(b)],$$

$Q(b)$ et $R(b)$ étant les variations totales de $q(t)$ et $r(t)$ dans (a, b). Quand le plus grand, ϵ, des intervalles $(t_i - t_{i-1})$ tend vers zéro, cette expression et par suite l'expression (34) tendent vers zéro. Les trois expressions telles que (33) sont bien nulles.

Reste à prouver que les fonctions u, v, w que nous venons de définir satisfont bien à la condition (24) sauf peut-être en un ensemble dénombrable de points. Il suffit évidemment de la prouver pour u_3, v_3, w_3, qui ne diffèrent de u, v, w qu'en deux points. Or la démonstration de la nullité de (33) s'applique évidemment au cas où la limite supérieure de l'intervalle d'intégration

serait t et non b. Elle s'applique quelles que soient les fonctions continues $\Delta x(t)$, $\Delta y(t)$, $\Delta z(t)$. On arrive alors au résultat énoncé en remplaçant dans l'égalité obtenue b par t et Δx, Δy, Δz par x_0', y_0', z_0'.

16. Mais à côté de l'avantage—présenté par la forme (29) de la différentielle sur la forme (23)—que nous venons de mentionner, il faut placer plusieurs inconvénients. D'abord nous avons dû supposer que la courbe L_0 est une courbe rectifiable à tangente continue, tandis que la forme (23) s'applique à une courbe continue quelconque. Et surtout le système des fonctions p, q, r est largement indéterminé pour la différentielle d'une fonction de ligne donnée. Ainsi l'expression (29) garde évidemment la même valeur quand on remplace p, q, r par $p + \rho x$, $q + \rho y$, $r + \rho z$, où ρ désigne une constante quelconque.

Parties régulières et irrégulières de la différentielle

17. Les deux formes précédentes (23) et (29) de la différentielle présentent celle-ci comme un bloc. Il y a avantage àemployer une autre forme qui manifeste l'existence dans la différentielle de parties plus ou moins régulières, plus ou moins simples.

Nous allons donc transformer le résultat précédent en utilisant une décomposition de l'intégrale de Stieltjes en trois parties de complications différentes. C'est cette décomposition à laquelle j'ai fait allusion au § 7 et dont j'ai démontré l'existence et l'unicité au Congrès des Sociétés Savantes tenu à Grenoble en Mai, 1913. J'y ai montré que toute intégrale de Stieltjes

$$\int_a^b x(t)\, d[u(t)]$$

peut se mettre sous la forme suivante

$$\int_a^b x(t)\, d[u(t)] = \sum_n A_n x(t_n) + \int_a^b \alpha(t)\, x(t)\, dt + \int_a^b x(t)\, d[\lambda(t)].$$

Pour former cette expression, on forme la " fonction des sauts " de $u(t)$

$$(35) \quad u_1(t) = \sum_{a < t_n \leqq t} [u(t_n) - u(t_n - 0)] + \sum_{a \leqq t_n < t} [u(t_n + 0) - u(t_n)],$$

où les t_n sont les points de discontinuité de $u(t)$. Puis, on pose $u_2(t) = u(t) - u_1(t)$; $u_2(t)$ est une fonction continue à variation bornée; elle a donc une dérivée presque partout; on prendra pour $\alpha(t)$ la valeur de cette dérivée là où elle existe et une valeur finie quelconque, par exemple zéro, ailleurs. La fonction $\alpha(t)$ sera donc sommable. Enfin, on posera

$$\lambda(t) = u_2(t) - \int_a^t \alpha(t)\, dt;$$

la fonction $\lambda(t)$ sera continue à variation bornée et elle aura presque partout une dérivée nulle. D'autre part, A_n est le saut de $u(t)$ au point t_n.

18. En opérant de même pour v et w, *on voit que la différentielle d'une fonction de ligne quelconque peut se mettre sous la forme suivante*

$$
\begin{aligned}
T_{\Delta x,\,\Delta y,\,\Delta z} = & \int_a^b \{ \Delta x(t)\,\alpha(t) + \Delta y(t)\,\beta(t) + \Delta z(t)\,\gamma(t) \}\,dt \\
& + \sum_n \{ A_n \Delta x(t_n) + B_n \Delta y(t_n) + C_n \Delta z(t_n) \} \\
(36) \qquad & + \Big[\int_a^b \Delta x(t)\,d[\lambda(t)] + \int_a^b \Delta y(t)\,d[\mu(t)] \\
& \qquad + \int_a^b \Delta z(t)\,d[\nu(t)] \Big].
\end{aligned}
$$

On voit de suite que cette formule diffère des formules ordinaires du Calcul des Variations non seulement en ce que les fonctions x, y, z, y entrent par leur accroissement et non par leur "variation" mais en ce qu'en outre elle comprend d'autres termes en dehors de l'intégrale ordinaire et des termes aux limites.

19. Nous devons remarquer que les trois termes du second membre de l'expression précédente peuvent aussi s'écrire, en posant

$$u_0(t) = \int_a^t \alpha(t)\,dt$$

et en raisonnant pour v, w comme pour $u(t)$, sous la forme d'intégrales de Stieltjes:

$$
\begin{aligned}
& \int_a^b \{ \Delta x(t)\,d[u_0(t)] + \Delta y(t)\,d[v_0(t)] + \Delta z(t)\,d[w_0(t)] \}, \\
(37) \qquad & \int_a^b \{ \Delta x(t)\,d[u_1(t)] + \Delta y(t)\,d[v_1(t)] + \Delta z(t)\,d[w_1(t)] \}, \\
& \int_a^b \{ \Delta x(t)\,d[\lambda(t)] + \Delta y(t)\,d[\mu(t)] + \Delta z(t)\,d[\nu(t)] \}.
\end{aligned}
$$

Nous allons montrer que *la condition* (23) *applicable à la somme des trois termes précédent, s'applique aussi à chacun d'eux séparément, et qu'elle prend une forme particulièrement simple.*

20. On a en effet d'après (24)

$$K(t) = \text{une certaine constante } k,$$

sauf peut-être en un ensemble dénombrable de points. Dès lors on a *partout*

$$(38) \qquad K(t \pm 0) = k,$$

et, d'après (22),

$$
\begin{aligned}
0 = x_0'(t)\,[u(t+0) - u(t-0)] + y_0'(t)\,[v(t+0) - v(t-0)] \\
+ z_0'(t)\,[w(t+0) - w(t-0].
\end{aligned}
$$

En écrivant cette égalité pour les points t_n—pour les autres, elle est évidente —*on aura*

$$A_n x_0'(t_n) + B_n y_0'(t_n) + C_n z_0'(t_n) = 0 \quad (n = 1, 2, 3, \cdots), \tag{39}$$

pourvu toutefois que t_n soit distinct de a et de b.

Or on a d'après (35)

$$\int_a^t x_0' \, d[u_1(t)] = x_0'(t)[u(t) - u(t-0)] + \sum_{a<t_n<t} A_n x_0'(t_n) + x_0'(a)[u(a+0) - u(a)]$$

et de même d'après (22), (38), et (39)

$$\int_a^t [x_0' \, du_1 + y_0' \, dv_1 + z_0' \, dw_1] = K(t) - K(a) = K(t), \tag{40}$$

puisque $K(a)$ est évidemment nul. On voit donc que la condition analogue à (24) est bien vérifiée pour le second terme de (37). De plus, *elle résulte des relations simples* (39) qui la remplacent.

21. Mais, en remplaçant, dans (40), $K(t)$ par son expression, il reste

$$\int_a^t [x_0' \, du_2 + y_0' \, dv_2 + z_0' \, dw_2] = 0, \tag{41}$$

et ceci pour toute valeur de t. Or soit $U_2(t)$ la variation totale de $u_2(t)$ de a à t. Les fonctions $u_2(t)$ et $U_2(t)$ sont à variation bornée et continues. Elles sont donc dérivables presque partout. Si t_0 est un point où elles sont toutes deux dérivables et si l'on écrit la formule analogue à (17) divisée par $t' - t_0$

$$\left| \frac{\int_{t_0}^{t'} x_0'(t) \, d[u_2(t)]}{t' - t_0} - x_0'(t_0) \frac{u_2(t') - u_2(t_0)}{t' - t_0} \right| < \Omega(\epsilon) \left| \frac{U_2(t') - U_2(t_0)}{t' - t_0} \right|,$$

on voit en passant à la limite que la fonction

$$\int_a^t x_0'(t) \, d[u_2(t)]$$

a une dérivée au point t_0 égale à $x_0'(t_0)$ multipliée par $d[u_2(t_0)]/dt_0$. En opérant de même pour v_2 et w_2 et se souvenant de la définition de α, β, γ, on voit que le premier membre de (41) a presque partout une dérivée égale à $x_0'(t)\alpha(t) + y_0'(t)\beta(t) + z_0'(t)\gamma(t)$. Comme le second membre est nul, *on a en définitive presque partout*

$$x_0'(t)\alpha(t) + y_0'(t)\beta(t) + z_0'(t)\gamma(t) = 0. \tag{42}$$

On peut même supposer que cette égalité a lieu partout en changeant au besoin les valeurs de $\alpha(t)$, $\beta(t)$, $\gamma(t)$ sur un ensemble de points de mesure nulle, ce qui n'altérera pas la formule (36). Si par exemple u_2, v_2, w_2 ont des dérivées continues partout, α, β, γ seront égaux à ces dérivées et l'équation (42) ayant lieu presque partout aura lieu nécessairement partout. D'ailleurs si l'on intégre (42) de a à t, en remarquant que

$$u_0(t) = \int_a^t \alpha(t)\,dt, \qquad v_0(t) = \int_a^t \beta(t)\,dt, \qquad w_0(t) = \int_a^t \gamma(t)\,dt,$$

on voit qu'on aura partout

$$(43) \qquad \int_a^t x_0'(t)\,d[u_0(t)] + y_0'(t)\,d[v_0(t)] + z_0'(t)\,d[w_0(t)] = 0.$$

Donc la condition (24) s'applique aussi au premier terme de (37). *Et elle se traduit par la condition plus simple* (42). Enfin en retranchant (43) de (41), on voit que la condition (24) s'applique aussi au troisième terme de (37), car on obtient partout

$$(44) \qquad \int_a^t x_0'(t)\,d[\lambda(t)] + y_0'(t)\,d[\mu(t)] + z_0'(t)\,d[\nu(t)] = 0.$$

Autre forme de la différentielle

22. L'expression (36) de la différentielle de U_L contient des constantes et des fonctions liées par les relations (39), (42) et (44). On peut alors en déduire une nouvelle forme de la différentielle où l'on a tenu compte d'avance de ces relations. Cette forme sera à la forme (36) ce qu'est la forme (29) à la forme (23). Nous ferons donc la même hypothèse que pour (29), à savoir que L_0 est rectifiable et à tangente continue.

Étant donnée la relation (39), on peut toujours choisir des constantes P_n, Q_n, R_n, telles que l'on ait

$$(45) \qquad \begin{gathered} A_n = Q_n z_0'(t_n) - R_n y_0'(t_n), \qquad B_n = R_n x_0'(t_n) - P_n z_0'(t_n), \\ C_n = P_n y_0'(t_n) - Q_n x_0'(t_n). \end{gathered}$$

Par suite le second terme de la différentielle (36) devient

$$(46) \qquad \begin{aligned} \sum_n \{ & P_n [y_0'(t_n)\Delta z(t_n) - z_0'(t_n)\Delta y(t_n)] \\ & + Q_n [z_0'(t_n)\Delta x(t_n) - x_0'(t_n)\Delta z(t_n)] \\ & + R_n [x_0'(t_n)\Delta y(t_n) - y_0'(t_n)\Delta x(t_n)]\}, \end{aligned}$$

où t_n est distinct de a et de b et *auquel il faut ajouter un terme de la forme*

$$(47) \quad A\Delta x(a) + B\Delta y(a) + C\Delta z(a) + A'\Delta x(b) + B'\Delta y(b) + C'\Delta z(b),$$

les relations (39) n'ayant pas lieu nécessairement aux extrémités.

De même, la relation (42) permet de choisir des fonctions sommables $P(t)$, $Q(t)$, $R(t)$ telles que l'on ait presque partout

$$\alpha(t) = Q(t)z_0'(t) - R(t)y_0'(t), \qquad \beta(t) = R(t)x_0'(t) - P(t)z_0'(t),$$

$$\gamma(t) = P(t)y_0'(t) - Q(t)x_0'(t).$$

Il nous suffit de prendre par exemple

$$P = \frac{\gamma y_0' - \beta z_0'}{x_0'^2 + y_0'^2 + z_0'^2}, \qquad Q = \frac{\alpha z_0' - \gamma x_0'}{x_0'^2 + y_0'^2 + z_0'^2}, \qquad R = \frac{\beta x_0' - \alpha y_0'}{x_0'^2 + y_0'^2 + z_0'^2}.$$

On peut supposer—en prenant au besoin pour variable t l'arc de la courbe—que $x'^2 + y'^2 + z'^2$ reste partout supérieur à un nombre $h > 0$, de sorte que P, Q, R sont bien sommables. Alors le *premier terme de la différentielle* (26) *peut s'écrire*

$$\int_a^b \{P(t)[y_0'(t)\Delta z(t) - z_0'(t)\Delta y(t)] + Q(t)[z_0'(t)\Delta x(t) - x_0'(t)\Delta z(t)]$$
$$+ R(t)[x_0'(t)\Delta y(t) - y_0'(t)\Delta x(t)]\}\, dt.$$

On retrouve ainsi l'expression qui constitue,—après remplacement des accroissements de x, y, z par leurs "variations,"—la dérivée fonctionnelle au sens de M. Volterra. Mais nous voyons qu'il s'y ajoute des termes d'espèce différentes, les termes (46), (47) et ceux qui résultent de la transformation, que nous allons opérer, du troisième terme de (36). Il nous suffit d'appliquer à ce troisième terme la transformation même qui a été appliquée dans le § 13 à la différentielle tout entière prise sous la forme (23). Seulement, nous remarquons que la fonction qui remplace $K(t)$ est d'après (44) *partout* égale à zéro. De sorte que l'expression qui correspond au δ du § 13 est cette fois nulle. D'autre part, les fonctions qui correspondent à $p(t)$, $q(t)$, $r(t)$ s'obtenant en remplaçant dans les formules (25) $u(t)$, $v(t)$, $w(t)$ par $\lambda(t)$, $\mu(t)$, $\nu(t)$ non seulement seront à variation bornée mais encore seront continues et avec une dérivée nulle presque partout. La démonstration de ce fait est identique aux démonstrations analogues des §§ 12 et 21.

En résumé, on voit *que pour toute courbe L_0 rectifiable à tangente continue, on peut mettre la différentielle d'une fonction de ligne U_L sous la forme*

$$
\begin{aligned}
&\int_a^b \{ (P(t)[y_0'(t)\Delta z(t) - z_0'(t)\Delta y(t)] \\
&+ Q(t)[z_0'(t)\Delta x(t) - x_0'(t)\Delta z(t)] \\
&+ R(t)[x_0'(t)\Delta y(t) - y_0'(t)\Delta x(t)]\}dt \\
&+ \int_a^b [y_0'(t)\Delta z(t) - z_0'(t)\Delta y(t)]\, d[X(t)] \\
&+ \int_a^b [z_0'(t)\Delta x(t) - x_0'(t)\Delta z(t)]\, d[Y(t)] \\
(48)\qquad &+ \int_a^b [x_0'(t)\Delta y(t) - y_0'(t)\Delta x(t)]\, d[Z(t)] \\
&+ \sum_n \{ P_n[y_0'(t_n)\Delta z(t_n) - z_0'(t_n)\Delta y(t_n)] \\
&+ Q_n[z_0'(t_n)\Delta x(t_n) - x_0'(t_n)\Delta z(t_n)] \\
&+ R_n[x_0'(t_n)\Delta y(t_n) - y_0'(t_n)\Delta x(t_n)]\} \\
&+ A\Delta x(a) + B\Delta y(a) + C\Delta z(a) \\
&+ A'\Delta x(b) + B'\Delta y(b) + C'\Delta z(b),
\end{aligned}
$$

où les A, B, C, A', B', C', P_n, Q_n, R_n sont des constantes quelconques, où les t_n sont des points fixes distincts des extrémités, où $P(t)$, $Q(t)$, $R(t)$ sont des fonctions sommables dans (a, b) et où $X(t)$, $Y(t)$, $Z(t)$ sont des fonctions continues à variation bornée et qui admettent presque partout une dérivée première nulle.

Transformation de la différentielle en intégrale de M. Lebesgue

23. On peut enfin transformer l'expression de la différentielle de façon à éliminer les intégrales de Stieltjes. Il suffit pour cela d'employer une méthode appliquée par M. Lebesgue* au cas d'une seule intégrale de Stieltjes. Je crois utile de développer cette méthode dans le cas un peu plus général actuel parce que M. Lebesgue n'est pas entré dans tous les détails et surtout parce que l'artifice analytique qu'il emploie parait assez subtil, tandis que dans le cas actuel il apparait sous une forme géométrique très naturelle.

Nous supposons seulement sur la courbe L_0 qu'elle est continue et nous partons de l'expression

$$(49)\qquad T_{\Delta x,\,\Delta y,\,\Delta z} = \int_a^b \Delta x(t)\, d[u(t)] + \int_a^b \Delta y(t)\, d[v(t)] + \int_a^b \Delta z(t)\, d[w(t)]$$

*Comptes Rendus t. 150 (1910), p. 86–88.

de la différentielle. Il s'agit d'effectuer sur cet intégrale de Stieltjes de simples changements de variables qui la ramènent à une intégrale ordinaire ou plus exactement à une intégrale de Lebesgue. Il y a lieu de modifier—très légèrement—la démonstration de M. Lebesgue, qui s'applique à chaque intégrale séparément, de façon à employer le même changement de variable pour les trois intégrales.

Tout d'abord, d'après la définition des intégrales de Stieltjes, on sait qu'on peut modifier u, v, w en un ensemble dénombrable de valeurs de t (autres que a et b) sans changer la valeur de l'intégrale. *Nous pourrons donc supposer que* $u(t)$, $v(t)$, $w(t)$ *sont partout continues à droite,*—sauf peut-être pour $t = a$.

Nous allons—opérant un peu différemment de M. Lebesgue—employer deux changements de variable successifs sur t, le changement intermédiaire donnant déjà un résultat qui peut être utile. Soit Σk_n une série convergente quelconque à termes positifs. Posons

$$\varphi(a) = a, \qquad \varphi(t) = t + \sum_{t_n \leqq t} k_n \qquad (a < t \leqq b),$$

où les t_n sont les points de discontinuité des fonctions u, v, w. La fonction $\varphi(t)$ est une fonction croissante qui n'est discontinue qu'aux points t_n. Soit maintenant $\theta(T)$ la borne inférieure des valeurs de t telles que $\varphi(t) \geqq T$. En posant $B = \varphi(b)$, on voit que $\theta(T)$ est une fonction continue non décroissante pour $a \leqq T \leqq B$. De plus $\theta(T)$ prend la valeur t_n pour $\varphi(t_n - 0) \leqq T \leqq \varphi(t_n)$. Nous ferons d'abord dans la formule (49) le changement de variable $t = \theta(T)$, mais avec une petite modification.

Appelons $\mathbf{U}(T)$, $\mathbf{V}(T)$, $\mathbf{W}(T)$ les fonctions définies de la manière suivante. Nous remarquons que si t croît de a à b, $\varphi(t)$ croît de a à B, mais il est discontinu aux points t_n de sorte que

$$\varphi(t_n - 0) < \varphi(t_n) = \varphi(t_n + 0)$$

(si l'un des t_n est égal à a, on a au contraire $\varphi(a) < \varphi(a + 0)$. Quand T n'est pas dans un de ces intervalles exceptionnels nous poserons

$$\mathbf{U}(T) = u[\theta(T)], \qquad \mathbf{V}(T) = v[\theta(T)], \qquad \mathbf{W}(T) = w[\theta(T)].$$

Si au contraire, on a par exemple $\varphi(t_n - 0) \leqq T \leqq \varphi(t_n)$, on a $\theta(T) = t_n$, d'où $u[\theta(T)] = u(t_n)$. Dans ce cas, on prendra pour $\mathbf{U}(T)$ par exemple non $u(t_n)$ mais une fonction linéaire entre $T'_n = \varphi(t_n - 0)$ et $T_n = \varphi(t_n)$, prenant aux extrémités les valeurs $\mathbf{U}(T'_n) = u(t_n - 0)$, $\mathbf{U}(T_n) = u(t_n)$; et de même pour $\mathbf{V}(T)$, $\mathbf{W}(T)$.

On démontre alors facilement que la différentielle (49) peut s'écrire sous la forme d'une somme d'intégrales de Stieltjes

$$(50)\quad T_{\Delta x,\,\Delta y,\,\Delta z} = \int_a^B \Delta x\,[\,\theta\,(T)\,]\,d\,[\,\mathbf{U}\,(T)\,] + \int_a^B \Delta y\,[\,\theta\,(T)\,]\,d\,[\,\mathbf{V}\,(T)\,] + \int_a^B \Delta z\,[\,\theta\,(T)\,]\,d\,[\,\mathbf{W}\,(T)\,],$$

où $\mathbf{U}\,(T)$, $\mathbf{V}\,(T)$, $\mathbf{W}\,(T)$ *sont des fonctions continues et à variation bornée et où* $\theta\,(T)$ *est une fonction continue non décroissante, ces quatre fonctions restant indépendantes de* Δx, Δy, Δz.

On voit qu'on a ainsi simplifié d'une part l'expression (49) en remplaçant les fonctions u, v, w par des fonctions de même nature mais continues, et qu'on l'a compliqué d'autre part par l'introduction de la fonction $\theta\,(T)$, qui peut être constante dans certains intervalles.

24. Considérons $\mathbf{U}\,(T)$, $\mathbf{V}\,(T)$, $\mathbf{W}\,(T)$ comme les coordonnées d'un point variable; ce point décrit une courbe rectifiable continue Γ. *Mis sous forme géométrique, en passant d'une à trois coordonnées, l'artifice de M. Lebesgue consiste au fond à prendre comme nouvelle variable l'arc de cette courbe* et paraît maintenant s'imposer. Comme il pourrait arriver que $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ fussent en même temps constants dans un même intervalle, cette circonstance nous oblige à préciser la substitution adoptée. Appelons $\Psi\,(T)$ l'arc de la courbe Γ de a à T. $\Psi\,(T)$ est une fonction continue non décroissante. Soit $\Sigma = \Psi\,(B)$; c'est la longueur de la courbe Γ. Nous appellerons $\psi\,(\sigma)$ la borne inférieure des valeurs de T telles que $\Psi\,(T) \geqq \sigma$ et nous poserons

$$\Phi\,(\sigma) = \theta\,[\,\psi\,(\sigma)\,]; \quad \mathbf{A}_1\,(\sigma) = \mathbf{U}\,[\,\psi\,(\sigma)\,]; \quad \mathbf{B}_1\,(\sigma) = \mathbf{V}\,[\,\psi\,(\sigma)\,];$$

$$\mathbf{C}_1\,(\sigma) = \mathbf{W}\,[\,\psi\,(\sigma)\,].$$

On voit alors comme au paragraphe précédent, que la différentielle (49) peut s'écrire sous la forme

$$(51)\quad T_{\Delta x,\,\Delta y,\,\Delta z} = \int_0^\Sigma \Delta x\,[\,\Phi\,(\sigma)\,]\,d\,[\,\mathbf{A}_1\,(\sigma)\,] + \int_0^\Sigma \Delta y\,[\,\Phi\,(\sigma)\,]\,d\,[\,\mathbf{B}_1\,(\sigma)\,] + \int_0^\Sigma \Delta z\,[\,\Phi\,(\sigma)\,]\,d\,[\,\mathbf{C}_1\,(\sigma)\,].$$

Or $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$ sont évidemment des fonctions continues à variation bornée et à nombres dérivés bornés. Elles ont donc presque partout chacune une dérivée, et même il existe un ensemble E de valeurs de σ de mesure Σ dans l'intervalle $(0, \Sigma)$ où les dérivées de $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$ existent et où la somme de leurs carrés est égale à 1. Définissons alors des fonctions $\mathbf{A}\,(\sigma)$, $\mathbf{B}\,(\sigma)$, $\mathbf{C}\,(\sigma)$ égales aux dérivées respectives de $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$ en chaque point de E et égales à $1 / \sqrt{3}$

partout ailleurs. On aura alors quel que soit σ dans l'intervalle $(0, \sigma)$

$$(52)\quad \begin{gathered} \mathbf{A}_1(\sigma) = \int_0^\sigma \mathbf{A}(\sigma)\,d\sigma, \quad \mathbf{B}_1(\sigma) = \int_0^\sigma \mathbf{B}(\sigma)\,d\sigma, \quad \mathbf{C}_1(\sigma) = \int_0^\sigma \mathbf{C}(\sigma)\,d\sigma, \\ [\mathbf{A}(\sigma)]^2 + [\mathbf{B}(\sigma)]^2 + [\mathbf{C}(\sigma)]^2 = 1. \end{gathered}$$

Les fonctions $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ étant sommables, $\Delta x[\Phi(\sigma)]$, $\Delta y[\Phi(\sigma)]$, $\Delta z[\Phi(\sigma)]$ étant mesurables et bornées, l'intégrale

$$(53)\quad \int_0^\Sigma \{\Delta x[\Phi(\sigma)]\,\mathbf{A}(\sigma) + \Delta y[\Phi(\sigma)]\,\mathbf{B}(\sigma) + \Delta z[\Phi(\sigma)]\,\mathbf{C}(\sigma)\}\,d\sigma$$

a un sens. Nous allons montrer qu'elle est égale au second membre de (51). Nous considérerons dans ce but les fonctions $\mathbf{F}(\sigma)$, $\mathbf{G}(\sigma)$ de σ obtenues en remplaçant dans (51) et (53) les limites supérieures d'intégration par σ. La seconde a comme on sait l'élément d'intégration pour dérivée presque partout. Un raisonnement analogue à celui de § 21 démontre que la première a la même dérivée en tout point où $\Delta x[\Phi(\sigma)]$, $\Delta y[\Phi(\sigma)]$, $\Delta z[\Phi(\sigma)]$ sont continues et où $\mathbf{A}_1(\sigma)$, $\mathbf{B}_1(\sigma)$, $\mathbf{C}_1(\sigma)$ et leurs variations totales ont une dérivée, c'est à dire aussi, presque partout. Ainsi $[\mathbf{F}(\sigma) - \mathbf{G}(\sigma)]$ a une dérivée nulle presque partout. On voit facilement que les nombres dérivés à droite de $\mathbf{F}(\sigma)$ et $\mathbf{G}(\sigma)$ sont bornés; donc aussi ceux de $\mathbf{F}(\sigma) - \mathbf{G}(\sigma)$. Par suite $\mathbf{F}(\sigma) - \mathbf{G}(\sigma)$ est l'intégrale indéfinie d'un de ces nombres dérivés là où il existe et comme il est nul presque partout, $\mathbf{F}(\sigma) - \mathbf{G}(\sigma)$ est une constante. Cette constante est nulle pour $\sigma = 0$. On a bien enfin $\mathbf{F}(\sigma) = \mathbf{G}(\sigma)$. En définitive, *la différentielle d'une fonction de ligne peut se mettre pour une ligne continue quelconque sous la forme d'une intégrale de Lebesgue:*

$$(54)\quad \begin{aligned} T_{\Delta x,\,\Delta y,\,\Delta z} = \int_0^\Sigma \{ &\Delta x[\Phi(\sigma)]\,\mathbf{A}(\sigma) + \Delta y[\Phi(\sigma)]\,\mathbf{B}(\sigma) \\ &+ \Delta z[\Phi(\sigma)]\,\mathbf{C}(\sigma)\}\,d\sigma, \end{aligned}$$

où $\Phi(\sigma)$ est une fonction non décroissante, où $\mathbf{A}(\sigma)$, $\mathbf{B}(\sigma)$, $\mathbf{C}(\sigma)$ sont des fonctions sommables bornées, et où ces quatre fonctions sont indépendantes de Δx, Δy, Δz.

25. Seulement, il faut remarquer qu'on n'obtiendra pas une forme admissible de la différentielle en choisissant pour $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ des fonctions sommables bornées arbitraires. Ces fonctions dérivent en effet, par une suite d'opérations déterminées, des fonctions u, v, w lesquelles doivent vérifier une certaine condition. Lorsque la courbe L_0 est rectifiable et à tangente continue, cette condition s'exprime sous la forme (24). Il n'est pas nécessaire d'obtenir la relation correspondante entre $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ pour transformer l'expression (55) sous une forme ne faisant intervenir que des fonctions sommables et bornées arbitraires.

Il nous suffit en effet d'utiliser le procédé qui nous a permis de passer de la forme (49) à la forme (54) et de l'appliquer à la forme explicite (29).

On voit alors que *pour une courbe rectifiable à tangente continue la différentielle peut se mettre sous la forme de la somme d'un terme fini et d'une intégrale de Lebesgue*

$$\begin{aligned} T_{\Delta x,\,\Delta y,\,\Delta z} = \int_0^H \{ & [y_0' [\Theta(\sigma)] \Delta z [\Theta(\sigma)] - z_0' [\Theta(\sigma)] \Delta y [\Theta(\sigma)]] P(\sigma) \\ & + [z_0' [\Theta(\sigma)] \Delta x [\Theta(\sigma)] - x_0' [\Theta(\sigma)] \Delta z [\Theta(\sigma)]] Q(\sigma) \\ & + [x_0' [\Theta(\sigma)] \Delta y [\Theta(\sigma)] - y_0' [\Theta(\sigma)] \Delta x [\Theta(\sigma)]] R(\sigma) \} \, d\sigma \\ & + R [x_0'(a) \Delta x(a) + y_0'(a) \Delta y(a) + z_0'(a) \Delta z(a)] \\ & + S [x_0'(b) \Delta x(b) + y_0'(b) \Delta y(b) + z_0'(b) \Delta z(b)] \end{aligned}$$

où $\Theta(\sigma)$ est une fonction non décroissante, $P(\sigma)$, $Q(\sigma)$, $R(\sigma)$ sont des fonctions sommables bornées, R, S, H sont des constantes, et où ces fonctions et ces constantes sont indépendantes de Δx, Δy, Δz.

POITIERS,
le 16 *Mai* 1913.

A TYPE OF PRIMITIVE ALGEBRA*

BY

J. H. M. WEDDERBURN

In a recent paper,† L. E. Dickson has discussed the linear associative algebra, A, defined by the relations

$$xy = y\theta(x), \qquad y^n = g,$$

where $\theta(x)$ is a polynomial in x which is rational in the field F in which A is defined. In this paper Dickson shows for $n = 2$ and $n = 3$ that, when θ and g are properly chosen, A is primitive, i. e., every element in it with the exception of zero has an inverse: A is in fact in the case $n = 2$ a direct generalization of quaternions.

So far as I am aware no algebra other than these two and fields has been proved to be primitive; hence it is of considerable interest to find that for any value of n, θ and g can be so chosen as to make Dickson's algebra primitive.

In discussing primitive algebras we may without any real loss of generality assume that any element commutative with every other element of the algebra is a scalar, i. e., an element of the field F. For, all such elements generate a commutative subalgebra B and, as in the theory of groups, we can find a complex C for which

$$A = BC = CB.$$

We can therefore regard A as an algebra whose coefficients lie in the field F extended by the elements of the commutative primitive algebra B and, in this algebra, scalars are the only elements commutative with every other element.

A primitive algebra will be called *normal* when reduced in this manner.

It follows from the general theory of linear associative algebras‡ that a normal primitive algebra is of order n^2 and that, when the field is sufficiently extended, it is equivalent to the simple matric algebra e_{pq} $(p, q = 1, 2, \cdots, n)$, so that its identical equation is of degree n. We shall now investigate the consequences that result from the assumption that the normal primitive

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† These Transactions, vol. 15 (1914), pp. 31–46.

‡ See, for instance, my paper in the Proceedings of the London Mathematical Society, ser. 2, vol. 6 (1907), pp. 77–118, where references are given.

algebra A contains an element for which the group of the corresponding identical equation is the cyclic group of order n.

1. Let A be a normal primitive algebra of order n^2 and let x be an element of it for which the group of the identical equation $f(x) = 0$ is cyclic.

If $\xi_1, \xi_2, \cdots, \xi_n$ are the roots of x, the corresponding primitive idempotent elements are

$$e_r \equiv e_{rr} = \frac{(x-\xi_1)\cdots(x-\xi_{r-1})(x-\xi_{r+1})\cdots(x-\xi_n)}{(\xi_r-\xi_1)\cdots(\xi_r-\xi_{r-1})(\xi_r-\xi_{r+1})\cdots(\xi_r-\xi_n)} \qquad (r = 1, 2, \cdots, n)$$

and* when F is extended by the roots of x, A contains a matric† algebra (e_{pq}) which does not reduce to $e_1, e_2, \cdots, e_n$, since then x would be commutative with every element of A. We may therefore suppose $e_{12} \neq 0$, so that there is an element y_1, rational in F, for which $e_1 y_1 e_2$ is not zero. The conjugates of $e_1 y_1 e_2$ in F are $e_r y_1 e_{r+1}$ $(r = 1, 2, \cdots, n)$, none of which can therefore be zero.

If we set

$$e_r y_1 e_{r+1} = \eta_{r,\, r+1}\, e_{r,\, r+1},$$

the element

$$y = \sum_{r=1}^{n} \eta_{r,\, r+1}\, e_{r,\, r+1}$$

is rational since it is the sum of the conjugates of $e_1 y_1 e_2$; its nth power is evidently the scalar $\eta_{12}\eta_{23}\cdots\eta_{n1} = g$, say, which is therefore rational. Since

$$x = \sum_{r=1}^{n} \xi_r e_r,$$

$$xy = \sum_{r=1}^{n} \xi_r \eta_{r,\, r+1}\, e_{r,\, r+1} = yx_1,$$

where

$$x_1 = \sum_{r=1}^{n} \xi_{r-1} e_r.$$

The element x_1 is commutative with x and is therefore a rational polynomial in x, say $\theta(x)$, since the identical equation, $f(x) = 0$, has no repeated roots. Evidently x and y generate A when expressed in its matric form and therefore also in its rational form.

We have therefore the following

THEOREM. *A normal primitive algebra which contains an element x whose group is cyclic is generated by x and an element y which satisfies the relations*

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* See Proceedings of the London Mathematical Society, *l. c.*, p. 97.

† A matric algebra is one for which a basis (e_{pq}) can be chosen for which $e_{pq}e_{rs} = 0$, $q \neq r$, $e_{pq}e_{qr} = e_{pr}$, $\Sigma e_{rr} = 1$. Some of the elements e_{pq} may be zero, but if so the algebra is reducible.

$$xy = y\theta(x), \qquad y^n = g,$$

where $\theta(x)$ and g are rational in the field of the coefficients and y^n is the first power of y which is commutative with x.*

Conversely if x and y satisfy these conditions, the group of x is the cyclic group of order n. This may be shown as follows.

Since

$$xy = y\theta(x),$$

therefore

$$xy^r = y^r\theta^r(x),$$

where $\theta^r(x)$ denotes the function θ iterated r times. If $\theta^r(x) = x$, then y^r is commutative with x, which is contrary to the given conditions unless $r = n$.

The matric form of y can be considerably simplified by choosing the units e_{pq} in such a way that $\eta_{r,\,r+1}\, e_{r,\,r+1}$ is replaced by $e_{r,\,r+1}$ for $r = 1, 2, \cdots, n-1$, e_{n1} being chosen so that

$$e_{12}\, e_{23} \cdots e_{n-1,\,n}\, e_{n1} = e_{11}.$$

This change in the basis of (e_{pq}) leaves e_{rr} $(e = 1, 2, \cdots n)$ unaltered so that x has the same form as before while y becomes

$$y = e_{12} + e_{23} + \cdots + e_{n-1,\,n} + ge_{n1}.$$

It is easily shown from the forms given above for x and y that any element, z, of A can be expressed uniquely in the form

$$y^{n-1}h_0 + y^{n-2}h_1 + \cdots + yh_{n-2} + h_{n-1},$$

where h_r $(r = 0, 1, \cdots, n-1)$ are rational polynomials in x.

Now

$$y^r = e_{1,\,r+1} + e_{2,\,r+2} + \cdots + e_{n-r,\,n} + g\,(e_{n-r+1,\,1} + \cdots + e_{nr})$$

and

$$y^r x^s = \xi^s_{r+1}\, e_{1,\,r+1} + \xi^s_{r+2}\, e_{1,\,r+2} + \cdots,$$

so that in the matric form of any element of A all the coefficients to the left of the principal diagonal are multiplied by g, and if any particular coefficient is zero all the coefficients in the same diagonal are zero.

2. We shall now determine the conditions which g must satisfy in order that an algebra of Dickson's type shall be primitive.

Let us consider an element of the form

$$z = y^r + y^{r-1}h_1 + \cdots + yh_{r-1} + h_r \qquad (h_r \neq 0)$$

* We may evidently modify y so that g is an algebraic integer of the field.

and let $k_1, k_2, \cdots, k_{n-r}$ be polynomials in x to be determined later, then

$$(y^{n-r} + y^{n-r-1} k_1 + \cdots + k_{n-r})z = g + y^{n-1} h_1 + y^{n-2} h_2 + \cdots + y^{n-r} h_r$$
$$+ y^{n-1} y^{-r} k_1 y^r + y^{n-2} y^{-r+1} k_1 y^{r-1} h_1 + \cdots + y^{n-2} y^{-r} k_2 y^r + \cdots + \cdots .$$

If we put $k_1 = - y^r h_1 y^{-r}$, which is a known rational polynomial in x, the coefficient of y^{n-1} is zero. Similarly the coefficient of y^{n-2} vanishes if

$$k_2 = - y k_1 y^{r-1} h_1 y^{-r} - y^r h_2 y^{-r},$$

which is also a known polynomial in x, and so on till the coefficient of y^r is reached, after which the process terminates. We can therefore in general determine the k's so that the product commences with a term $y^{r-s} h(x)$ $(s < r)$ and, after multiplying on the right by $1/h(x)$, which exists if $h \neq 0$, we have a number which begins with the term y^{r-s}. If this number has an inverse, z will also have one, since, by a well-known theorem on matrices, if a product has an inverse, the same is true of each factor. We have therefore only two types of elements to consider: first, those which may be reduced to the form $y + \varphi(x)$ by repeated applications of the process given above; and second elements for which at some stage the product is independent of y.

If $y + \varphi(x)$ is expressed in its matric form, its determinant is easily seen to be

$$\varphi(\xi_1)\varphi(\xi_2)\cdots\varphi(\xi_n) - g,$$

so that, if g is not the norm of any rational function of ξ, any element of this type has an inverse.

Suppose now that z is an element of the second of the two types mentioned above, so that there exists an element

$$z_1 = y^{n-r} + y^{n-r-1} k_1 + \cdots + k_{n-r},$$

such that $z_1 z$ is independent of y. The determination of the coefficients $k_1, \cdots, k_{n-r}$ is wholly independent of g, as the transform of any polynomial in x by a power of y depends solely on $\theta(x)$. We may therefore set

$$z_1 z = g + k,$$

where k is independent of g, which may be regarded as a variable scalar.

If k is not itself a scalar, $g + k$ certainly has an inverse and therefore z has also. We may assume therefore that k is a scalar, so that the matrix corresponding to z_1 differs from the adjoint of the matrix corresponding to z merely by a scalar factor which must be an integral function of g since z_1, z and $g + k$ are all integral in g. Since the determinant of the adjoint of a matrix is a power of the determinant of that matrix, the determinant of z,

say $|z|$, must be a power of $g + k$, and as $|z|$ is obviously of the rth degree in g, beginning with the term $(-1)^{n-r} g^r$, we have

$$|z| = (-1)^{n-r} (g + k)^r .$$

But, when $g = 0$, $|z|$ becomes the norm of h_r so that

$$(-1)^{n-r} k^r = N(h_r) .$$

Unless the scalar $g + k$ is zero, z has an inverse. In the contrary case, $g^r = N(-h_r)$. Hence, *if no power of g less than the nth is the norm of a rational polynomial in x, every element of the algebra, except zero, has an inverse.**

*The existence of such rational numbers g follows from Satz 33 of Hilbert's *Bericht*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 4 (1897), p. 198.

PROPERTIES OF SURFACES WHOSE ASYMPTOTIC CURVES BELONG TO LINEAR COMPLEXES*

BY

CHARLES T. SULLIVAN

Introduction

In this paper a study is made of the geometrical properties of surfaces whose asymptotic curves belong to linear complexes. The treatment is based on the methods developed by E. J. Wilczynski in his book on projective differential geometry† and a series of memoirs published in these Transactions.‡

So far as I am aware, the only papers on this subject are a note by Sophus Lie§ and a thesis by Arnold Peter.‖ In his thesis, Peter gives the analytical details of Lie's note and establishes the theorem that "*The determination of the surfaces can be reduced to quadratures.*"

One of the theorems established below, namely, "The ruled surfaces of the problem have straight line directrices" is due to Peter. The method of proof employed here, however, is entirely different from that used by Peter and is, moreover, essentially connected with the subsequent study of non-ruled surfaces.

In my work, as well as in that of Peter, a certain quadric surface plays an important rôle. From my point of view this quadric is the locus of the directrices of the osculating ruled surfaces associated with the two families of

* Presented to the Society (Chicago), April, 1912.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces.* B. G. Teubner, Leipzig, 1906. We shall hereafter refer to this book as W.

‡ Vol. 8 (1907), pp. 223–260; vol. 9 (1908), pp. 79–120, 293–315. We shall hereafter refer to these as M_1, M_2, M_3.

§ Sophus Lie, Christiania Videnskabsselskabs Forhandlinger, 1882, Nr. 21.

‖ Arnold Peter, *Die Flächen deren Haupttangentenkurven linearen Komplexen angehören* Leipzig, Dissertation, 1895. Since the presentation of this paper to the Society, some points of its subject matter have been touched upon in two other papers, viz., one by Enrico Bompiani, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912); the other by Corrado Segre, Reale Accademia delle Scienze di Torino, vol. 49 (1913). I have recently learned that a certain aspect of the subject of this paper was treated by M. Keraval, Bulletin de la Societé Mathématique de France, vol. 39 (1911). The methods used in these papers are entirely unlike those employed here, and the results are only distantly related to those of this paper.

asymptotic curves. If the asymptotic curves of only one of the two families belong to linear complexes, this quadric is replaced by a directrix-ruled surface of a higher order. On the other hand, the quadric introduced by Peter is characteristic of the canonical differential equations of the surfaces. However there is nothing in Peter's thesis concerning the geometrical significance of this quadric, nor is there anything analogous to my discussion of the osculating ruled surfaces and the directrix-ruled surface. All of the other results in the two papers are entirely unrelated and therefore require no further comparison.* The two principal results obtained in this paper are:

A geometric construction for the surfaces of the problem (§ 4), and a normal form for their defining equations (§ 3). In the normal form of these equations the coefficients, and therefore all the invariants of the surfaces, are given explicitly as functions of the two parameters u and v of the asymptotic curves.

I take this opportunity of expressing my indebtedness to Professor Wilczynski for advice generously given me from time to time during the preparation of this paper.

§ 1. Theorems concerning ruled surfaces whose asymptotic curves belong to linear complexes

Any non-developable ruled surface may be defined by a system of differential equations of the form†

$$(1)\qquad \begin{aligned} y'' + p_{11}\, y' + p_{12}\, z' + q_{11}\, y + q_{12}\, z &= 0, \\ z'' + p_{21}\, y' + p_{22}\, z' + q_{21}\, y + q_{22}\, z &= 0, \end{aligned}$$

where

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2 y}{dx^2}, \text{ etc.}$$

Its generators are the lines that join those points on the integral curves C_y and C_z which correspond to the same values of the independent variable x.

The differential equations of the integral curves C_y and C_z are‡

$$(2)\qquad \begin{aligned} y^{(4)} + 4p_1\, y^{(3)} + 6p_2\, y'' + 4p_3\, y' + p_4\, y &= 0, \\ z^{(4)} + 4q_1\, z^{(3)} + 6q_2\, z'' + 4q_3\, z' + q_4\, z &= 0, \end{aligned}$$

where

* Certain theorems relating to the asymptotic curves on ruled surfaces with straight line directrices have been established by Cremona, Annali di matematiche, 1867–68, Halphen, Bulletin de la Societé mathématique de France, vol. 5 (1877), and Snyder, Bulletin of the American Mathematical Society, vol. 5 (1899).

†W, p. 126.

‡ W, p. 230.

$$
\begin{aligned}
&p_1 = \frac{1}{4\Delta_1}(q_{12}\, l_{12} - p_{12}\, m_{12}), \qquad p_2 = \frac{1}{6\Delta_1}(s_{12}\, l_{12} - r_{12}\, m_{12}),\\
&q_1 = \frac{1}{4\Delta_2}(q_{21}\, l_{21} - p_{21}\, m_{21}), \qquad q_2 = \frac{1}{6\Delta_2}(s_{21}\, l_{21} - r_{21}\, m_{21}),\\
&p_3 = \frac{1}{4\Delta_1}[\, m_{12}(p_{12}\, r_{11} - p_{11}\, r_{12}) + l_{12}(p_{11}\, s_{12} - q_{12}\, r_{11}) - l_{11}\Delta_1\,],\\
(3)\qquad &p_4 = \frac{1}{\Delta_1}\;[\, m_{12}(p_{12}\, s_{11} - q_{11}\, r_{12}) + l_{12}(q_{11}\, s_{12} - q_{12}\, s_{11}) - m_{11}\Delta_1\,],\\
&q_3 = \frac{1}{4\Delta_2}\;[\, m_{21}(p_{21}\, r_{22} - p_{22}\, r_{21}) + l_{21}(p_{22}\, s_{21} - q_{21}\, r_{22}) - l_{22}\Delta_2\,],\\
&q_4 = \frac{1}{\Delta_2}\;[\, m_{21}(p_{21}\, s_{22} - q_{22}\, r_{21}) + l_{21}(q_{22}\, s_{21} - q_{21}\, s_{22}) - m_{22}\Delta_2\,];
\end{aligned}
$$

and where we shall assume $\Delta_1 \neq 0$, $\Delta_2 \neq 0$; i. e., that neither C_y nor C_z is a plane curve.

The quantities Δ_1, Δ_2, l_{ij}, m_{ij}, s_{ij} are defined as follows:

$$
\begin{aligned}
&\Delta_1 = p_{12}\, s_{12} - q_{12}\, r_{12}, \qquad \Delta_2 = p_{21}\, s_{21} - q_{21}\, r_{21},\\
&r_{11} = p_{11}^2 + p_{12}\, p_{21} - p'_{11} - q_{11}, \qquad s_{11} = p_{11}\, q_{11} + p_{12}\, q_{21} - q'_{11},\\
&r_{12} = p_{12}(p_{11} + p_{22}) - p'_{12} - q_{12}, \qquad s_{12} = p_{11}\, q_{12} + p_{12}\, q_{22} - q'_{12},\\
&r_{21} = p_{21}(p_{11} + p_{22}) - p'_{21} - q_{21}, \qquad s_{21} = p_{21}\, q_{11} + p_{22}\, q_{21} - q'_{21},\\
&r_{22} = p_{22}^2 + p_{12}\, p_{21} - p'_{22} - q_{22}, \qquad s_{22} = p_{21}\, q_{12} + p_{22}\, q_{22} - q'_{22},\\
&l_{11} = -\, p_{11}\, r_{11} - p_{21}\, r_{12} + r'_{11} + s_{11}, \qquad m_{11} = -\, r_{11}\, q_{11} - r_{12}\, q_{21} + s'_{11},\\
&l_{12} = -\, p_{12}\, r_{11} - p_{22}\, r_{12} + r'_{12} + s_{12}, \qquad m_{12} = -\, r_{11}\, q_{12} - r_{12}\, q_{22} + s'_{12},\\
&l_{21} = -\, p_{11}\, r_{21} - p_{21}\, r_{22} + r'_{21} + s_{21}, \qquad m_{21} = -\, r_{21}\, q_{11} - r_{22}\, q_{21} + s'_{21},\\
&l_{22} = -\, p_{12}\, r_{21} - p_{22}\, r_{22} + r'_{22} + s_{22}, \qquad m_{22} = -\, r_{21}\, q_{12} - r_{22}\, q_{22} + s'_{22}.
\end{aligned}
$$

The fundamental seminvariants and invariants of the integral curve C_y are given by the following expressions*:

$$
\begin{aligned}
&P_2 = p_2 - p'_1 - p_1^2, \qquad P_3 = p_3 - p''_1 - 3p_1\, p_2 + 2p_1^3,\\
(4)\qquad &P_4 = p_4 - 4p_1\, p_3 - 3p_2^2 + 12p_1^2\, p_2 - 6p_1^4 - p_1^{(3)},\\
&\theta_3 = P_3 - \tfrac{3}{2}P'_2, \qquad \theta_4 = P_4 - 2P'_3 + \tfrac{6}{5}P''_2 - \tfrac{6}{25}P_2^2.
\end{aligned}
$$

On replacing p, P, and θ by q, Q, and θ' respectively, we obtain the corresponding expressions for the integral curve C_z.

* W, p. 239.

Let the curves C_y and C_z be two of the *curved* asymptotic lines on the integrating ruled surface S of system (1). Then we have $p_{12} = p_{21} = 0$, and we may also assume that $p_{11} = p_{22} = 0$.*

It is easy to show and, moreover, it is well known, that an asymptotic curve on a ruled surface is never a plane curve unless it is a straight line. If we leave aside the case of a quadric surface, there will be at most two asymptotic curves of the second kind which are plane curves, and these will be straight lines. It follows that the curves C_y and C_z are distinct from these since, by hypothesis, they are not plane curves. The functions q_{12} and q_{21} must therefore be different from zero.

The tangents to the integral curves C_y and C_z will belong to linear complexes if, and only if, the invariants θ_3 and θ_3' vanish identically.† Let us assume, then, that system (1) has been so transformed that $p_{ij} = 0$ $(i = 1, 2; j = 1, 2)$; and let us express θ_3 and θ_3' in terms of the remaining coefficients of (1) by means of (2), (3), (4). We find the following values for θ_3 and θ_3';

$$(5) \qquad \theta_3 = q_{11}' - q_{22}' - \frac{q_{12}'}{q_{12}}(q_{11} - q_{22}), \qquad \theta_3' = q_{11}' - q_{22}' - \frac{q_{21}'}{q_{21}}(q_{11} - q_{22}).$$

Thus, the conditions $\theta_3 = \theta_3' = 0$ reduce to

$$(6) \qquad q_{11}' - q_{22}' = \frac{q_{12}'}{q_{12}}(q_{11} - q_{22}), \qquad q_{11}' - q_{22}' = \frac{q_{21}'}{q_{21}}(q_{11} - q_{22}).$$

It may happen either that C_y and C_z are the *only* asymptotic curves on the surface S which belong to linear complexes, or else that there exists a third asymptotic curve on S which has the same property. In the latter case all the asymptotic curves on S have this property and, if the function $q_{11} - q_{22}$ should happen to vanish for the original pair of asymptotic curves, we may avoid this complication by choosing another pair of fundamental curves. We may therefore assume $q_{11} - q_{22}$ to be different from zero. If, however, C_y and C_z constitute the only pair of asymptotic curves on the surface which belong to linear complexes, they both belong to the same complex and we cannot avoid the vanishing of $q_{11} - q_{22}$ if C_y and C_z are to be asymptotic curves. In this case the invariant‡

$$\theta_9 = \begin{vmatrix} u_{11} - u_{22} & u_{12} & u_{21} \\ v_{11} - v_{22} & v_{12} & v_{21} \\ w_{11} - w_{22} & w_{21} & w_{21} \end{vmatrix}$$

also vanishes while its second order minors are not all equal to zero. The

* W, pp. 114, 142.

† W, p. 254.

‡ W, pp. 96, 167.

integral ruled surface itself belongs to a linear complex but has at most one straight line directrix.

Let us return to the case when $q_{11} - q_{22}$ is different from zero. The integration of (6) gives

$$q_{11} - q_{22} = aq_{12}, \qquad q_{11} - q_{22} = bq_{21},$$

where a and b are constants different from zero. In this case not only does the invariant θ_9 vanish, but so do all of its second order minors. Therefore the surface has a pair of straight line directrices coincident or distinct according as the invariant θ_4 of the surface S vanishes or not, i. e., according as the flecnodes of each generator are coincident or distinct.* The converse of this is also true.† Hence we have the theorem:

In order that all of the asymptotic curves of a ruled surface may belong to linear complexes the surface must have two straight line directrices, distinct or coincident.

If the invariant θ_4 of the ruled surface S does not vanish, the independent variable may be so chosen as to make θ_4 equal to any non-vanishing constant.‡ We may, therefore, assume the independent variable so chosen as to make the function $q_{11} - q_{22}$ equal to unity. The equations of the surfaces S will then assume the form

$$(7) \qquad y'' + q_{11}\, y + az = 0, \qquad z'' + by + q_{22}\, z = 0,$$

where a and b are constants and where $q_{11} - q_{22} = 1$.

The differential equations (2) now become

$$y^{(4)} + (q_{11} + q_{22})\, y'' + 2q'_{11}\, y' + (q''_{11} + q_{11}\, q_{22} - ab)\, y = 0,$$

$$z^{(4)} + (q_{11} + q_{22})\, z'' + 2q'_{22}\, z' + (q''_{22} + q_{11}\, q_{22} - ab)\, z = 0.$$

From these it follows that *any curved asymptotic line on S is a projection of any other.*

In order to determine the ruled surfaces S all of whose asymptotic curves are twisted cubics, we must satisfy the further conditions $\theta_4 = \theta'_4 = 0$.§ We thus obtain the following equations for the functions q_{11} and q_{22}:

$$(8) \qquad q_{11} - \tfrac{1}{2} = q_{22} + \tfrac{1}{2} = \varphi, \qquad \varphi'' + \tfrac{8}{5}\varphi^2 + c_1 = 0,$$

where

$$c_1 = -\tfrac{5}{8}(1 + 4ab).$$

If $\varphi' = 0$, then

$$q_{11} = k = \tfrac{1}{2} \pm \tfrac{5}{8}\sqrt{1 + 4ab}, \qquad q_{22} = k - 1,$$

so that q_{11} and q_{22} are constants.

* W, p. 169.
† W, p. 288.
‡ W, p. 117.
§ W, p. 242.

From these equations it follows that *the first derivative surface* is a projective transformation of the surface S itself, and the second derivative surface coincides with the surface S.*

The surface turns out to be the quartic

$$x_1^3 x_4 - x_2^3 x_3 = 0,\dagger$$

whose invariant θ_4 is equal to $16\,(4ab + 1)$.

It therefore follows that the invariants θ_3 and θ_4 of the curve C_y may vanish, while the invariant θ_4 of the surface S is different from zero.

If $\varphi' \neq 0$, we can integrate the equation of condition (8) by means of elliptic functions. In fact, if we multiply both members of (8) by φ' and integrate, we find

$$\varphi'^2 = -\tfrac{4}{15}(4\varphi^3 - g_2\varphi - g_3),$$

where $g_2 = -\tfrac{15}{2}C_1$ and g_3 is a constant of integration. Now it is always possible to construct a Weierstrass $\wp$-function with given invariants g_2, g_3 to satisfy this equation. We then find

$$q_{11} = \wp(u; g_2, g_3) + \tfrac{1}{2}; \qquad q_{22} = \wp(u; g_2, g_3) - \tfrac{1}{2}.$$

Equations (7) may be reduced to the form‡

$$(7a) \qquad \frac{d^2\eta}{du^2} + [A\wp + B]\,\eta = 0 \qquad \frac{d^2\zeta}{du^2} + [A\wp + B_1]\,\zeta = 0,$$

where A, B, B_1 are constants.

These equations are of the Picard type, and precisely those of such great importance in the analytical theory of heat. In fact, an integral of (7a) leads at once to an integral of the heat equation in elliptic coördinates. Since A is not of the form $n\,(n+1)$, the equations are not of the Lamé type.

If the directrices of the surface S coincide, i. e., if the invariant θ_4 vanishes, we can write equations (1) in the form

$$(9) \qquad \frac{d^2 y}{dx^2} + q(x)\,y = 0, \qquad \frac{d^2 z}{dx^2} + y + q(x)\,z = 0,$$

so that C_y is the straight line directrix and C_z is any curved asymptotic line on the surface. The condition (8) becomes

* W, pp. 187–188.

† To get this result, we transform (7) so that the fundamental curves coincide with the flecnode curves and then integrate the resulting equations.

‡ This reduction is effected by means of the following transformations:

$$u = \frac{2i}{\sqrt{15}}\,x, \qquad y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta,$$

where η and ζ are the linear factors of a certain quadratic covariant C (W, p. 124 et seq.).

$$\frac{d^2 q}{dx^2} + \tfrac{8}{5} q^2 = 0. \tag{8a}$$

If $q'' = 0$, q itself must vanish. In this case equations (9) show that the coördinates of any point on the surface S are given by the equations

$$x_1 = \alpha + \beta x^2, \qquad x_2 = \alpha x + \frac{\beta}{3} x^3, \qquad x_3 = -2\beta x, \qquad x_4 = -2\beta,$$

where α and β are arbitrary constants. Hence the equation of the surface is

$$x_3^3 + 3x_4 (x_1 x_3 - x_2 x_4) = 0,$$

which is that of a Cayley cubic scroll. If $q'' \neq 0$, equation (8a) gives on integration

$$\left(\frac{dq}{dx}\right)^2 = -\frac{4}{15}(4q^3 - g_3),$$

where g_3 is a constant. As before we construct the Weierstrass $\wp$-function satisfying this equation; so that equations (9) may be written

$$\frac{d^2 y}{du^2} + A\wp y = 0, \qquad \frac{d^2 z}{du^2} + A(\wp z + y) = 0, \tag{9a}$$

where A is a constant but not of the form $n(n+1)$.

Therefore *every ruled surface whose asymptotic curves are twisted cubics may be represented by a system of equations of the form (7a) or (9a), where $\wp$ is the Weierstrass elliptic function. If the coefficients of these equations reduce to constants, we obtain either a certain quartic or a Cayley cubic scroll.*

§ 2. Analytic criteria for non-ruled surfaces whose asymptotic curves belong to linear complexes

It is known that any non-developable surface S may be regarded as an integral surface of a non-involutory, completely integrable system of partial differential equations of the form*

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a' y_u + gy = 0, \tag{10}$$

where the asymptotic curves are parametric, and where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \qquad \text{etc.}, \quad \cdots.$$

The integrability conditions of this system of equations are

$$\begin{gathered} a'_{uu} + g_u + 2ba'_v + 4a' b_v = 0, \qquad b_{vv} + f_v + 2a' b_u + 4ba'_u = 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a' f_u + 4gb_v + 2bg_v = 0. \end{gathered} \tag{11}$$

* M_1, p. 241.

The asymptotic curves $v =$ const. and $u =$ const. will be denoted by Γ' and Γ'' respectively.

The differential equations of Γ' and Γ'' are*

$$
\begin{aligned}
&y_{uuuu} + 4p_1' \, y_{uuu} + 6p_2' \, y_{uu} + 4p_3' \, y_u + p_4' \, y = 0, \\
&y_{vvvv} + 4p_1'' \, y_{vvv} + 6p_2'' \, y_{vv} + 4p_3'' \, y_v + p_4'' \, y = 0,
\end{aligned} \tag{12}
$$

where

$$p_1' = -\frac{b_u}{2b}, \qquad p_2' = \frac{1}{3}(f + b_v) - \frac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right),$$

$$p_3' = \frac{1}{2}\left(f_u + 4a' b^2 - f\frac{b_u}{b}\right),$$

$$p_4' = f_{uu} - f^2 + 4b^2 g - 2bf_v + \frac{f}{b}(2bf + 2bb_v - b_{uu}) - 2\frac{b_u}{b^2}(bf_u - b_u f).$$

The fundamental seminvariants of Γ' have the values

$$
\begin{aligned}
P_2' &= p_2' - \frac{\partial p_1'}{\partial u} - p_1'^2, \\
P_3' &= p_3' - \frac{\partial^2 p_1'}{\partial u^2} - 3p_1' p_2' + 2p_1'^3, \\
P_4' &= p_4' - 4p_1' p_3' - 3p_2'^2 + 12p_1'^2 p_2' - 6p_1'^4 - \frac{\partial^3 p_1'}{\partial u^3}.
\end{aligned} \tag{13}
$$

Finally the fundamental invariants of Γ' are

$$\theta_3' = P_3' - \frac{3}{2}\frac{\partial P_2'}{\partial u}, \qquad \theta_4' = P_4' - 2\frac{\partial \theta_3'}{\partial u} - \frac{9}{15}\left(3\frac{\partial^2 P_2'}{\partial u^2} + \frac{2}{5}P_2'^2\right). \tag{14}$$

If we replace

$$a', \; b, \; f, \; g, \; p_i', \; P_i', \; \theta', \; u, \; v$$

by

$$b, \; a', \; g, \; f, \; p_i'', \; P_i'', \; \theta'', \; v, \; u$$

respectively, we obtain the corresponding expressions for Γ''.

In the subsequent applications of these equations we shall generally assume that $a' \not\equiv 0$, $b \not\equiv 0$, so that ruled surfaces are to be excluded unless the contrary is stated.

Let us consider the integral surfaces S for which the asymptotic curves Γ' and Γ'' belong to linear complexes. For these surfaces the invariants θ_3' and θ_3'' must vanish identically. By direct calculation from equations (12), (13), (14), these conditions are found to reduce to

* M_2, pp. 90, 93.

$$(15)\qquad \begin{aligned} \Omega' &\equiv 4a'b - \frac{\partial^2 \log a'}{\partial u\,\partial v} = 0, \\ \Omega'' &\equiv 4a'b - \frac{\partial^2 \log b}{\partial u\,\partial v} = 0, \end{aligned}$$

whence, by subtraction,

$$(16)\qquad \frac{\partial^2 \log (a'/b)}{\partial u\,\partial v} = 0.$$

Hence, by integration,

$$(17)\qquad b = \varphi(u)\,\theta(v)\,a'.$$

The transformation

$$(18)\qquad y = \lambda(u, v)\,\bar{y}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v)$$

leaves the parametric curves of equations (10) unchanged. Moreover the function $\lambda(u, v)$ may be chosen in such a way that the transformed system of equations shall again have the same (canonical) form as the system (10). The transformed coefficients $\bar{a}'$, $\bar{b}$, $\bar{f}$, $\bar{g}$ will then be given by the equations

$$(19)\qquad \begin{aligned} \bar{a}' &= \frac{\alpha_u}{\beta_v^2}\,a', \qquad & \bar{f} &= \frac{1}{\alpha_u^2}\left(f - b\zeta - \tfrac{1}{2}\mu\right), \\ \bar{b} &= \frac{\beta_v}{\alpha_u^2}\,b, \qquad & g &= \frac{1}{\beta_v^2}\left(g - a'\eta - \tfrac{1}{2}\nu\right), \end{aligned}$$

where

$$\eta = \frac{\alpha_{uu}}{\alpha_u}, \qquad \zeta = \frac{\beta_{vv}}{\beta_v}, \qquad \mu = \eta_u - \tfrac{1}{2}\eta^2, \qquad \nu = \zeta_v - \zeta^2.*$$

In particular, the transformation

$$y = \frac{1}{\sqrt{\dfrac{d\bar{u}}{du}\cdot\dfrac{d\bar{v}}{dv}}}\cdot\bar{y}, \qquad \bar{u} = \int \sqrt[3]{\varphi(u)}\,du, \qquad \bar{v} = \int \frac{1}{\sqrt[3]{\theta(v)}}\,dv,$$

will replace equations (10) by a system of the same form in which the fundamental invariants a' and b are equal to each other. Let us assume that this transformation has been effected already, so that we have

$$a' = b.$$

The conditions

$$\Omega' = \Omega'' = 0$$

will then reduce to

$$(20)\qquad \frac{\partial^2 \log b}{\partial u\,\partial v} = 4b^2.$$

If we put $b = e^{\phi/2}$, the above equation becomes

* M_1, pp. 249–250.

$$\frac{\partial^2 \phi}{\partial u \, \partial v} = 8e^{\phi}. \tag{20a}$$

But (20a) is the form known as Liouville's equation.* Its general integral is

$$e^{\phi} = \frac{1}{4} \frac{U' V'}{(U + V)^2},$$

where U and V are functions of the single variables u and v respectively, and where the accents indicate differentiation with respect to these variables. Hence the general integral of (20) is

$$a' = b = \frac{1}{2} \frac{\sqrt{U' V'}}{(U + V)}, \tag{21}$$

where we assume $U' \not\equiv 0$, $V' \not\equiv 0$ when we wish to exclude ruled surfaces. We shall refer to equations (10) as being in the *normal form*† when the invariants a' and b are given by (21). We therefore have the theorem:

Both families of asymptotic curves of a non-ruled surface belong to linear complexes, if and only if, the differential equations of the surface can be put in the normal form characterized by the values (21) *for the invariants a' and b.*‡

§ 3. Properties of the osculating ruled surfaces, and explicit determination of the differential equations of the surfaces of § 2

The differential equations of R_1 (one of the osculating ruled surfaces of the first kind) are§

$$\begin{aligned} y_{vv} + p_{11} y_v + p_{12} z_v + q_{11} y + q_{12} z &= 0, \\ z_{vv} + p_{21} y_v + p_{22} z_v + q_{21} y + q_{22} z &= 0, \end{aligned} \tag{22}$$

where $z = y_u$ and

$$p_{11} = p_{12} = p_{22} = 0, \qquad q_{11} = g, \qquad q_{12} = 2a',$$

$$p_{21} = -4a' b, \qquad q_{21} = g_u - 2a' f, \qquad q_{22} = g + 2a'_u.$$

Since p_{12} is equal to zero, Γ'' is an asymptotic curve on R_1 as well as on the surface S.

The invariants of weights four and nine of R_1 are

$$\theta = (u_{11} - u_{22})^2 + 4u_{12} u_{21}, \tag{23}$$

and

$$\theta_9 = \begin{vmatrix} u_{11} - u_{22} & u_{12} & u_{21} \\ v_{11} - v_{22} & v_{12} & v_{21} \\ w_{11} - w_{22} & w_{12} & w_{21} \end{vmatrix},$$

* Darboux *Leçons sur la Théorie des Surfaces*, t. IV, pp. 419–424.

† The complete *normal form* will be given later.

‡ The invariants Ω' and Ω'' vanish for these surfaces only.

§ M_2, p. 81 et seq.

where

$$u_{11} - u_{22} = 8a'_u, \qquad u_{12} = -8a', \qquad u_{21} = 4a'_{uu} + 8a'(f + b_v),$$

$$v_{11} - v_{22} = 16(a'_{uv} - 4a'^2 b), \qquad v_{12} = -16a'_v,$$

$$v_{21} = 8(a'_{uuv} - 4a'a'_u b + 2a'_v f + 2a'f_v + 2a'_v b_v + 2a'b_{vv})$$

$$= 8[a'_{uuv} + 2a'_v(f + b_v) - 4a'^2 b_u - 12a'ba'_u],$$

$$w_{11} - w_{22} = 32(a'_{uvv} - 4a'^2 b_v - 12a'ba'_v), \qquad w_{12} = -32a'_{vv},$$

$$w_{21} = 16[a'_{uuvv} + 2a'_{vv}(f + b_v) - 4a'^2 b_{uv} - 12a'(a'_v b_u + a'_u b_v)$$

$$- 16a'(ba'_{uv} - a'^2 b^2) - 20a'_u a'_v b],$$

the final forms of u_{21}, v_{21}, w_{21} resulting from an application of the integrability conditions (11).

Analogous equations may be obtained for R_2 (an osculating ruled surface of the second kind).

If we differentiate the conditions $\Omega' = 0$, $\Omega'' = 0$, we obtain the following equations:

$$(24)\qquad \begin{aligned} &a'a'_{uuv} - a'_v a'_{uu} - 12a'^2 ba'_u - 4a'^3 b_u = 0,\\ &a'a''_{uvv} - a'_u a'_{vv} - 12a'^2 ba'_v - 4a'^3 b_v = 0,\\ &b\,b_{uuv} - b_v b_{uu} - 12a'b^2 b_u - 4b^3 a'_u = 0,\\ &b\,b_{uvv} - b_u b_{vv} - 12a'b^2 b_v - 4b^3 a'_v = 0,\\ &a'a'_{uuvv} - a'_{uu} a'_{vv} - 24a'ba'_u a'_v - 12a'^2(a'_u b_v + a'_v b_u)\\ &\qquad - 12a'^2 ba'_{uv} - 4a'^3 b_{uv} = 0,\\ &bb_{uuvv} - b_{uu} b_{vv} - 24a'bb_u b_v - 12b^2(a'_u b_v + a'_v b_u)\\ &\qquad - 12a'b^2 b_{uv} - 4b^3 a'_{uv} = 0, \end{aligned}$$

which enable us to reduce the elements of θ_9 to

$$u_{11} - u_{22} = 8a'_u, \qquad u_{12} = -8a', \qquad u_{21} = 4[a'_{uu} + 2a'(f + b_v)],$$

$$v_{11} - v_{22} = 16\frac{a'_u a'_v}{a'}, \quad v_{12} = -16a_v, \quad v_{21} = \frac{8a'_v}{a'}[a'_{uu} + 2a'(f + b_v)],$$

$$w_{11} - w_{22} = 32\frac{a'_u a'_{vv}}{a'}, \quad w_{12} = -32a'_{vv}, \quad w_{21} = \frac{16a'_{vv}}{a'}[a'_{uu} + 2a'(f + b_v)].$$

If we substitute these simplified expressions in θ_9, we find that θ_9 and all of its second order minors vanish. In the same way we find that θ'_9, the invariant of

weight nine for R_2, and all of its second order minors vanish. Consequently we have the theorem:

*If both families of asymptotic curves of a surface S belong to linear complexes, then both families of osculating ruled surfaces R_1 and R_2 have straight line directrices, distinct or coincident according as the corresponding invariant θ or θ' is different from zero or not.**

We shall need several other formulæ connected with the theory of the osculating ruled surfaces. In developing these formulæ we shall indicate another proof of the above theorem.

If P_η and P_ζ are the flecnodes on the generator $P_y P_z$ of R_1, we have†

$$y = 16a'(\eta + \zeta), \qquad z = (8a'_u + \sqrt{\theta})\,\eta + (8a'_u - \sqrt{\theta})\,\zeta.$$

When we make this transformation the equations of R_1, referred to its flecnode curves, are found to be

$$(25)\qquad \begin{aligned} &\eta_{vv} + \pi_{11}\eta_v + \pi_{12}\zeta_v + \kappa_{11}\eta + \kappa_{12}\zeta = 0,\\ &\zeta_{vv} + \pi_{21}\eta_v + \pi_{22}\zeta_v + \kappa_{21}\eta + \kappa_{22}\zeta = 0, \end{aligned}$$

where

$$\pi_{11} = \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta}, \qquad \pi_{12} = \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta},$$

$$\pi_{21} = \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta}, \qquad \pi_{22} = \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta},$$

$$C = 8a'_{uv} - 8\frac{a'_u a'_v}{a'} - 32a'^2 b.$$

But since we are dealing with the case when $\Omega' = 0$, we find $C = 0$. Consequently

$$\pi_{12} = \pi_{21} = \frac{a'_v}{a'} - \frac{\theta_v}{2\theta} = \frac{2^5}{a'\theta}\,[a'(a'a'_{uuv} - a'_{uu}a'_v - 12a'^2 ba'_u - 4a'^3 b) + a'_u(a'_u a'_v - a'a'_{uv} + 4a'^3 b)] = 0,$$

$$\pi_{11} = \pi_{22} = 2\frac{a'_v}{a'},$$

and therefore

$$\theta = \varphi(u)\,a'^2,$$

where φ is an arbitrary function of u. We find further

* Several months after this paper was presented to the American Mathematical Society, a memoir containing this theorem was published by Enrico Bompiani, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912).

† M_2, p. 83 et seq.

$$\kappa_{11} = \frac{1}{\Delta}\left[-4a'\theta - 32ga'\sqrt{\theta} - 32a'a'_u\sqrt{\theta} - \frac{8}{\sqrt{\theta}}\left(2a'_{vv}\theta + a'\theta_{vv} - \frac{a'\theta_v^2}{2\theta}\right)\right],$$

$$\kappa_{12} = \frac{-8}{\Delta\sqrt{\theta}}\left[2a'_{vv}\theta + a'\theta_{vv} - \frac{a'\theta_v}{2\theta}\right],^* \qquad \text{etc.,}$$

which reduce to

$$\kappa_{11} = a'_u + g + \frac{\sqrt{\theta}}{8}, \qquad \kappa_{12} = 0, \qquad \kappa_{21} = 0, \qquad \kappa_{22} = a'_u + g - \frac{\sqrt{\theta}}{8}.$$

The equations of the osculating ruled surface R_1 may therefore be reduced to the form

$$(25a) \qquad \begin{aligned} &\eta_{vv} + \frac{\theta_v}{\theta}\eta_v + \left(a'_u + g + \frac{\sqrt{\theta}}{8}\right)\eta = 0, \\ &\zeta_{vv} + \frac{\theta_v}{\theta}\zeta_v + \left(a'_u + g - \frac{\sqrt{\theta}}{8}\right)\zeta = 0, \end{aligned}$$

where $\theta / a'^2 = \varphi(u)$, a function of u alone.

The corresponding equations for R_2 may be obtained from the above by replacing θ, a', g by θ', b, f respectively, and interchanging u and v. Combining these results we have the following theorem:

If the invariants θ and θ' are different from zero, and if both families of asymptotic curves on S belong to linear complexes, the invariants θ / a'^2 and θ' / b^2† are functions of the single variables u and v respectively. The equations of the osculating ruled surfaces may be reduced to the form (25a), which makes evident the already established theorem that the osculating ruled surfaces have straight line directrices.

Suppose $\theta = 0$. Let P_ζ coincide with the double flecnode on $P_y P_z$. We make the transformation

$$y = \eta, \qquad z = \frac{a'_u}{2a'}\eta - \zeta,^{\ddagger}$$

so that the equations of R_1 referred to Γ'' and its flecnode curve are§

$$(26) \qquad \begin{aligned} &\eta_{vv} + \frac{b_v^2 - 2bb_{vv}}{4b^2}\eta - 2a'\zeta = 0, \\ &\zeta_{vv} + \frac{b_v^2 - 2bb_{vv}}{4b^2}\zeta = 0, \end{aligned}$$

which are characteristic of ruled surfaces with coincident directrices.

* In order to obtain these results we must use relations (24), (11) and the relation

$$2a'_{vv}\theta + a'\theta_{vv} - a'_v\theta_v = 0$$

obtained by differentiating $\pi_{12} = \pi_{21} = 0$ with respect to v.

† $\theta' / b^2 = \psi(v)$, ψ being an arbitrary function of v.

‡ M_2, p. 86.

§ The reductions are effected by means of (24), (11), and the condition $\theta = 0$. The equations of R_2 referred to Γ' and its flecnode curve have a similar form when $\theta' = 0$.

We can now complete the determination of the *normal form* of the equations of S. We have

$$a' = b, \qquad \frac{\partial^2 \log b}{\partial u \partial v} = 4b^2,$$

and the integrability conditions (11), which are reducible to

$$b_{uu} + g_u + 6bb_v = 0, \qquad b_{vv} + f_v + 6bb_u = 0,$$

$$g_{uu} - f_{vv} - 4fb_u - 2bf_u + 4gb_v + 2bg_v = 0.$$

We have further (from the previous theorem and the definition of θ and θ'),

$$\theta = 64\,[b_u^2 - 2b\,(b_{uu} + 2b \cdot \overline{f + b_v})] = \varphi(u)\,b^2,$$

$$\theta' = 64\,[b_v^2 - 2b\,(b_{vv} + 2b \cdot \overline{g + b_u})] = \psi(v)\,b^2.$$

Whence

$$f = -b_v - \frac{1}{2}\frac{\partial^2 \log b}{\partial u^2} - \frac{1}{4}\frac{b_u^2}{b^2} + \varphi_1(u),$$

$$g = -b_u - \frac{1}{2}\frac{\partial^2 \log b}{\partial v^2} - \frac{1}{4}\frac{b_v^2}{b^2} + \varphi_2(v),$$

where

$$\varphi_1(u) = -\frac{\varphi(u)}{256}, \qquad \varphi_2(v) = -\frac{\psi(v)}{256}.$$

Substituting these values of f and g in the integrability conditions, we find that the first two are satisfied identically and the third becomes

$$-4b_u\,\varphi_1(u) + 4b_v\,\varphi_2(v) - 2b\,\varphi_1'(u) + 2b\,\varphi_2'(v) = 0.$$

Now b is given by equation (21), so that

$$\frac{b_u}{b} = \frac{1}{2}\frac{U''}{U'} - \frac{U'}{U+V}, \qquad \frac{b_v}{b} = \frac{1}{2}\frac{V''}{V'} - \frac{V'}{U+V}.$$

We must therefore have

$$2\frac{\varphi_1(u)\,U' - \varphi_2(v)\,V'}{U+V} + \varphi_2(v)\frac{V''}{V'} - \varphi_1(u)\frac{U''}{U'} = \varphi_1'(u) - \varphi_2'(v).$$

Thus the left hand member of this equation reduces to a function of u minus a function of v. Now U' and V' can not vanish, for then b also would vanish. Hence U and V are not constants, and we must have

$$\frac{\varphi_1(u)\,U' - \varphi_2(v)\,V'}{U+V} = H_1(u) - H_2(v)$$

or

$$(\alpha) \quad \varphi_1(u)\,U' - \varphi_2(v)\,V' = UH_1(u) - VH_2(v) + VH_1(u) - UH_2(v),$$

where φ_i, H_i are functions to be determined. These must be such that $VH_1(u) - UH_2(v)$ is of the form $G_1(u) - G_2(v)$, that is we must have

$$\frac{\partial^2}{\partial u \partial v}(VH_1(u) - UH_2(v)) = 0,$$

whence

$$\frac{1}{U'}\frac{dH_1}{du} = \frac{1}{V'}\frac{dH_2}{dv}.$$

Since the left hand member is a function of u alone and the right hand member is a function of v alone, their equality implies

$$\frac{1}{U'}\frac{dH_1}{du} = k_0, \quad \frac{1}{V'}\frac{dH_2}{dv} = k_0,$$

where k_0 is a constant. Therefore

$$H_1(u) = k_0 U + k_1, \qquad H_2(v) = k_0 V + k_2,$$

where k_1 and k_2 are additional constants. If these values for H_1 and H_2 be substituted in (α), we find

$$\varphi_1(u) = \frac{aU^2 + bU + c}{U'}, \qquad \varphi_2(v) = \frac{aV^2 - bV + c}{V'},$$

where $a = k_0$, $b = k_1 - k_2$, and c is a further constant. If we substitute these values of $\varphi_1(u)$ and $\varphi_2(v)$ in the expressions for f and g, we shall find

$$(27)\quad \begin{aligned} f = &-\frac{1}{2}\frac{\sqrt{U'V'}}{(U+V)}\left(\frac{V''}{2V'} - \frac{V'}{(U+V)}\right) - \frac{1}{4}\left(\frac{U''}{U'} - \frac{3}{4}\left(\frac{U''}{U'}\right)^2\right) \\ &+ \frac{3}{4}\left(\frac{U''}{U+V} - \frac{U'^2}{(U+V)^2}\right) + \frac{aU^2 + bU + c}{U'}, \\ g = &-\frac{1}{2}\frac{\sqrt{U'V'}}{(U+V)}\left(\frac{U''}{2U'} - \frac{U'}{(U+V)}\right) - \frac{1}{4}\left(\frac{V''}{V'} - \frac{3}{4}\left(\frac{V''}{V'}\right)^2\right) \\ &+ \frac{3}{4}\left(\frac{V''}{U+V} - \frac{V'^2}{(U+V)^2}\right) + \frac{aV^2 - bV + c}{V'}. \end{aligned}$$

If the osculating ruled surfaces are characterized by equations of the form (26), it is readily seen that $a = b = c = 0$, in the above equations. A discussion of some of the geometrical configurations corresponding to this case will be given below. We may recapitulate these results in the following theorem:

Every non-ruled surface all of whose asymptotic curves belong to linear complexes is defined by a system of equations of the form (10), *where the fundamental invariants a', b and seminvariants f, g are given explicitly as functions of u and v*

by equations (21) *and* (27), *and conversely. Since all the invariants of* S *are functions of* a', b, f, g *and their derivatives, they can be determined as explicit functions of* u *and* v.

§ 4. Properties of the linear complexes of the asymptotic curves, and a geometric construction for the surfaces of the problem

The directrices δ_1 and δ_2 of the surface R_1 are found to have the equations*

$$(28)\qquad \begin{aligned} 2^4 a' x_1 + Ax_2 + 2^5 a'^2 bx_4 &= 0, \\ 2^4 a' x_3 + Ax_4 &= 0, \end{aligned}$$

and

$$\begin{aligned} 2^4 a' x_1 + Bx_2 + 2^5 a'^2 bx_4 &= 0, \\ 2^4 a' x_3 + Bx_4 &= 0, \end{aligned}$$

where

$$A = 8a'_u - \sqrt{\theta}, \qquad B = 8a'_u + \sqrt{\theta}.$$

The equations of the directrices δ'_1 and δ'_2 of R_2 are found to be

$$(28a)\qquad \begin{aligned} 2^4 bx_1 + A_1 x_3 + 2^5 a' b^2 x_4 &= 0, \\ 2^4 bx_2 + A_1 x_4 &= 0, \end{aligned}$$

and

$$\begin{aligned} 2^4 bx_1 + B_1 x_3 + 2^5 a' b^2 x_4 &= 0, \\ 2^4 bx_2 + B_1 x_4 &= 0, \end{aligned}$$

where

$$A_1 = 8b_v - \sqrt{\theta'}, \qquad B_1 = 8b_v + \sqrt{\theta'}.$$

These equations are of fundamental importance in the succeeding investigation.

It is clear that, when Ω' and Ω'' vanish, the osculating linear complexes of the curves Γ' and Γ'' coincide with the linear complexes which contain all of the tangents to these curves. Hence, referred to the moving tetraedron $P_y P_z P_\rho P_\sigma$, the linear complexes determined by Γ' and Γ'' are given by the equations†

$$(29)\qquad \begin{aligned} &(a)\quad C' \equiv -b_v \omega_{34} - b\omega_{14} + b\omega_{23} = 0, \\ &(b)\quad C'' \equiv -a'_u \omega_{42} + a' \omega_{14} + a' \omega_{23} = 0. \end{aligned}$$

The invariants of C' and C'' are $-b^2$ and a'^2 respectively.

We shall now determine the linear complexes to which belong the asymptotic curves through the neighbouring points $P_{u+du,\, v}$ and $P_{u,\, v+dv}$. Their

* $P_y P_z P_\rho P_\sigma$ is the tetrahedron of reference. In this system the coördinates of the point $P \equiv \alpha y + \beta z + \gamma\rho + \delta\sigma$ are, by definition, $(\alpha, \beta, \gamma, \delta)$.

† M_2, pp. 92–94.

equations will, of course, have the same form as (29), the tetrahedra of reference being those associated with the points considered. Now when u receives an increment du, v remaining constant, the semi-covariants y, z, ρ, σ change. Let $\bar{y}$, $\bar{z}$, $\bar{\rho}$, $\bar{\sigma}$ be their new values.* We shall find

$$
(30)\qquad
\begin{aligned}
\bar{y} &= y + z du + \cdots,\\
\bar{z} &= z - (fy + 2b\rho)\, du + \cdots,\\
\bar{\rho} &= \rho + \sigma du + \cdots,\\
\bar{\sigma} &= \sigma + [(2bg - f_v)\, y + 4a' bz - (2b_v + f)\, \rho]\, du + \cdots.
\end{aligned}
$$

Let the coördinates of any point Q referred to T and $\bar{T}$ be (x_1, x_2, x_3, x_4) and $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ respectively. Then the point Q is given by each of the expressions

$$
x_1 y + x_2 z + x_3 \rho + x_4 \sigma, \qquad \bar{x}_1 \bar{y} + \bar{x}_2 \bar{z} + \bar{x}_3 \bar{\rho} + \bar{x}_4 \bar{\sigma},
$$

which must therefore be identical except for a factor. Making use of this fact and equations (30), we find (on neglecting du^2 and higher powers of du)

$$
(31)\qquad
\begin{aligned}
\omega x_1 &= \bar{x}_1 + (-f\bar{x}_2 + \alpha\bar{x}_4)\, du,\\
\omega x_2 &= \bar{x}_1\, du + \bar{x}_2 + 4a' b\bar{x}_4\, du,\\
\omega x_3 &= -2b\bar{x}_2\, du + \bar{x}_3 - \beta\bar{x}_4\, du,\\
\omega x_4 &= \bar{x}_3\, du + \bar{x}_4.
\end{aligned}
$$

Whence

$$
\begin{aligned}
\omega' \bar{x}_1 &= x_1 + (fx_2 - \alpha x_4)\, du,\\
\omega' \bar{x}_2 &= -x_1\, du + x_2 - 4a' bx_4\, du,\\
\omega' \bar{x}_3 &= 2bx_2\, du + x_3 + \beta x_4\, du,\\
\omega' \bar{x}_4 &= -x_3\, du + x_4,
\end{aligned}
$$

where ω and ω' are factors of proportionality and where

$$
\alpha = 2bg - f_v, \qquad \beta = 2b_v + f.
$$

If ω_{ij} and $\bar{\omega}_{ij}$ are the coördinates of a line referred to T and $\bar{T}$ respectively, we shall have

$$
\begin{aligned}
\bar{\omega}_{14} &= \omega_{14} + (-\omega_{13} - f\omega_{42})\, du,\\
\bar{\omega}_{23} &= \omega_{23} + (-\omega_{14} + 4a' b\omega_{34} - \beta\omega_{42})\, du,\\
\bar{\omega}_{42} &= \omega_{42} + (\omega_{14} + \omega_{23})\, du.
\end{aligned}
$$

* The tetrahedron $T \equiv P_y P_z P_\rho P_\sigma$ is displaced to $\bar{T} \equiv P_{\bar{y}} P_{\bar{z}} P_{\bar{\rho}} P_{\bar{\sigma}}$.

The linear complex C''_{u+du}, containing the asymptotic curve Γ''_{u+du}, is given by the equation

$$(32) \qquad -\bar{a}'_u\,\bar{\omega}_{42} + \bar{a}'\,(\bar{\omega}_{14} + \bar{\omega}_{23}) = 0,$$

referred to $\bar{T}$, where

$$\bar{a}' = a' + a'_u\,du + \cdots,$$

$$\bar{a}'_u = a'_u + a'_{uu}\,du + \cdots.$$

Therefore the equation of C''_{u+du}, referred to T, is

$$(32a) \qquad \begin{aligned} &(-a'_u\,\omega_{42} + a'\,\omega_{14} + a'\,\omega_{23}) \\ &\qquad + (-2a'\,\omega_{13} + 4a'^2\,b\omega_{34} + \delta\omega_{42})\,du = 0, \end{aligned}$$

where

$$\delta = -a'_{uu} - 2a\,(f + b_v) = \frac{\theta - 2^6\,a'^2_u}{2^7\,a'}.$$

We shall need the invariant $\bar{I}''$ of the complex

$$-2a'\,\omega_{13} + 4a'^2\,b\omega_{34} + \delta\omega_{42} = 0,$$

and the mutual invariant $(I'', \bar{I}'')$ of this complex and C''_u. We find

$$\bar{I}'' = -2a'\,\delta, \qquad (I'', \bar{I}'') = 2a'\,a'_u.$$

The complexes C''_u an dC''_{u+du} determine the pencil

$$(33) \qquad \lambda\,(-a'_u\,\omega_{42} + a'\omega_{14} + a'\,\omega_{23}) + \mu\,(-2a'\,\omega_{13} + 4a'^2\,b\omega_{34} + \delta\omega_{42}) = 0.$$

The special complexes of this pencil are given by the values of λ and μ which satisfy the equation

$$\lambda^2\,I'' + \lambda\mu\,(I'', \bar{I}'') + \mu^2\,\bar{I}'' = 0.$$

Hence

$$\frac{\lambda}{\mu} = \frac{-a_u \pm \dfrac{\sqrt{\theta}}{2^3}}{a'}.$$

Introducing these values of λ/μ into equation (33), we find the line coördinates of the directrices of the special complexes of the pencil to be

$$(34) \qquad \begin{aligned} &\omega_{12} = 4a'^3\,b, \qquad \omega_{13} = a'^2_u \mp \frac{a'_u\sqrt{\theta}}{2^3} + a'\,\delta, \qquad \omega_{14} = a'\left(-a'_u \pm \frac{\sqrt{\theta}}{2^3}\right), \\ &\omega_{23} = a'\left(-a'_u \pm \frac{\sqrt{\theta}}{2^3}\right), \qquad \omega_{34} = 0, \qquad \omega_{42} = -2a'^2. \end{aligned}$$

Their point equations are

$$(34a) \qquad \begin{aligned} &2^4\,a'\,x_1 + (8a'_u \mp \sqrt{\theta})\,x_2 + 2^5\,a'^2\,bx_4 = 0, \\ &2^4\,a'\,x_3 + (8a'_u \mp \sqrt{\theta})\,x_4 = 0. \end{aligned}$$

If we substitute the coördinates (34) in equation (29a), we find that the directrices of the congruence determined by the complexes C''_u and C'_{u+du} (i. e., according to equation (28) the directrices of R_1) belong to the complex C'_v. Similarly we prove that the directrices of R_2 belong to C''_u. Hence the theorem:

The directrices of the osculating ruled surfaces R_1 are contained in the complexes C'_v determined by Γ', and the directrices of R_2 are contained in the complexes C''_u determined by Γ''.

It is of interest to obtain this result in another way. Let H be the common osculating hyperboloid of R_1 and R_2. Its equation is

$$x_1 x_4 - x_2 x_3 + 2a' b x_4^2 = 0 .* \tag{35}$$

The lines (34a) and the corresponding lines of R_2 are easily seen to be on H. The complex C''_u contains three generators of the first kind on H and therefore all the generators of this set. The complex C'_v contains three and therefore all the generators of the second set on H. The directrices (δ_1, δ_2) and (δ'_1, δ'_2) are among these generators, as we have just seen. This proves the theorem.

Let v remain fixed and let u vary. Then R_2 remains fixed as P_y describes the asymptotic curve Γ'_v on S. Hence the directrices (δ'_1, δ'_2) of R_2 belong to all of the complexes C''_u. But R_2 is any surface of the one parameter family of osculating ruled surfaces of the second kind. Hence all the directrices of R_2 are generators of a ruled surface S_2 contained in all of the complexes C''_u of S. Similarly the directrices of R_1 are generators of a ruled surface S_1 contained in all of the complexes C'_v of S. We shall now prove that S_1 is identical with S_2 and therefore a quadric, the two sets of directrices being complementary reguli.

Since the four directrices (δ_1, δ_2) and (δ'_1, δ'_2) of the osculating ruled surfaces R_1 and R_2 determined by Γ' and Γ'' are situated upon the osculating hyperboloid H of the point P_y, and since this is true for every point of Γ' when v is fixed, and for every point of Γ'' when u is fixed, it follows that all the directrices of R_1 intersect all the directrices of R_2. Therefore the generators of S_1 and S_2 constitute complementary reguli on the same quadric surface. We shall call this surface the *directrix quadric* and denote it, for brevity, by the letter Q.

If the integrating surface S is not ruled, all the complexes C' can not be those of a pencil of complexes. For, if all the complexes C' were contained in a pencil, the congruence determined by C'_v and C'_{v+dv} would have the same pair of directrices (δ'_1, δ'_2) for every v. Therefore all the osculating ruled surfaces

* M_2, p. 82.

of one family would belong to the same linear congruence, viz., the one having (δ_1', δ_2') as directrices. But this implies that S itself is a ruled surface of the congruence. Similarly we show that the complexes C'' are not all contained in a pencil.

Therefore *if all the linear complexes of the asymptotic curves of either family belong to a pencil, the surface must be ruled.*

If Γ' or Γ'' belong to a one parameter family of linear complexes not contained in a net, then these complexes could not intersect in more than two lines.* Therefore, there could not be more than one pair of directrices for the osculating ruled surfaces of the other family. But by the above theorem the surface S would, in this case, be ruled. We therefore have the theorem:

If both families of asymptotic curves of a non-ruled surface S belong to linear complexes, these complexes must form two one-parameter families belonging to two involutory nets.†

This theorem, together with one which we shall now prove, leads to an important result, namely, *a geometric construction for the surfaces having the property in question.*

The osculating hyperboloid $H_{u+du,\,v}$ of the surfaces R_1 and R_2 determined by the asymptotic curves through the point $P_{u+du,\,v}$ has the equation

$$\bar{x}_1\,\bar{x}_4 - \bar{x}_2\,\bar{x}_3 + 2\bar{a}'\,\bar{b}\bar{x}_4^2 = 0\,,$$

referred to $\bar{T}$. This becomes, when referred to T,

$$(x_1\,x_4 - x_2\,x_3 + 2a'\,bx_4^2) + (-\,2bx_2^2 - 2b_v\,x_2\,x_4 + \gamma x_4^2)\,du = 0\,,$$

where

$$\gamma = 2\,[\,(a'\,b)_u - bg\,]\,.$$

Now $H_{u,\,v}$ and $H_{u+du,\,v}$ intersect along two generators of R_2 consecutive to $P_y\,P_\rho$, and two generators of the second set upon H in the planes

$$(36) \qquad 2^4\,bx_2 - (8b_v \pm \sqrt{\theta'}\,)\,x_4 = 0\ddagger.$$

The tangent plane to $H_{u,\,v}$ at the point $(y_1, 0, y_3, 0)$ is

$$(36a) \qquad x_2\,y_3 - x_4\,y_1 = 0\,.$$

Therefore the generators of intersection of $H_{u,\,v}$ and $H_{u+du,\,v}$, apart from those consecutive to $P_y\,P_\rho$, pass through the points

$$(36b) \quad \omega\eta = (8b_v - \sqrt{\theta'}\,)\,y - 16b\rho\,, \qquad \omega\zeta = -\,(8b_v + \sqrt{\theta'}\,)\,y + 16b\rho.$$

These two generators are the flecnode tangents to R_2 at the flecnodes on $P_y\,P_\rho$.

* The complexes of a hypernet have, at most, two lines in common.

† The mutual invariant (I'', I') vanishes and therefore the complexes are in involution.

‡ This follows from the definition of θ' and the second integrability condition.

Let us now assume that the asymptotic curves Γ'' belong to a one parameter family of linear complexes contained in a net, but not contained in a pencil. The complexes of this net have in common a regulus ρ'' of a quadric Q, as we have seen. The directrices $(\delta_1(u, v), \delta_2(u, v))$ of R_1 are situated upon the complementary regulus ρ' of Q, and also upon the osculating hyperboloid $H_{u,v}$. Hence Q and $H_{u,v}$ intersect in another pair of generators $(g_1(u), g_2(u))$ of Q and $H_{u,v}$ which belong to the regulus ρ''. Again, the directrices $(\delta_1(u+du, v), \delta_2(u+du, v))$ of $R_{1_{u+du,v}}$ are on Q and belong to ρ'; hence Q and $H_{u+du,v}$ intersect in another pair of generators $(g_1(u+du), g_2(u+du))$ which belong to ρ''. Now the generators, not consecutive to $P_y P_\rho$, common to $H_{u,v}$ and $H_{u+du,v}$ are given by equations (36), (36*a*), (36*b*). These being on $H_{u,v}$ and $H_{u+du,v}$ must intersect the four generators $(\delta_1(u, v), \delta_2(u, v))$ and $(\delta_1(u+du, v), \delta_2(u+du, v))$ of Q and are, therefore, generators of the regulus ρ''. If we call these lines (l_1, l_2), it follows that the pairs $(g_1(u), g_2(u))$ and $(g_1(u+du), g_2(u+du))$ both coincide with the pair (l_1, l_2). For otherwise the total intersection of Q and $H_{u,v}$ would be of higher order than the fourth, i. e., Q and $H_{u,v}$ would coincide and S would be itself a quadric surface. In precisely the same way we see that $H_{u+du,v}$, etc., $\cdots$, intersect in two generators belonging to ρ'' of Q. These must, therefore, be (l_1, l_2). Proceeding in this manner we find that all of the osculating hyperboloids of $R_2(v_0)$ (say) have in common a fixed pair of lines of the regulus ρ'' of Q, viz., the pair (l_1, l_2), and therefore R_2 has a pair of straight line directrices* $(\delta_1(u, v_0), \delta_2(u, v_0))$. Hence all of its asymptotic curves belong to linear complexes. In particular Γ' belongs to a linear complex, and this is true for every value of v. We therefore have the theorem:

If the asymptotic curves of one family of a non-ruled surface belong to a one-parameter family of linear complexes contained in a net, then the asymptotic curves of the second family must also belong to a one-parameter family of linear complexes contained in an involutory net.

To construct the surfaces having the property in question we may now proceed as follows: Select a one-parameter family of pairs of lines $(\delta_1(u), \delta_2(u))$ on Q. Each of these pairs of lines determines a pencil of linear complexes, $(\delta_1(u), \delta_2(u))$ being the axes of the special complexes of this pencil. In each of these pencils pick out (according to some arbitrary law) one complex $C(u)$. We thus obtain a one-parameter family of complexes contained in a net. Every line which intersects $(\delta_1(u), \delta_2(u))$ belongs to $C(u)$. Let us begin with $u = u_0$. Let $R_1(u_0)$ be an arbitrary ruled surface having $(\delta_1(u_0), \delta_2(u_0))$ as directrices. $R_1(u_0)$ will then belong to $C(u_0)$, and one of its asymptotic curves $\Gamma(u_0)$ intersecting every generator twice will likewise belong to

* $(\delta_1(u, v_0), \delta_2(u, v_0))$ are the lines (l_1, l_2).

the complex $C(u_0)$. Let $g_{u_0}^{(0)}$, $g_{u_0}^{(1)}$, $g_{u_0}^{(2)}$, etc., be the generators of $R_1(u_0)$ corresponding to $v = v_0$, $v_0 + dv$, $v_0 + 2dv$, etc., and intersecting $\Gamma(u_0)$ in the points A_0, A_1, A_2, etc.

Now let $(\delta_1(u_0 + du), \delta_2(u_0 + du))$ be the pair of lines of Q which correspond to $u = u_0 + du$, and $C(u_0 + du)$ the corresponding linear complex. Let P_0 be a point of $g_{u_0}^{(0)}$ ultimately to be allowed to approach the corresponding point A_0 of $\Gamma(u_0)$ as its limit. The line $g_{u_0+du}^{(0)}$ through P_0 which intersects $(\delta_1(u_0 + du), \delta_2(u_0 + du))$ belongs to the complex $C(u_0 + du)$. Hence the null-plane of P_0 in the complex $C(u_0 + du)$ contains $g_{u_0+du}^{(0)}$. Let its intersection with $g_{u_0}^{(1)}$ be called P_1. Then $P_0 P_1$ also belongs to $C(u_0 + du)$. Denote by $g_{u_0+du}^{(1)}$ the line through P_1 which intersects $(\delta_1(u_0 + du), \delta_2(u_0 + du))$, and by P_2 the point where $g_{u_0}^{(2)}$ is cut by the null-plane of P_1 in the null-system of $C(u_0 + du)$. Then $g_{u_0+du}^{(1)}$ and $P_1 P_2$ are lines of the complex $C(u_0 + du)$. As we continue this process and proceed to the limit (i. e., $dv \doteq 0$) we obtain a curve on $R_1(u_0 + du)$ whose tangents obviously belong to the complex $C(u_0 + du)$. Moreover, the null-planes of the points of this curve in the null-system of the complex $C(u_0 + du)$ are clearly tangent to the surface $R_1(u_0 + du)$. But for any curve of a linear complex the osculating plane at any point of the curve coincides with its null-plane. Therefore the curve $R_1 R_2 R_3 \cdots$ is an asymptotic curve of $R_1(u_0 + du)$.

By starting anew with the lines $(\delta_1(u_0 + 2du), \delta_2(u_0 + 2du))$ and the surface $R_1(u_0 + du)$, etc., we can construct a new ruled surface $R_1(u_0 + 2du)$ with an asymptotic curve on it belonging to the complex $C(u_0 + 2du)$, etc. If now we allow du to approach zero, we (finally) obtain a surface S as the envelope of the ruled surfaces constructed in this way; and the asymptotic curves of one family on S clearly belong to the given one parameter family of linear complexes. But by a previous theorem it follows that the asymptotic curves of the second family on S will also belong to linear complexes.

If the asymptotic curves of a single family belong to linear complexes (i. e., if $\Omega' = 0$, $\Omega'' \neq 0$), we select a one parameter family of pairs of lines $(\delta_1(u), \delta_2(u))$ on a directrix ruled surface of the third or higher order so as to preclude the possibility of the second family of asymptotic curves belonging to linear complexes. The remaining steps in this construction are precisely the same as in the case just considered and need not be repeated. Our construction shows that this directrix ruled surface may be chosen arbitrarily.

§ 5. The lines common to the complexes C'_v, C''_u, C'_{v+dv}, C''_{u+du}*

The complexes C'_v, C''_u, C'_{v+du}, C''_{u+du} have in common the same lines as the complexes

* Cf. M_2, p. 97.

$$- a'_u \omega_{42} + a' \omega_{14} + a' \omega_{23} = 0,$$

$$- 2a' \omega_{13} + 4a'^2 b\omega_{34} + \delta\omega_{42} = 0,$$

$$- b_v \omega_{34} - b\omega_{14} + b\omega_{23} = 0,$$

$$+ 2b\omega_{12} + 4a' b^2 \omega_{42} + \delta' \omega_{34} = 0,$$

where

$$\delta = \frac{\theta - 2^6 a'^2_u}{2^7 a'}, \qquad \delta' = \frac{\theta' - 2^6 b_v^2}{2^7 b}.$$

All of the lines common to these complexes intersect the lines

$$2a' bx_1 + a'_u bx_2 + a' b_v x_3 = 0, \qquad x_4 = 0;$$

$$2bx_2 + b_v x_4 = 0, \qquad 2a' x_3 + a'_u x_4 = 0,$$

which are the directrices d and d' of the first and second kinds of the point $P_{y(u,\,v)}$.* If

$$p = - a'_u y + 2a' z, \qquad q = - b_v y + 2b\rho,$$

then $\lambda p + \mu q$ is an arbitrary point on the directrix of the first kind. Similarly, if we write

$$r = - a' b_v z - ba'_u \rho + 2a' b\sigma,$$

then $\lambda' y + \mu' r$ is an arbitrary point on the directrix of the second kind. The coördinates of the line joining these points are

$$\omega_{12} = - 2a' \lambda\lambda' + a' a'_u b_v \lambda\mu' + a' b_v^2 \mu\mu',$$

$$\omega_{13} = - 2b\mu\lambda' + ba'^2_u \lambda\mu' + bb_v a'_u \mu\mu',$$

$$\omega_{14} = - 2a' ba'_u \lambda\mu' - 2a' bb_v \mu\mu',$$

$$\omega_{23} = - 2a' ba'_u \lambda\mu' + 2a' bb_v \mu\mu',$$

$$\omega_{34} = 4a' b^2 \mu\mu', \qquad \omega_{42} = - 4a'^2 b\lambda\mu'.$$

In order that this line may belong to the complexes C'_{v+dv} and C''_{u+du}, λ, μ, λ', μ' must satisfy the equations

$$\theta\lambda\mu' - 2^7 \lambda' \mu + 2^6 (a'_u b_v - 2^3 a'^2 b^2) \mu\mu' = 0,$$

$$2^7 \lambda\lambda' - 2^6 (a'_u b_v - 2^3 a'^2 b^2) \lambda\mu' - \theta' \mu\mu' = 0.$$

Hence

$$\frac{\lambda}{\mu} = \pm \sqrt{\frac{\theta'}{\theta}}, \qquad \frac{\lambda'}{\mu'} = \frac{(a'_u b_v - 2^3 a'^2 b^2) \pm \dfrac{\sqrt{\theta\theta'}}{2^6}}{2}.$$

* M_2, pp. 95, 96.

Thus we have two lines* common to these four complexes, and they are obviously the diagonals of the skew quadrilateral $(\delta_1, \delta_2, \delta_1', \delta_2')$. Since $\lambda/\mu = \pm\sqrt{\theta'/\theta}$, l_1 and l_2 intersect the directrix d in two points which are harmonic conjugates with respect to p and q. Again d' intersects the osculating hyperboloid H in the points $P(1, 0, 0, 0)$ and $Q(x_1, x_2, x_3, x_4)$, where

$$x_1 = (a_u' b_v - 2^3 a'^2 b^2), \qquad x_2 = -2a' b_v, \qquad x_3 = -2a_u' b, \qquad x_4 = 4a' b.$$

Hence l_1 and l_2 intersect d' in two points which are harmonic conjugates with respect to P and Q. We therefore have the theorem:

If the asymptotic curves of a non-ruled surface belong to linear complexes, and if the directrices of R_1 and R_2 are distinct, the diagonals of the net-work of skew quadrilaterals formed by the directrices of R_1 and R_2 on the directrix quadric are the lines common to C_v', C_u'', C_{v+dv}', C_{u+du}''. These lines intersect the directrices of the first kind in points which are harmonic conjugates with respect to the intersections of the latter with the asymptotic tangents. They intersect the directrices of the second kind in points which are harmonic conjugates with respect to the intersections of the latter with the osculating hyperboloid.

§ 6. Applications to the theory of ruled surfaces

If the integrating surface S is a ruled surface not a quadric, we may assume a' equal to zero, and b different from zero. Then from equations (11) and

$$\Omega' \equiv b_{uv} - \frac{b_u b_v}{b},$$

it follows that we can find by quadratures a transformation of the type (18) which replaces equations (10) by the following system:

$$y_{uu} + 2y_v + fy = 0, \qquad y_{vv} - ay = 0, \tag{37}$$

where a is a constant and f is a function of u alone, since, by equations (11), $g_u = g_v = f_v = 0$. Therefore *any non-developable ruled surface (not a quadric) whose asymptotic curves belong to linear complexes may be represented by a system of equations of the form* (37), *and conversely, every such system of equations defines a projectively equivalent family of non-developable surfaces having the property in question.*

From (37) we find that

$$\theta' = 2^8 a, \qquad \theta_4 = \frac{2}{5} f_{uu} + \frac{16}{25} f^2 - 4a. \dagger$$

* We shall refer to these as l_1 and l_2.

† θ' denotes the invariant of weight four for the surface S and θ_4 the invariant of weight four for Γ''.

If we impose the additional condition that the asymptotic curves be twisted cubics, we find precisely the same equations as we obtained from the point of view of ordinary differential equations. Corresponding to the first case under (9) we find

$$\theta' = a = f = 0 .$$

Equations (37) now become

$$y_{uu} + 2y_v = 0 , \qquad y_{vv} = 0 .$$

On integrating the second equation we find

$$y = U_1 v + U_2 ,$$

where U_1 and U_2 are functions of u alone. Substituting this expression for y in the first equation and integrating the resulting equations

$$\frac{d^2 U_1}{du^2} = 0 , \qquad \frac{d^2 U_2}{du^2} + 2U_1 = 0 ,$$

we find

$$y = (au + b)v + \left(- \frac{au^3}{3} - bu^2 + cu + d \right) ,$$

where a, b, c and d are arbitrary constants. Therefore the equation of the surface S is

$$3y_4 (y_2 y_3 - y_1 y_4) - 2y_3^3 = 0 ,$$

which is the equation of a Cayley cubic scroll. If we notice that the conditions imposed upon equations (37) in this case are precisely that *the invariants θ', C', and h shall vanish, we have a simple direct proof of the theorem* that* "If these invariants vanish identically, the surface must be a Cayley cubic scroll." By the classical method we find the line $y_3 = y_4 = 0$ is the nodal line, and the point $P(1, 0, 0, 0)$ is the pinch-point.

In his third memoir,† Wilczynski has proved an interesting and important theorem for the case when θ' and a certain invariant $\mathfrak{B}$‡ are each different from zero. Now for the surfaces of the type we are considering [i. e., $\Omega' = 0$] the osculating linear complexes are indeterminate, and the invariant $\mathfrak{B}$ vanishes whether θ' vanishes or not. The theorem just referred to undergoes extensive modifications in this case. In fact, we can find the locus of the pinch-points and singular tangent planes not only of the ∞^1 osculating Cayley cubic scrolls belonging to a single generator g of S but also of all the ∞^2 Cayley cubic scrolls associated with S.

An arbitrary point P_Y on the generator g of S will be represented by

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* M_3, p. 307.

† M_3, p. 315.

‡ $\mathfrak{B} = - 2^6 ((b\theta'_u - 2b_u\theta')^2 - 4b^2 c'^2 \theta')$.

$$Y = y + l\rho,$$

where l is an arbitrary constant.

Since the invariant Ω' vanishes, we may assume that the equations of S have been reduced to the form (37). We shall find

$$Y_{uu} + 2Y_v + fY = 0, \qquad Y_{vv} - aY = 0. \tag{38}$$

The semi-covariants of this system of equations are

$$Y = y + l\rho, \qquad Z = z + l\sigma, \qquad P = aly + \rho, \qquad \Sigma = alz + \sigma.$$

The pinch-point π of the Cayley cubic scroll, osculating S at the point P_Y, is given by the expression

$$\pi = 2^3\Sigma = 2^3(alz + \sigma).$$

Hence as P_Y moves along the generator g, the pinch-point of the osculating Cayley cubic scroll moves along the line joining the semi-covariant points P_z and P_σ. We shall now set up the equations of S in the form (1). They are found to be

$$Y_{uu} + fy + 4\rho = 0, \qquad \rho_{uu} + 4ay + f\rho = 0. \tag{39}$$

The semi-covariants of this system are

$$r = 2z, \qquad s = 2\sigma, \qquad y = y, \qquad \rho = \rho.$$

For uniformity we shall use the notation of the theory of ruled surfaces and write these equations

$$y_{uu} + fy + 4z = 0, \qquad z_{uu} + 4ay + fz = 0, \tag{39a}$$

and denote their semi-covariants by y, z, ρ, σ. If we denote the semi-covariants of (37) by y', z', ρ', σ' we shall find

$$y' = y, \qquad z' = \frac{\rho}{2}, \qquad \rho' = z, \qquad \sigma' = \frac{\sigma}{2}.$$

Therefore the equations of transformation between the coördinate systems determined by $P_{y'}\,P_{z'}\,P_{\rho'}\,P_{\sigma'}$ and $P_y\,P_z\,P_\rho\,P_\sigma$ are

$$\begin{aligned} &\omega' x_1' = x_1, \qquad \omega' x_2' = 2x_3, \qquad \omega' x_3' = x_2, \qquad \omega' x_4' = 2x_4, \\ &\omega x_1 = 2x_1', \qquad \omega x_2 = 2x_3', \qquad w x_3 = x_2', \qquad \omega x_4 = x_4', \end{aligned} \tag{40}$$

where ω and ω' are factors of proportionality. From these equations we find that the coördinates of the pinch-point π of the Cayley cubic scroll osculating S at P_y are

$$x_1 = x_2 = 0, \qquad x_3 : x_4 = a,$$

the coördinate system being that determined by $P_y P_z P_\rho P_\sigma$. Equations (37) show that the invariant θ' is a constant; so that the first derived ruled surface of S is the principal surface* of the flecnode congruence of S. We therefore have the theorem:

The locus of the pinch-points of the osculating Cayley cubic scrolls of a ruled surface whose asymptotic curves belong to linear complexes is the principal surface of the flecnode congruence of the surface itself.

An interesting special case arises when $f(u)$ reduces to a constant. It follows from the general theory of ruled surfaces that the surface S then gives rise to a surface of pinch-points S_1 which is a projective transformation of S; and S_1 gives rise to a surface of pinch-points which is the surface S itself. As P_Y describes an asymptotic curve on S, the corresponding pinch-point describes an asymptotic curve on the surface of pinch-points.

From equations (40) it follows that the nodal line of the Cayley cubic scroll of the point P_Y intersects the generator g in the point

$$Q \equiv at_1 y + t_2 z,$$

where $t_1 : t_2 = l$. The flecnodes on g are the points

$$\eta = \sqrt{ay} - z, \qquad \zeta = \sqrt{ay} + z.$$

The cross ratio of the points (P, Q, η, ζ) is -1. The singular tangent plane of the Cayley cubic scroll is tangent to the osculating hyperboloid at the pinch-point.† Its coördinates are

$$(t_2, -at_1, 0, 0).$$

If we denote that generator of S_1 (the surface of pinch-points), which corresponds to g, by g', we see that the singular tangent planes of the Cayley cubic scrolls of g form a pencil whose axis is g'.

Therefore *as the point P moves along the generator g of S, the singular tangent planes of the osculating Cayley cubic scrolls rotate about the corresponding generator of the locus of pinch-points as axis. If $f(u)$ reduces to a constant the point P, the point Q where the nodal line of the Cayley cubic scroll of P intersects the generator g, and the flecnodes η and ζ on g form a harmonic group.*

§ 7. The directrix curves on the surfaces S of the problem can be determined by quadratures

The linear complexes determined by the asymptotic curves through P_y have a congruence in common. The directrix d (the directrix of the first kind)

* W, pp. 216, 218, 220.

† M_3, p. 303.

of this congruence is situated in the tangent plane of P_y; the directrix d' (the directrix of the second kind) passes through P_y. As P_y moves over S, d and d' generate the directrix congruences of the first and second kinds respectively.* The differential equation of the directrix curves on S, which correspond to the developables of these congruences, is

$$bLdu^2 + 2Mdudv - a'Ndv^2 = 0, \tag{41}$$

where

$$L = -2a'(2a'bf + 2a'bb_v + ba'_{uu}) + ba'^2_u = \frac{\theta}{64},$$

$$M = 2a'b(a'b_{uv} - ba'_{uv}) + 2(b^2 a'_u a'_v - a'^2 b_u b_v),$$

$$N = -2b(2a'bg + 2a'ba'_u + a'b_{vv}) + a'b_v^2 = \frac{\theta'}{64}.$$

Since Γ' and Γ'' belong to linear complexes, we have

$$\Omega' = \Omega'' = 0,$$

and therefore

$$M = 2[a'^2(bb_{uv} - b_u b_v) - b^2(a'a'_{uv} - a'_u a'_v)] = 0.$$

Hence equation (41) becomes

$$b\theta du^2 - a'\theta' dv^2 = 0. \tag{41a}$$

If we introduce the transformation†

$$\bar{u} = \alpha(u) = \int \sqrt[3]{\frac{\theta_1(u)}{\varphi_1(u)}}\,du, \qquad \bar{v} = \beta(v) = \int \sqrt[3]{\frac{\theta_2(v)}{\varphi_2(v)}}\,dv, \qquad y = \frac{1}{\sqrt{\alpha_u \beta_v}}\bar{y},$$

we see, from equations (17), (18), (19) and a previous theorem, that (41*a*) becomes

$$d\bar{u}^2 - d\bar{v}^2 = 0,$$

which gives (on integration)

$$\bar{u} + \bar{v} = \text{const.}, \qquad \bar{u} - \bar{v} = \text{const.},$$

as the finite equations of the two families of directrix curves on S. *Since M vanishes the directrix curves form a conjugate system on the surface S.*

If Ω', Ω'', θ, θ' all vanish (i. e., if the normal forms of the seminvariants f and g are characterized by the vanishing of the constants a, b, c), equation (41) *is satisfied for every direction $du : dv$.* In this case we find that the point

* M_2, p. 114 et seq.

† The origin of this transformation is clear if we recall that $b/a' = \theta_1(u)/\theta_2(v)$, $\theta/a'^2 = \varphi_1(u)$, $\theta'/b^2 = \varphi_2(v)$.

$$P = \lambda\,(y + y_u\,du + y_v\,dv) + \mu\,(\tau + \tau_u\,du + \tau_v\,dv)^*$$

is on the line

$$2bx_2 - b_v\,x_4 = 0\,, \qquad 2a'\,x_3 + a'_u\,x_4 = 0$$

for every value of $du : dv$, if

$$\lambda : \mu = -\,(2bQ + b_v S) : 2b = -\,(2a'\,R + a'_u\,S') : 2a'\,.$$

It follows that under these circumstances all the directrices of the second kind in the vicinity of a point P have a point in common. Referred to T the coördinates of this point are found to be

$$(42) \quad x_1 = -\,(2bQ + b_v S)\,, \quad x_2 = -\,2a'\,bb_v\,, \quad x_3 = -\,2a'_u b^2\,, \quad x_4 = 4a'\,b^2\,.$$

Using the fact that the invariants Ω' and Ω'' vanish, we find

$$x_1 : x_2 : x_3 : x_4 = 2^3\,a'^2\,b^2 - a'_u\,b_v : 2a'\,b_v : 2a'_u\,b : -\,4a'\,b\,.$$

To prove that this point is fixed in space we consider the following expression

$$\pi = -\,(2bQ + b_v\,S)y - 2a'\,bb_v\,z - 2a'_u\,b^2\,\rho + 4a'\,b^2\,\sigma$$

for the point (42), and calculate

$$\pi + \frac{\partial \pi}{\partial u}\,du + \frac{\partial \pi}{\partial v}\,dv\,,$$

making use of the relations

$$\Omega' = \Omega'' = \theta = \theta' = 0\,.$$

We find

$$\pi + \frac{\partial \pi}{\partial u}\,du + \frac{\partial \pi}{\partial v}\,dv = \left(1 + \frac{a_u\,b + 4a'\,b_u}{2a'\,b}\,du + \frac{a'\,b_v + 4a'_v\,b}{2a'\,b}\,dv\right)\pi\,,$$

which shows that π is a fixed point.

Let us now consider the directrices of the first kind. The point

$$P = \lambda\,(r + r_u\,du + r_v\,dv) + \mu\,(s + s_u\,du + s_v\,dv)$$

will be on the line joining

$$r = -\,a'_u\,y + 2a'\,z \quad \text{and} \quad s = -\,b_v\,y + 2b\rho\,,$$

if the equations

$$(8a'^2\,b^2 + a'_u\,b_v)\,(\lambda a'\,dv + \mu bdu) = 0\,, \qquad \lambda a'\,dv + \mu bdu = 0\,,$$

are satisfied. If we choose

$$\lambda : \mu = -\,b\,du : a'\,dv\,,$$

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* $\tau = -\,a'\,b_v\,z - ba'_u\,\rho + 2a'\,b\sigma\,, \quad Q = 8a'^2\,b^2 - a'\,b_{uv} - a'_u\,b_v\,, \quad S = a'_u\,b + 2a'\,b_u\,,$
$S' = a'\,b_v + 2a'_u\,b, \qquad R = -\,(2\,a'\,bf + 2a'\,bb_v + ba'_{uu} + a'_u\,b_u)\,.$

we see that two consecutive directrices intersect for every value of the ratio $du : dv$. It therefore follows that all the directrices of the first kind in the vicinity of the point P_y lie in the same plane. We proceed to determine this plane. By using the conditions

$$\Omega' = \Omega'' = \theta = \theta' = 0$$

we can show that the four points

$$r, \; s, \; r + r_u\, du + r_v\, dv, \; s + s_u\, du + s_v\, dv$$

are coplanar. After some reductions we find the equation of their plane, which we shall call the *directrix plane*, to be

$$2^2\, a'\, b x_1 + 2a'_u\, b x_2 + 2a'\, b_v\, x_3 + (2^3\, a'^2\, b^2 + a'_u\, b_v)\, x_4 = 0\,.$$

It can be readily shown that this *directrix plane* is the null-plane of π in both of the complexes C' and C''. If we introduce the transformation*

$$\frac{\lambda\mu}{k} x_1 = -\alpha\mu X - \lambda\beta Y + \alpha\beta Z + \lambda\mu\omega\,, \qquad \frac{\lambda\mu}{k} x_3 = \lambda Y - \alpha Z\,,$$

$$\frac{\lambda\mu}{k} x_2 = \mu X - \beta Z\,, \qquad \frac{\lambda\mu}{k} x_4 = Z\,,$$

which refers the plane to the canonical tetrahedron of the point P_y, the above equation becomes

$$Z + 8\omega = 0\,,$$

and the point π referred to this tetrahedron becomes

$$X = Y = 0\,, \qquad Z = -2^3\,, \qquad \omega = 1\,.$$

These results may be combined in the following theorem:

If both families of asymptotic curves of the surface S belong to linear complexes and if the invariants θ and θ' vanish, the directrix congruences degenerate. The directrix congruence of the first kind consists of the net of lines in the directrix plane, while the directrix congruence of the second kind consists of the sheaf of lines through the null-point of the directrix plane in the complexes C' and C''.

There remains to be considered the case of a degenerate *directrix quadric.* This case is closely allied to that of the degenerate directrix congruence referred to above. There also exist a number of other curves and congruences organically connected with the configurations of this problem. The discussion of all of these questions will be reserved for a future occasion.

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* M_2, p. 104.

RELATIVELY UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS*

BY

E. W. CHITTENDEN

E. H. Moore† has introduced the notion of uniform convergence of a sequence of functions relative to a scale function. It is the purpose of this paper to study this type of convergence in the field of functions of a real variable.

1. The following definition of relatively uniform convergence is, for the case of a sequence of functions of a real variable, equivalent to the definition given by Moore.

A sequence $\{ \mu_n \}$ of single-valued, real-valued functions μ_n of a variable p, ranging over a set $\mathfrak{P}$ of real elements p, converges *relatively uniformly* on $\mathfrak{P}$ in case there exist functions θ and σ, defined on $\mathfrak{P}$, and for every m an integer n_m (dependent on m), such that for every n greater than or equal to n_m the inequality

$$m \mid \theta - \mu_n \mid \leqq \mid \sigma \mid \ddagger \tag{1}$$

holds for every element p of $\mathfrak{P}$.

The function σ of the definition above is called a *scale function.* The sequence $\{ \mu_n \}$ is said to converge *uniformly relative to the scale function* σ; or more simply, *relatively uniformly.*

2. The following propositions are immediate consequences of the definition of relatively uniform convergence.§

I. Uniformity of convergence relative to a constant scale function different from zero is equivalent to uniform convergence.

II. Uniformity relative to σ implies uniformity relative to every function τ such that $\mid \tau \mid \overline{\overline{>}} \mid \sigma \mid$.

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* Presented to the Society March 22, 1913.

† *An Introduction to a Form of General Analysis, The New Haven Mathematical Colloquium,* Yale University Press, New Haven, 1910. This memoir will be cited as I. G. A.

‡ The relation (1) holds identically in p. In such cases we follow the usage of Moore (cf. I. G. A., p. 27) and omit the variable.

§ Cf. I. G. A., p. 33, et seq. All propositions and theorems of this paper are stated for sequences of functions. The corresponding propositions for series of functions are readily inferred.

III. Uniformity as to a function σ such that $A \leqq |\sigma| \leqq B$, where A and B are positive, implies uniform convergence.

IV. If a sequence converges uniformly relative to a scale function σ, but does not converge uniformly, then σ is not bounded.

V. If a sequence of functions is defined on a class $\mathfrak{P}$ and if $\mathfrak{P}$ may be divided into a finite number of parts such that the sequence converges relatively uniformly on each part, then the sequence converges relatively uniformly on $\mathfrak{P}$.*

VI. If a sequence converges relatively uniformly on $\mathfrak{P}$, $\mathfrak{P}$ may be divided into a sequence $\{\mathfrak{P}_n\}$ such that no two sets $\mathfrak{P}_{n_1}$, $\mathfrak{P}_{n_2}$ $(n_1 \neq n_2)$ have a common element, and such that on each $\mathfrak{P}_n$ the sequence converges uniformly.

3. The following examples are illustrative of relative uniformity of convergence.

(*a*) The class $\mathfrak{P}$ is the interval $0 \leqq p \leqq 1$; the sequence $\{\mu_n\}$ such that $\mu_n(p) = 1/np$ $(p \neq 0)$, $\mu_n(0) = 0$. This sequence does not converge uniformly, but the function $\sigma(p) = 1/p$ is effective as a scale function.

(*b*) The class $\mathfrak{P}$ is the infinite segment, $1 \leqq p$; $\mu_n(p) = 1/np$. The sequence converges uniformly, but satisfies the stronger condition of uniform convergence relative to the scale function $\sigma(p) = 1/p$.†

4. Using the notation of § 1, we denote by $\phi_n(p)$ the least upper bound of $|\theta(p) - \mu_{n'}(p)|$ for all $n' \geqq n$. If the sequence $\{\mu_n\}$ converges to θ uniformly, relative to a scale function $\sigma(p)$, then for every m there exists an n_m such that the inequality (1) is satisfied for all $n \geqq n_m$. Hence we may write, in view of the definition of $\phi_n(p)$,

$$m\phi_{n_m}(p) \leqq |\sigma(p)|. \tag{2}$$

As an immediate consequence of this result we have the following theorem:

THEOREM 1. *A necessary and sufficient condition that a sequence $\{\mu_n\}$ of functions μ_n converges relatively uniformly on $\mathfrak{P}$ to a limit function θ is that there exist a sequence $\{n_m\}$ of positive integers such that the sequence $m\phi_{n_m}(p)$ has an upper bound $B(p)$ for every p.*

5. Let $\mathfrak{P}$ be any interval (a, b), and (r, q) a sub-interval of $\mathfrak{P}$ in which the only point of non-uniform convergence‡ of a sequence $\{\mu_n\}$ is the end point q. The sequence $\{\mu_n\}$ being supposed convergent on (r, q), we have a function $A(p) \geqq 1$ such that for every p in (r, q)

* Cf. § 6 of this paper, proposition VIII, which is an extension of proposition V and the converse of VI.

† Other examples are given later in this paper. Cf. also I. G. A.

‡ A point q is a point of non-uniform convergence in case the measure of non-uniformity of convergence of the sequence is greater than zero at q. Cf. W. H. Young, Proceedings of the London Mathematical Society, ser. 2, vol. I (1903–4) p. 91; also E. W. Hobson, *Theory of Functions of a Real Variable*, Cambridge University Press, p. 474, § 342; p. 484, § 349.

$$|\theta(p) - \mu_n(p)| \leqq A(p).$$

We will now show that the sequence $\{\mu_n\}$ converges uniformly on (r, q) relative to the scale function $\sigma(p) = A(p)/(q-p)$; $\sigma(q) = A(q)$. For each m greater than $1/(q-p)$ choose p_m so that $q - p_m = 1/m$. Then in the interval (r, p_m), which contains no point of non-uniformity, the sequence $\{\mu_n\}$ converges uniformly, and therefore n_m exists so that for every $n \geqq n_m$, p in (r, p_m), and $p = q$,

$$m|\theta(p) - \mu_n(p)| \leqq 1.$$

But for every n, and p in the segment $p_m \leqq p < q$,

$$m|\theta(p) - \mu_n(p)| \leqq A(p)/(q-p).$$

The combination of the last two inequalities gives the desired convergence relative to σ.

The result just obtained is stated in the following theorem:

THEOREM 2. *If q is on the left an isolated point of non-uniformity of convergence of a convergent sequence of functions, there exists a left neighborhood of q on which the sequence converges relatively uniformly.*

A similar statement holds if q is isolated on the right.

6. We have occasion to use the following two propositions from general analysis.*

VII. If a sequence of functions is defined on an enumerable set $\mathfrak{P}$ and converges on $\mathfrak{P}$ it converges relatively uniformly on $\mathfrak{P}$.

VIII. If a sequence of functions is defined and converges relatively uniformly on each of a sequence of classes $\mathfrak{P}_n$ it converges relatively uniformly on the class $\mathfrak{P}$ of all elements which are in some class $\mathfrak{P}_n$, the least common superclass of the classes $\mathfrak{P}_n$.

7. Denote by $\mathfrak{Q}$ the set of all points q of non-uniform convergene of a sequence of functions converging on an interval $\mathfrak{P}$ to a definite limit function; by $\mathfrak{Q}'$ the derived set of $\mathfrak{Q}$, and by $\mathfrak{Q}^\circ$ the aggregate of the points of $\mathfrak{Q}$ and $\mathfrak{Q}'$. $\mathfrak{Q}^\circ$ is closed. The set complementary to $\mathfrak{Q}^\circ$ consists of the interior points of an enumerable set of non-overlapping intervals, in each of which the sequence of functions converges relatively uniformly (Theorem 2). It follows from proposition VIII that the sequence converges relatively uniformly on the complement of $\mathfrak{Q}^\circ$. Since the points of $\mathfrak{Q}^\circ$ which are not in $\mathfrak{Q}'$ form an enumerable set, we may state, in view of the preceding remarks and propositions V and VII, the following theorem:

THEOREM 3. *A necessary and sufficient condition that a sequence of functions converges relatively uniformly on an interval $\mathfrak{P}$ is that it converges relatively uni-*

* Proposition VII is from I. G. A., p. 87. A simple proof may be given following the lines of the proof of Theorem 2 above. Proposition VIII is an immediate extension of VII.

formly on the derived set $\mathfrak{Q}'$ of the set $\mathfrak{Q}$ of the points of $\mathfrak{P}$ which are points of non-uniformity.

COROLLARY I. *If $\mathfrak{Q}'$ is enumerable the sequence converges relatively uniformly on $\mathfrak{P}$.*

COROLLARY II. *If a sequence of functions does not converge relatively uniformly on $\mathfrak{P}$, the corresponding set $\mathfrak{Q}'$ is not enumerable, that is, $\mathfrak{Q}$ is dense on a perfect set.*

8. The converse of corollary II above is not true, as is shown by the following example.* $\mathfrak{P}$ is the interval $(0, 1)$. We define the sequence $\{ \mu_n \}$ as follows: $\mu_n(p) = 0$, if p is irrational, zero, or equal to m/k (m and k relatively prime, and $k \neq n$); $\mu_n(p) = 1$, if $p = m/n$ (m, n relatively prime). For this sequence of functions every p is a point of non-uniformity of convergence. However, the functions of the sequence are all zero on the irrational points. Hence by propositions V and VII the sequence converges relatively uniformly.

Osgood† has given an example of a sequence of *continuous* functions converging to zero as a limit for which the set $\mathfrak{Q}$ and its derivative form a perfect set, but which converges relatively uniformly because the sequence is identically zero on $\mathfrak{Q}$ and $\mathfrak{Q}'$.

9. For sequences of continuous functions we have the following theorem:

THEOREM 4. *If a sequence of continuous functions converges on an interval $\mathfrak{P}$ to a continuous limit in such a way that the set $\mathfrak{Q}$ of all points of non-uniformity of convergence is dense on $\mathfrak{P}$, the sequence does not converge relatively uniformly.*

To establish this theorem it is sufficient to show that for every sequence $\{ n_m \}$ of positive integers n_m there exists a p determined by the sequence $\{ n_m \}$ such that the sequence $\{ m\phi_{n_m}(p) \}$‡ is not bounded. Theorem 4 follows in view of Theorem 1.

Let $\{ n_m \}$ be any sequence of integers with index m, and $\{ \mu_n \}$ denote the sequence of continuous functions of the theorem, converging to a continuous limit θ. Since the sequence $\{ \mu_n \}$ does not converge uniformly in any subinterval of $\mathfrak{P}$ there exists an integer m_1 with the following property: for every N there exists an n_1 greater than N and a p_1 (interior to $\mathfrak{P}$) such that

$$m_1 \,|\, \theta(p_1) - \mu_{n_1}(p_1) \,| > 1 .$$

Since θ and μ_{n_1} are continuous, we may find an interval I_1 interior to $\mathfrak{P}$ such that for every p in I_1

$$m_1 \,|\, \theta(p) - \mu_{n_1}(p) \,| > 1 .$$

* Cf. W. H. Young, Proceedings of the London Mathematical Society, Ser. 2, vol. I (1903–4), p. 94; also E. W. Hobson, *Theory of Functions of a Real Variable*, p. 487.

† *Non-Uniform Convergence and the Integration of Series Term by Term*, American Journal of Mathematics, vol. 19 (1897), p. 168.

‡ Cf. § 4 for explanation of this notation.

Consequently, if we take N equal to n_{m_1} of the sequence $\{ n_m \}$, we have

$$m_1 \phi_{n_{m_1}}(p) > 1$$

for every p in I_1.

By similar reasoning we may show that there exists an m_2 and an interval I_2 interior to I_1 such for every p in I_2

$$m_2 \phi_{n_{m_2}}(p) > 2 .$$

Proceeding step by step in the manner indicated we may obtain a sequence $\{ m_j \}$ and a sequence $\{ I_j \}$ of closed intervals, each interior to the preceding, such that for every j we have on I_j

$$m_j \phi_{n_{m_j}}(p) > j .$$

There must exist an element p° common to the intervals I_j. For such an element we have for every j

$$m_j \phi_{n_{m_j}}(p^\circ) > j .$$

The sequence $m \phi_{n_m}(p^\circ)$ is therefore unbounded, which was to be proved.

Osgood* has given an example of a sequence of continuous functions converging to a continuous limit for which the set $\mathfrak{Q}$ is dense on $\mathfrak{P}$. It follows from the theorem just proved that this sequence does not converge relatively uniformly.

10. A point of non-uniformity of convergence of a sequence of functions relative to a subset $\mathfrak{R}$ of $\mathfrak{P}$ is a point whose measure of non-uniformity of convergence relative to $\mathfrak{R}$ is greater than zero.†

Evidently a point of non-uniformity of convergence relative to $\mathfrak{R}$ is a point of non-uniformity of convergence relative to any set containing $\mathfrak{R}$, and therefore of $\mathfrak{P}$.

A slight extension of the proof of Theorem 4 serves to establish the following generalization of that theorem.

THEOREM 5. *If a sequence of functions defined on $\mathfrak{P}$ and continuous on $\mathfrak{R}$, a perfect subset of $\mathfrak{P}$, converges on $\mathfrak{P}$ to a limit which is continuous on $\mathfrak{R}$ in such a fashion that the class $\mathfrak{Q}$ of all points of $\mathfrak{R}$ which are points of non-uniformity of convergence relative to $\mathfrak{R}$ is dense on $\mathfrak{R}$, then the sequence of functions does not converge relatively uniformly on $\mathfrak{R}$ or on $\mathfrak{P}$.*

URBANA, ILL.,
May 7, 1913.

* Osgood, loc. cit., p. 171.

† Hobson, loc. cit., p. 484, § 349.

NOTE ON FERMAT'S LAST THEOREM*

BY

H. S. VANDIVER

1. If x, y and z are integers prime to each other, and

$$x^p + y^p + z^p = 0, \tag{1}$$

where p is a prime, and

$$q(r) = \frac{r^{p-1} - 1}{p},$$

Furtwängler † has shown that

$$q(r) \equiv 0 \pmod{p}$$

for each factor r of x, in case $x \not\equiv 0 \pmod{p}$, and for each factor r of $x^2 - y^2$, in case $x^2 - y^2$ is prime to p.

By applying this theorem, Furtwängler deduces the criterion of Wieferich $q(2) \equiv 0 \pmod{p}$ and the criterion of Mirimanoff $q(3) \equiv 0 \pmod{p}$ for the solution of (1) in integers prime to p. I shall here extend these results and show that in addition we have, provided that $q(2) \not\equiv 0 \pmod{p^3}$, the criteria $q(5) \equiv 0 \pmod{p}$ for $p \equiv 1 \pmod{3}$ and $q(5) \equiv q(7) \equiv 0 \pmod{p}$ for $p \equiv 2 \pmod{3}$.

2. Assume that x, y and z are prime to each other and to p and that $p > 5$. If one of the integers x, y, z is divisible by 5, then $q(5) \equiv 0 \pmod{p}$ by Furtwängler's theorem. If none of them is so divisible, then, modulo 5, x^p, y^p, z^p have the residues ± 2, ± 2, ± 1 or ± 1, ± 1, ∓ 2 in some order. We may therefore take $x^p \equiv y^p \pmod{5}$. Then $x \equiv y \pmod{5}$, since every integer has a unique cube root modulo 5. Thus 5 is a divisor of $x^2 - y^2$. Hence (§ 1) $q(5) \equiv 0 \pmod{p}$, unless $x^2 \equiv y^2 \pmod{p}$, i. e., unless $x \equiv y \pmod{p}$, since $x \equiv -y$ and $x + y + z \equiv 0 \pmod{p}$ would imply $z \equiv 0 \pmod{p}$, contrary to hypothesis. Using $x + y + z \equiv 0$, we may state the result:

If (1) *is satisfied by integers prime to* p, *then the congruence*

$$q(5)(t-1)(t+2)(t+1/2) \equiv 0 \pmod{p} \tag{2}$$

is satisfied by each of the following values of t:

$$\frac{x}{y}, \frac{y}{x}, \frac{x}{z}, \frac{z}{x}, \frac{y}{z}, \frac{z}{y}. \tag{3}$$

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* Presented to the Society, February 28, 1914.

† Sitzungsberichte K. Akademie der Wissenschaften, Wien, vol. 121 (1912), p. 589.

3. From (1) we have

$$\frac{x^p + y^p}{x + y} = v^p,$$

when v is an integer, since the quotient is relatively prime to $x + y$ and hence is a pth power. Since v is a factor of z, it is not divisible by p, and is of the form $1 + kp$, since the fraction is congruent to $-z^p/(-z)$ modulo p. Furthermore,

$$(1 + kp)^{p-1} \equiv 1 \pmod{p^2}$$

by Furtwängler's theorem. Multiply the members by $1 + kp$ and apply $(1 + kp)^p \equiv 1 \pmod{p^2}$. Hence $k \equiv 0 \pmod{p}$, and $v^p \equiv 1 \pmod{p^3}$. Hence

$$x^p + y^p \equiv x + y,$$

$$x^p + z^p \equiv x + z, \pmod{p^3}, \tag{4}$$

$$y^p + z^p \equiv y + z$$

$$x^p \equiv x, \qquad y^p \equiv y, \qquad z^p \equiv z \pmod{p^3}. \tag{5}$$

Hence by (1),

$$x + y + z \equiv 0 \pmod{p^3}. \tag{6}$$

4. Suppose that $y = x + p\mu$. Substituting in the first relation (4), we have

$$x^p + (x + p\mu)^p \equiv 2x + p\mu \pmod{p^3},$$

$$2x^p + p^2 \mu x^{p-1} \equiv 2x + p\mu \pmod{p^3}.$$

Hence, by (5),

$$p\mu\,(px^{p-1} - 1) \equiv 0 \pmod{p^3}, \qquad \mu \equiv 0 \pmod{p^2}.$$

We may therefore set $y = x + p^3 \mu$. Then, from (6), $z = -2x + p^3 \nu$. Hence, from (1),

$$x^p + (x + p^3 \mu)^p + (-2x + p^3 \nu)^p = 0,$$

$$2x^p - 2^p x^p \equiv 0 \pmod{p^4},$$

$$q(2) \equiv 0 \pmod{p^3}.$$

5. Now consider the criteria given by Mirimanoff* for the solution of (1). He showed that if (1) is satisfied by integers prime to p, then the ratios (3) satisfy

$$F(t) = \prod_{i=1}^{m-1} (t + \alpha^i) \sum_{i=1}^{m-1} \frac{R_i}{t + \alpha^i} \equiv 0 \pmod{p} \tag{7}$$

when $m = 2, 3, \cdots, p - 1$ and

$$R_i = \frac{\varphi_{p-1}(-\alpha^i)}{(1 - \alpha^i)^{p-1}}, \qquad \alpha = e^{2\pi\sqrt{-1}/m},$$

* Journal für Mathematik, vol. 139 (1911), p. 309 et seq.

$$\varphi_i(t) = t - 2^{i-1}t^2 + 3^{i-1}t^3 - \cdots - (p-1)^{i-1}t^{p-1}.$$

He also showed that

$$F(-1) \equiv (-1)^m m q(m) \pmod{p}. \tag{8}$$

Let $m = 7$ in (7). Assume $p > 7$. The resulting congruence is of degree 5 in t. The ratios (3) have 6 incongruent values unless one of them is a root of

$$(t-1)(t+2)(t+1/2) \equiv 0 \text{ or } t^2 + t + 1 \equiv 0 \pmod{p}.$$

If $p \equiv 2 \pmod{3}$, the latter is not possible for t rational. Hence $t \equiv 1$, -2 or $-1/2$ and therefore, by § 4, $q(2) \equiv 0 \pmod{p^3}$ unless (7) is an identity. In the latter case we may set $t \equiv -1$ and obtain $q(7) \equiv 0 \pmod{p}$ by reason of (8). Hence the criteria:

If (1) *is satisfied by integers prime to* p, *then either*

$$q(2) \equiv 0 \pmod{p^3}, \qquad q(3) \equiv 0 \pmod{p},$$

or else

$$q(2) \equiv q(3) \equiv q(5) \equiv 0 \pmod{p};$$

and if $p \equiv 2 \pmod{3}$,

$$q(7) \equiv 0 \pmod{p}.$$

6. There are no primes p at present known such that $q(2) \equiv 0 \pmod{p^3}$. Meissner* observes that $q(2) \equiv 0 \pmod{1{,}093}$, but finds $q(2) \not\equiv 0 \pmod{1{,}093^2}$. He also states that $q(2) \not\equiv 0 \pmod{p}$ for every $p < 2{,}000$ excepting 1,093.

7. If any one of the forms

$$2^\alpha 3^\beta \pm 1, \qquad 2^\alpha \pm 3^\beta,$$

where α and β are positive integers or zero, is divisible by a prime p but is not divisible by p^2, then p is excluded as an exponent in (1), if x, y and z are prime to each other and to p.† For, if p is admissible in (1), then $q(2) \equiv q(3) \equiv 0 \pmod{p}$, and

$$(2^\alpha)^{p-1} \equiv (3^\beta)^{p-1} \equiv 1, \qquad (2^\alpha 3^\beta)^{p-1} \equiv 1 \pmod{p^2}. \tag{9}$$

But if $2^\alpha 3^\beta \pm 1 \equiv 0$ or $2^\alpha \pm 3^\beta \equiv 0 \pmod{p}$ but $\not\equiv 0 \pmod{p^2}$, then

$$(2^\alpha 3^\beta)^{p-1} \not\equiv 1 \pmod{p^2}, \qquad (2^\alpha)^{p-1} \not\equiv (3^\beta)^{p-1} \pmod{p^2},$$

which contradict (9). As an example, the integer $p = 2^{61} - 1$ is known to be prime, hence it is excluded as an exponent in (1).

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* Sitzungsberichte der Preuss. Akademie der Wissenschaften, 1913, no. 35, p. 663.

† See also Mirimanoff, Paris Comptes Rendus, vol. 150 (1910), p. 206.

A SET OF AXIOMS FOR LINE GEOMETRY*

BY

E. R. HEDRICK AND LOUIS INGOLD

1. Introduction. This paper proposes a set of axioms for line geometry† based upon the *line* as an undefined element and *intersection* as an undefined relation between unordered pairs of elements. These same undefined concepts have been used in a set of axioms by Pieri.‡

The Pieri system, however, does not seem to accomplish all that could be desired in the way of simplicity, and there are several rather obvious redundancies in it which the authors of the present paper have been able to avoid.§

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* Presented to the Society in a somewhat different form, December 1, 1911. At that time the paper of Pieri mentioned below was unknown to the authors.

† The precise types of geometry studied are discussed below.

‡ *Sui principi che regono la geometria delle rette*, Atti della Accademia di Torino, vol. 36 (1901), pp. 335–351. The relation used by Pieri is introduced first as a relation between ordered pairs of elements; but he states an axiom later to make the relation independent of the order.

§ For reference, we give here the system of Pieri:

Postulº I. Il raggio e una classe non illusoria.

Postulº II. Ogni raggia incontra se stesso.

Postulº III. Essendo a un raggio, esiste almeno un raggio, che taglia a senza coincider con a.

Postulº IV. Posto che a, b siano raggi distinti, e che a tagli b, si deduce che b taglia a.

Df. Date le rette a, b che s'incontrino senza confondersi, per fascio ab, s'intende la classe di tutti quei raggi, ognuno de quali s'incontra con ogni retta, che tagli al medesimo tempo a e b. La stessa figura si rappresenta col simbolo (ab).

Postulº V. Dal supposto, che due raggi a e b s'incontrino senza confondersi, e c sia un raggio di (ab) non coincidente con a, si deduce che b appartiene ad (ac).

Postulº VI. Qualunque volta due raggi a e b si tagliano senza confondersi, nel fascio (ab) giace almeno un raggio diverso da a e da b.

Postulº VII. Dato che a, b siano raggi distinti e l'un l'altro incidenti, dovra esistere un raggio che li tagli ambedue senza giacere in (ab).

Df. Si chiama triraggio, o trilatispigolo, la figura costituita in tre rette, di cui ciascuna incontri ciascuna sepero non esiste alcun fascio di raggi che le contenga.

Postulº VIII. Posto che i raggi a, b e c siano elementi d'un trilatispigolo, se a' e un (bc) diverso da b e da c, similmente b' un (ca) diverso da c e da a, deve esisteri almeno un raggio comune ai due fasci (aa') e (bb').

Postulº IX. Se di due rette a e b, che s'incontrano senza coincidere, ognuna e tagliata da due rette sghembe c e d; allora ogni raggio che incontri al medisimo tempo a e b, dovra tagliare uno almeno dei raggi c, d.

In this paper a set of five axioms is introduced which are essentially equivalent to Pieri's system, and which seem to be quite as simple in statement and in application as those of Pieri. It is shown also that the resulting space is a *general projective space*, that is, the space is either *properly* or *improperly* projective according to the classification introduced by Veblen and Young.* This follows from the fact that the five axioms of this paper are equivalent to the two groups A and E of assumptions given by them.

The axioms are stated in terms of the undefined concepts *line* and *intersection*, and two defined secondary concepts, viz., *a pencil of lines*, and what we have called *a doublet of lines*. These secondary concepts are defined in terms of line and intersection in what follows. After the axioms, a third secondary concept, *a field of lines*, is defined.

For the purpose of easy orientation, the reader may think of a field of lines either as a set of lines lying in a given projective plane, or as a set of lines passing through a given point of space. He may regard a pencil of lines as a set of lines of a plane through a given point of that plane. A doublet may be thought of as a set consisting of all the lines of a plane together with all the lines of space which pass through a given point of that plane. It should be noticed, however, that the concepts *plane* and *point* are not used in setting up the system; rather we have been able to define these concepts in terms of the fundamental notions *line* and *intersection*. The concepts *field*, *pencil*, and *doublet* seem particularly adapted to a treatment of line geometry and their early introduction makes possible a great simplification of the axioms.

The plan of the paper lends itself to extension to higher dimensions, although the generalization is not immediate. On some other occasion this extension will be treated.

2. The Axioms and their Independence. In what follows, the undefined elements (lines) will be denoted by small Roman letters. We shall presuppose nothing regarding the character of the undefined relation (intersection) except

Postul^{o} X. E ogni raggio, che incontri tutte e quattro le rette a, b, c, d giace nel fascio (ab).

Postul^{o} XI. Dati ancora due raggi a e b che s'incontrino senza coincidere, e date una retta g che non tagli a ne b, sara necessario che questa g incontri un qualche raggio di (ab).

Of these, Nos. I, III, V, VI, VII, VIII, IX, X, XI, are consequences of our axioms. No. II is unnecessary in the present system, since the relation of intersection is not defined to apply for a pair of identical lines. No. IV is not necessary in the system unless it is presumed that the relation of intersection depends on the order of the two elements as well as on the elements themselves.

It may be added that Pieri's V and VIII can be proved from his II, IV, VII, IX, X, and XI, and that the major content of his III and his VI can be proved from his XI.

* *A set of assumptions for projective geometry*, American Journal of Mathematics, vol. 30 (1908), pp. 347–380; in particular, p. 352. These axioms are stated below in § 6. See also Veblen and Young, *Projective Geometry*, Ginn & Co., 1910.

that it holds or fails for any two distinct elements.* If the relation holds for two lines a and b we shall say that they *intersect*, or that *a meets b*, or that b meets a. If the relation fails, we shall say that a and b are *skew*, or that they are *a pair of skew lines*.

DEFINITION 1. If a and b are a pair of intersecting lines, the set of lines consisting of a and b together with all the lines that meet both a and b is called a *doublet*, and is denoted by the symbol $[ab]$.

DEFINITION 2. The set of lines of a doublet $[ab]$ each of which meets every other line of the doublet $[ab]$ is called a *pencil*, and is denoted by the symbol (ab).

In any doublet $[a, b]$ there is evidently just one pencil (a, b).

AXIOM I. *There exists at least one pair of intersecting lines.*

AXIOM II. *In every doublet $[ab]$ there is at least one line not belonging to the pencil (ab).*

AXIOM III. *Every line l of every doublet $[ab]$ which is not in the pencil (ab) meets one and only one of any two skew lines of $[ab]$ both different from l.*

AXIOM IV. *Both of any two intersecting lines are skew to some third line.*†

AXIOM V. *Every line meets at least one line of every pencil.*‡

We give the following examples to show the technical independence of these five axioms, in their order.

1. The elements are two lines; they do not meet. All the axioms except I are satisfied (vacuously).

2. The elements are the lines of one plane together with the five perpendiculars to that plane at the vertices of a certain regular pentagon of the plane. Intersection shall be interpreted as usual (projectively), except that the perpendiculars to the plane shall all be skew to each other. Axiom II is the only axiom violated.

3. The elements are the lines of an ordinary projective space and one extra line l'. Intersection shall be interpreted as usual for all lines except l', while l' shall meet all lines that meet a given line l of space and be skew to all other lines of space, including l itself. Axiom III alone is violated.

4. The elements are the edges of a tetrahedron. Intersection is to be understood in the ordinary sense. All the axioms except IV are satisfied.

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* It should be noticed that in this paper the undefined relation holds or fails for the set of objects (a, b), not for the sequence (a, b).

† It is necessary to assume this only for every pair of lines of which at least one belongs to a certain fixed pencil. It seems actually to defeat the desire for simplicity to remove this redundancy, but it may be done by stating IV as follows:

IV′. If there exists a pair of intersecting lines, then there are two intersecting lines a and b such that, if either of any two intersecting lines m and n belongs to (ab), a line k skew to both m and n exists.

‡ It is evident that this last axiom contains a redundancy, in that we already know that any line which meets either a or b meets at least one line of the pencil (ab).

5. The elements are the lines of an ordinary projective space and one extra line l'. Intersection shall have the usual meaning, except that l' shall be skew to every other line. All the axioms except V are satisfied.

Axiom I may be broken into two parts, as follows:

I_1. *There exists at least one line.* (Pieri's I.)

I_2. *If there exists one line, there exists a pair of intersecting lines.**

These statements are independent of each other, and the system I_1, I_2, II, III, IV, V form a mutually independent system. For, the examples (2), (3), (4), (5), above, need not be altered; and (1) serves as an independence example for I_2. If there are no lines whatever, all but I_1 are satisfied.

Again, Axiom I_2 may be divided into two parts, as follows:

I_2'. *If there exists one line, there exists at least one other line.*

I_2''. *If there exist two lines, there exist two intersecting lines.*†

Axiom III can be readily broken into two parts by replacing in it the words *one and only one* by (III_1) the word *one*, or (III_2) the words *not more than one.*

3. Preliminary Definitions and Theorems. Before proceeding to any theorems, we shall define another secondary concept, a *field of lines.*

Definition 3. A set of lines of a doublet is called a *field* if (*a*) the set contains no pair of skew lines; (*b*) every line of the doublet not in the set is skew to at least one line of the set.

Fields will sometimes be denoted by small Greek letters. As an illustration of the usefulness of this concept, we may state, by Axiom III, that the lines of a given doublet $[ab]$ not in a given field α of $[ab]$ are all skew to any line of α not in the pencil (ab).

Theorem I. *In any doublet there are two and only two fields.*

By Axiom II, there exists at least one pair of skew lines p and q in any given doublet. All the lines of the doublet that meet p meet each other, by Axiom III. Now p is skew to all other lines of the doublet. Hence p together with the lines of the doublet that meet p constitute a field.

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* It should be noticed that I_2 is by no means equivalent to Pieri's III. But our V (or Pieri's XI) and I_2 suffice to prove his III.

† The possibility of dividing most axioms into component parts seems not to have been sufficiently emphasized. For example, the statement of A_1 of the Veblen-Young list (§6, below) is rather obviously composed of the two parts:

A_1'. If A and B are distinct points, there is at least one line.

A_1''. If A and B are distinct points and if there is at least one line, there is a line containing both A and B.

It is evident that A_1' is also a portion of E_1. A_1'' can be split into two parts in a similar manner, and so on *ad infinitum.* In fact it seems to be true that the existence of an infinite number of elements in any system at once renders it possible to arrange at least a countably infinite set of independent axioms whose content does not overlap, and which are all provable from any axioms that define that set. Hence it seems difficult to define conveniently the difference between single and multiple statements.

Another field consists of q and all the lines of the given doublet that meet q. There is no other field, since there is no line skew to both p and q, by Axiom III.

THEOREM 2. *If a line p of a doublet $[ab]$ meets two skew lines c and k of that doublet, it belongs to the pencil (ab).*

For, since every pair of the set a, b, p, c intersect, they belong to a field; likewise, a, b, p, k belong to a field. These fields are different, since c and k cannot belong to the same field. Hence p lies in both fields and therefore belongs to the pencil (ab).

COROLLARY. *The pencil (ab) contains the lines a and b.*

THEOREM 3. *If p and q are distinct lines of the pencil (ab), the doublet $[pq]$ is identical with the doublet $[ab]$.*

For, p and q belong to both fields of the doublet $[ab]$; hence a line meeting a and b meets p and q; and therefore the doublet $[ab]$ is contained in the doublet $[pq]$.

Again, a and b belong to $[pq]$ and each meets at least one pair of skew lines that belongs to $[ab]$ and therefore also to $[pq]$; hence, by Theorem 2, a and b belong to the pencil (pq).

Interchanging a and b with p and q in the argument above, it is seen that any line of $[pq]$ is also a line of $[ab]$.

COROLLARY 2. *If a line meets two lines of a pencil, it meets every line of the pencil.*

COROLLARY 3. *If c is a line of the pencil (ab) and $c \neq b$, then a is a line of the pencil (bc).*

THEOREM 4. *If c is a line of the doublet $[ab]$ which does not belong to the pencil (ab), that field of the doublet $[ab]$ which contains c consists of a, b, c and all the lines that meet a, b, and c.*

Let p and q be any two lines that meet a, b, and c; suppose that p and q are skew. Then, by Theorem 2, since c meets p and q, c belongs to the pencil (ab), contrary to hypothesis. Hence p and q intersect and therefore belong to the field of the doublet $[ab]$ which contains c.

DEFINITION 4. If c is a line of the doublet $[ab]$ which does not belong to the pencil (ab), the three lines a, b, c form a *triad*.

THEOREM 5. *There is one and only one field containing the lines of a given triad.*

This follows immediately from Theorem 4 and Definition 2.

It will be convenient to denote the field determined by the lines a, b, c of a triad by (abc).

THEOREM 6. *If two distinct pencils belong to the same field, they have one and only one line in common.*

Let the distinct pencils (ab) and (cd) belong to a field α; then there is a line l meeting a and b but not belonging to α. The lines c and d cannot both

belong to the pencil (ab); for, by Theorem 3, (ab) would then coincide with (cd). Hence l does not meet all lines of (cd). By Axiom V, l meets at least one line m of (cd). Hence by Corollary 2, l meets one and only one line m of (cd). The line m belongs to (ab), since l meets only those lines of α which are lines of (ab). No other line can be common to the two pencils, since l meets every line of (ab) and only the line m of (cd).

THEOREM 7. *The lines of a field that meet a line not belonging to the field form a pencil.*

Let p be a line not belonging to the field α of the doublet $[ab]$. If p meets a and b, then p meets every line of the pencil (ab), and the theorem is proved.

If p does not meet both a and b, there is at least one line c of the pencil (ab) which p meets, by Axiom V; also there is a line l of the field α which does not belong to the pencil (ab). Then α is identical with the field (abl) by Theorem 5. Suppose p does not meet a. The line p meets one line d of the pencil (al). Hence p meets two lines c and d of the field α, so that p meets all lines of the pencil (cd).

There is no other line of the field α which meets p. For if p meets all lines of the pencil (cd) and one other line of the field α, then p belongs to the field α.

4. Classification of Fields. We shall now show that fields fall into two main classes, for which a principle of duality holds. Later we shall identify these two classes of fields with points and planes of ordinary geometry.*

DEFINITION 5. Two distinct fields are said to be *incident* to one another if they have in common all the lines of a pencil.

THEOREM 8. *If two distinct fields are incident to the same field they have in common one and only one line.*

If α and β are both incident to ϕ, there is a pencil P common to α and ϕ and a pencil Q common to β and ϕ.

The pencils P and Q have one and only one line l in common. Let p and q be two other lines belonging to P and Q, respectively. Then l, p, and q are three lines of ϕ forming a triad; also p belongs to α and q belongs to β. Let k be any line common to α and β. Then k meets l, p, and q. Hence k belongs to ϕ, and therefore to each of the pencils P and Q. It follows that k is identical with l.

DEFINITION 6. Two fields which have one and only one line in common are said to be *similar* to each other; otherwise they are not similar, except that a field is said to be similar to itself.

THEOREM 9. *If a field α is not similar to a field ψ and if ψ is incident to a field ϕ, then α is similar to ϕ.*

* Such a classification, whose purpose is to realize the well-known duality of point and plane in line geometry, will obviously be a part of any treatment of the subject. Thus Pieri, *loc. cit.*, gives a treatment similar to that of this paper, but his proofs are of course essentially different.

1. If α is incident to ψ, there is one and only one line common to α and ϕ, by Theorem 8; hence α is similar to ϕ.

2. If α is not incident to ψ, let l be any line of α which does not belong to ϕ (if there is no such line, α and ϕ are identical and therefore similar). Denote by P the pencil common to ϕ and ψ. Then l meets one and only one line m of P; let r be another line of P. The lines of α which meet m form a pencil A; r meets at least one line s of A; s meets m and r; hence s belongs to the field ϕ. For, the line s cannot belong to ψ, since it belongs to α. The lines l, m, and s form a triad; α is incident to the field (lms). The field ϕ also is incident to (lms). Hence α and ϕ have only the line s in common, so that α is similar to ϕ, as was to be proved.

THEOREM 10. *If each of two fields α and β is similar to a field ϕ, α is similar to β.*

Let a be common to α and ϕ, and b common to β and ϕ. If a coincides with b, there is no other line common to α and β. For, suppose k is such a line, so that α and β are incident. Then, since k does not belong to ϕ, k meets ϕ in a pencil A. Let a' be a line of this pencil different from a. Then a, a', and k form a triad. The field $(aa'k)$ contains the pencil (ak) which is common to α and β. Hence $(aa'k)$ is identical with either α or β, and therefore α or β has the pencil $(aa') = A$ in common with ϕ, contrary to hypothesis.

If a is different from b, then at least a meets b, since they belong to the same field ϕ. Let ψ be the field incident to ϕ containing a and b. Then a meets β in a pencil B. Let b' denote a line of B different from b. The line b' meets a and b and does not belong to ϕ, since b' belongs to β. Hence b' belongs to ψ and therefore β has the pencil (bb') in common with ψ. Similarly, α has a pencil in common with ψ. Hence, by Theorem 8, α is similar to β.

THEOREM 11. *If each of two fields α and β is not similar to a field ϕ, α is similar to β.*

Let ψ be a field incident to ϕ. Then, by Theorem 9, α and β are both similar to ψ. Hence by Theorem 10, α is similar to β.

THEOREM 12. *If a field α is similar to a field ϕ and if another field β is not similar to ϕ, then α is not similar to β.*

For, if α were similar to β, then, by Theorem 10, we would have β similar to ϕ, contrary to hypothesis.

By Theorems 10, 11, 12, we can separate all fields into two classes, those in one class being all that are similar to a given field ϕ, and those of the other class being the fields which are not similar to ϕ. Any two belonging to the same class are similar to each other, any two belonging to different classes are not similar to each other.

5. Identification of Fields with Points and Planes. We shall call the fields of one of the classes just mentioned *planes*, and those of the other class *points*. Since these two classes of fields possess a complete duality, a distinction as to which fields are to be called points and which are to be called planes is impossible.

If a point and a plane are incident to one another, we say that the point *lies in* the plane or that the plane *passes through* the point, or *contains* the point.

If a line belongs to a plane, we say that the line *lies in* the plane or that the plane *passes through* the line.

We say that two intersecting lines a and b *meet in a point* and also that they *determine a plane*.

Other expressions commonly used in geometry, such as "the line joining two points," "the line of intersection of two planes," etc., may also be defined in an obvious way.

It is now possible to prove many theorems of projective geometry relating to the intersections of lines and planes and the collinearity of points; for example, Desargues's theorem regarding two perspective triangles could be proved; and, again, the quadrangle construction for the harmonic conjugate of a given point with respect to two others on the same line is clearly possible.

6. Comparison with Other Systems of Axioms. In conclusion we compare the axioms of this paper with a complete system of axioms for projective geometry, so that it may be seen what further axioms are needed to complete the usual theory. The system chosen for this purpose is that of Veblen and Young cited above. The following groups A and E of their assumptions will be proved to be equivalent to our Axioms I–V.

A_1. If A and B are distinct points, there is at least one line containing both A and B.

A_2. If A and B are distinct points, there is not more than one line containing both A and B.

A_3. If A, B, C are points not belonging to the same line, and if a line l contains a point D of a line joining B and C and a point E, distinct from D, of a line joining C and A, then the line l contains a point F of a line joining A and B.

E_0. There are at least three points on every line.

E_1. There exists at least one line.

E_2. It is not true that every point lies on every line.

E_3. It is not true that every point lies on every plane.*

* Under at least one interpretation of these statements, E_1 is a logical consequence of E_2, and both E_1 and E_2 are consequences of E_3. The somewhat altered statements in Veblen and Young, *Projective Geometry*, p. 24, appear to be independent, however.

E_3'. If S is a three-space, every point lies in S.

In this system the plane and the three-space are defined in terms of the undefined elements point and line as follows:

DEFINITION. If P, Q, R are three points not on the same line, and l is a line joining Q and R, the class S_2 of all points such that every point of S_2 is collinear with P and some point of l is called the *plane* determined by P and l. If P, Q, R, T are four points not in the same line or plane, and if α is a plane containing Q, R, and T, the class S_3 of all points such that every point of S_3 is collinear with P and some point of α is called the *three-space* determined by P and α.

Assumptions A_1 and A_2 follow from our axioms by the definition of similar fields.

To prove A_3, let a be common to the (point) fields C and B, and let b and c be the lines common to the (point) fields C and A, and A and B, respectively. Then a, b, and c form a triad (triangle). Let ϕ be the (plane) field determined by a, b, and c. If a line l meets a and b, and if the (point) field D determined by a and l and the (point) field E determined by b and l are distinct, then a, b, and l do not all three belong to a (point) field. Hence l belongs to the (plane) field (abc), so that l meets c, and the (point) field determined by l and c is a point of a line c joining A and B.

E_0 follows from I, II, IV, and V in an obvious manner.

E_1 follows from I; or indeed from E_2, as remarked above.

E_2 follows from E_3; or it may be proved directly as follows. Consider a line l on which there is a point P, and let a and b be two lines passing through P. We shall denote the field which defines the point P by α. The lines a and b determine also a (plane) field β. Let c be a line of this plane not belonging to the pencil (ab). Then c meets both a and b; the three points determined by a and b, b and c, c and a, respectively, do not all three lie on the same line. Hence E_2 is proved.

To prove E_3, consider the plane α of any plane triad a, b, c. There is a line p, not lying in α, which passes through the point P determined by a and b. The lines p and a determine a plane which contains a line q not passing through the point P. The lines p and q determine the point R. Now R is distinct from P, for otherwise q would pass through P; hence R does not lie in α, for then p would also lie in α.

To prove E_3, let S_3 be a three-space determined by a point P and a plane α containing the intersecting lines a and b; and let Q denote a point. Let l be a line common to P and Q. By V, l meets at least one line c of the pencil (ab). Hence Q is collinear with P and the point determined by l and c of the plane α. Therefore Q is a point of S_3 by definition.

It can also be shown that I, II, III, IV, and V follow from the assumptions A and E.

7. Conclusion. In order to complete the entire theory, it is only necessary to follow precisely the steps taken by Veblen and Young, for example, beyond their assumptions A and E, in the paper referred to above, and those taken by Veblen in a later paper.* The remaining assumptions of the Veblen-Young system are evidently independent of our set. Hence there seems to be no essential reason for pursuing the details of this discussion further.

COLUMBIA, MO.,
January, 1914.

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* *On the definition of the multiplication of irrational numbers*, American Journal of Mathematics, vol. 34 (1912), pp. 211–214.

THE CAUCHY PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATIONS.*

BY

GRIFFITH C. EVANS

Introduction

1. A large part of the theory of integral and integro-differential equations may be reduced to the corresponding theory of algebraic and differential equations by the introduction of convenient symbolism. For direct operations this analogy is well expressed by merely considering as a product the combination

$$AB = \int A(r, \tau)\, B(\tau, s)\, d\tau, \tag{1}$$

taking the integral with constant limits a, b, or variable limits s, r, as the case may be.† In this symbolism, however, the vanishing of the "product" of two functions does not imply necessarily that one of the functions vanishes. And therefore for the treatment of the inverse operations and the construction of a complete *algebra*, it is more convenient to consider, instead of the combination written above, the following combination of certain complex quantities. Let

$$\xi = u + jU(r, s), \qquad \eta = v + jV(r, s).$$

By $\xi\eta$ we understand the quantity

$$\xi\eta = uv + j\{uV(r, s) + vU(r, s) + \int U(r, \tau)\, V(\tau, s)\, d\tau\}, \tag{2}$$

taking as the limits of the integral a, b or s, r, according as the algebra is of one kind or the other.‡ The complex unit j is merely used to separate the

* Presented to the Society September, 1913.

† V. Volterra, *Questioni generali sulle equazioni integrali ed integro-differenziali*, Rendiconti della R. Accademia dei Lincei, vol. 19 (1910), p. 169 and p. 178.

‡ In order not to introduce special parameter values into the formal consideration of the problem, it is convenient to make the assumption that for constant limits of integration the j-coefficients contain a parameter λ in such a way that $|U(r, s)| < \lambda M$, and to consider small values of λ. That convention will not generally be necessary here, since we are to introduce explicitly parameters x, y. Where the introduction of such a parameter is necessary it will be specified.

quantity ξ into two parts, which may be called respectively ordinary coefficient, denoted by Ord ξ, and j-coefficient, written j-coefficient ξ. Each coefficient may be complex in the usual sense. By $\xi = 0$ is understood, of course, the pair of equations $u = 0$, $U = 0$, and by $\xi = \eta$, the pair $u = v$, $U = V$.

If the functions U, V are permutable, *i. e.*, if

$$\int U(r, \tau) V(\tau, s)\, d\tau = \int V(r, \tau) U(\tau, s)\, d\tau,$$

then the operation defined by (2) is commutative, and ξ and η are said to be permutable.

If $u = 0$, ξ is said to be a function of nullity. We see, by referring to the condition for equality, that division by a function not of nullity is equivalent to solving a linear integral equation of the second kind of Fredholm or Volterra type, and that division by a function of nullity is often impossible. With this distinction, if our quantities are permutable among themselves we may apply *all* the laws of algebra, and if the quantities are not permutable, all except the commutative law.* The function of nullity, which includes zero as a special case, takes much the same place in this algebra as the quantity zero in the ordinary algebra. In fact the two algebras are merihedrically isomorphic.† If we do not divide by zero in the ordinary algebra, we do not need to divide by a function of nullity in the symbolic correspondence.

2. If now we introduce into the quantities u and U which define ξ extra parameters or variables $(x_1, x_2, \cdots)$ we have obviously the formula

$$\frac{\partial}{\partial x_i}(\xi\eta) = \xi\frac{\partial \eta}{\partial x_i} + \frac{\partial \xi}{\partial x_i}\eta \tag{3}$$

irrespective of the commutativity or non-commutativity of the product $\xi\eta$. To every differential equation corresponds then an integro-differential equation by means of this symbolism, and a large part of the theory of differential equations may be carried over word for word to these other equations, even when the multiplication is not commutative.

It may be noticed, however, that this symbolism, in particular equation (3), is valid for those integro-differential equations only in which the variables of differentiation $(x_1, \cdots)$ are different from those of integration r, s. These equations were called by their inventor "*of static type*," on account of

* Evans, *L'algebra delle funzioni permutabili e non permutabili*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 1–28, and ibid. vol. 35, Errata Corrigi.

† Evans: *Sopra l'algebra delle funzioni permutabili*, Atti delle R. Accademia dei Lincei, ser. 5*a*, vol. 8 (1911), p. 710.

their application to the theory of slow motion.* It is the object of this paper to extend Cauchy's theorem for differential equations to integro-differential equations of this type.†

Theorem V, in the paper of Volterra first cited, deals with equations of this kind, and demonstrates the existence of what correspond, in the theory of differential equations, to the general and complete integrals of the equation. Boundary problems for this class of equations have been discussed by Volterra, Lauricella, recently by J. Pérès and others. Integro-differential equations and functional equations of non-static type have also been discussed by Volterra and others, and some corresponding existence theorems have been obtained.

Systems of symbolic differential equations with permutable coefficients

3. Let us limit ourselves to two variables of differentiation x, y besides the variables of integration r, s, and let us consider the system of symbolic equations

$$
(4) \qquad \begin{aligned}
\frac{\partial \xi_1}{\partial y} &= F_1\left(\frac{\partial \xi_1}{\partial x}, \cdots, \frac{\partial \xi_n}{\partial x}; \xi_1, \cdots, \xi_n; y, x \mid r, s\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial \xi_n}{\partial y} &= F_n\left(\frac{\partial \xi_1}{\partial x}, \cdots, \frac{\partial \xi_n}{\partial x}; \xi_1, \cdots, \xi_n; y, x \mid r, s\right).
\end{aligned}
$$

The variables r, s are to be restricted to a single region T of the complex plane, such that if r and s are any two points of T the whole of the straight line joining them lies in T. If for $x, y, \xi_1, \cdots, \xi_n, \partial\xi_1/\partial x, \cdots, \partial x_n/\partial x$ we substitute variables $x_1, \cdots, x_{2n+2}$, the functions $F_1, \cdots, F_n$ are to be analytic in these variables in a $(2n+2)$-dimensional neighborhood of the origin, which may be chosen in such a way as to be independent of r, s. The coefficients in the developments of these F_i are continuous functions of r, s in T, permutable among themselves.

The developments in (4) will be convergent and have meaning provided that the ordinary coefficients of the ξ_i, with their first derivatives in x, lie in the given $(2n+2)$-dimensional neighborhood.‡

* V. Volterra: *Sulle equazioni della elettrodinamica*, Rendiconti della R. Accademia dei Lincei, vol. 23 (1909), p. 209.

† In the interim of the writing of this paper and its publication, an article has appeared by M. Paul Lévy, in which the notion of characteristic is applied to another kind of functional equation. See: *Sur l'intégration des équations aux dérivées fonctionelles partielles*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 113–169.

‡ See: *A theorem of convergence*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), p. 10. To insure this convergence when the limits of integration are constant, parameters $\lambda_1, \cdots, \lambda_n$ must be introduced into the j-coefficients of $\xi_1, \cdots, \xi_n$. In Theorem 1, however, on account of the initial conditions these parameters are provided for by the variable y.

THEOREM 1. *There is one and only one system of solutions* $\xi_1, \cdots, \xi_n$ *of the system of equations* (4) *analytic in* x *and* y *in the neighborhood of the origin, and such that for* $y = 0$ *the conditions* $\xi_1 = 0, \cdots, \xi_n = 0$ *are satisfied. These solutions are themselves permutable with the coefficients of* $F_1, \cdots, F_n$ *and with each other.*

From the fact that the region of analyticity of the F_i is independent of r, s we are able to get inequalities for the values of the coefficients in the development of the F_i. We have then a complete identity of operations with the corresponding theory for differential equations; whence is established, step by step, the permutability of every coefficient in the development of the ξ_i with those of the F_i, and also the fact that they are permutable with each other, because they are, symbolically, integral rational expressions in the coefficients of the F_i. Likewise is established the uniqueness of the formal solutions so developed. The convergence of the developments follow immediately from the convergence of the corresponding power series for the solution of the analogous differential equations if we replace the length of the rectangle of convergence ρ, in the y direction, in the theory of differential equations, by the distanec ρ / c, where c is some quantity greater than unity, and greater also than the interval of integration, ab, or rs, as the case may be.

In fact, if

$$| \xi | = | u + jU | = | u | + | U | < A,$$

$$| \eta | = | v + jV | = | v | + | V | < B,$$

it follows that

$$| \xi\eta | < ABc,$$

and these are the inequalities that it is necessary to use in the demonstration of the convergence of the j-coefficients and ordinary coefficients in the developments of the ξ_i.

4. Let us say that a function $u(x \mid r, s)$

$$u(x \mid r, s) = \sum_0^\infty {}_i A_i(r, s)(x - x_0)^i$$

s *permutably analytic,* or permutably analytic in x, if it is analytic in x and all its coefficients are mutually permutable functions of r, s. We notice that if such a function is permutably analytic about the point x_0, it is permutably analytic when developed about any other point x_1 in the region of analytic extension; for the new coefficients are linear expressions in terms of the old, the coefficients in these expressions being monomials in $x_1 - x_0$.

A necessary and sufficient condition that $u(x \mid r, s)$ be permutably analytic in a given region is that it be analytic in that region and satisfy the condition

$$\int u(x_1 \mid r, \tau)\, u(x_2 \mid \tau, s)\, d\tau = \int u(x_2 \mid r, \tau)\, u(x_1 \mid \tau, s)\, d\tau,$$

where the limits of integration are constant, or s, r, as we deal with one kind of permutability or the other, and where x_1 and x_2 are any two values within the region of analyticity.*

The solutions $\xi_1, \cdots, \xi_n$ *in Theorem* 1 *are permutably analytic.*

We say that two functions are *analytically permutable*, or analytically permutable with each other, when every coefficient in the development of one is permutable with each of the coefficients of the other. As before, this definition is independent of the point x_0, or (x_0, y_0), about which the functions are developed.

Two functions are permutable if they are analytically permutable.

A necessary and sufficient condition that $u(x \mid r, s)$ and $v(x \mid r, s)$ be analytically permutable in x is that they be analytic in x in two regions σ_1 and σ_2 respectively, and that

$$\int u(x_1 \mid r, \tau)\, v(x_2 \mid \tau, s)\, dt = \int v(x_2 \mid r, \tau)\, u(x_1 \mid \tau, s)\, d\tau,$$

where x_1 is any value in σ_1 and x_2 any value in σ_2.

5. We may generalize Theorem 1 to the case where $\xi_1, \cdots, \xi_n$ are, for $y = 0$, arbitrary functions of x, r, s provided that these functions are permutably analytic in x, and analytically permutable with each other and with the coefficients in the developments of the F_i.† The F_i are assumed to be developable throughout a $(2n + 2)$-dimensional neighborhood that includes the ordinary coefficients of the given values of the ξ_i. *There is a unique system of solutions, and these solutions are permutably analytic, and analytically permutable with each other and with the coefficients of the* F_i.

Systems of symbolic differential equations whose coefficients are not necessarily permutable

6. As has been remarked by M. Pérès in regard to certain integral equations,‡ the method of procedure may be extended to take care of the case where the coefficients in the analytic developments are not necessarily permutable among themselves. We have, as before, a unique formal determination of the coefficients, and the convergence of the resulting series is established§ by

* Compare the concept of "continuous group of permutable functions," V. Volterra: *Sopra equazioni di tipo integrale*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1913, vol. 1, p. 405.

† When the limits of integration are constants, the given values of the j-coefficients of $\xi_1, \cdots, \xi_n$ are supposed to contain parameters $\lambda_1, \cdots, \lambda_n$ respectively, unless the F_i are entire functions of the $2n$ arguments consisting of the ξ_i and their derivatives.

‡ J. Pérès: *Sulle equazioni integrali*, Rendiconti della R. Accademia dei Lincei, ser. 5*a*, vol. 22 (1913), pp. 66–70.

§ The system (4) may be replaced by one linear in the derivatives in regard to x by the introduction of n more equations and n more unknowns. The dominant series is most simply stated in terms of the linear system.

means of exactly the same dominant series that is used in Theorem 1. In fact, the less-or-equal relation between the terms of series and dominant series (all of whose coefficients are positive) holds even to the partial terms whose sum is the coefficient of $x^m y^n$.

Let F be analytic in its $2n+2$ arguments in a $(2n+2)$-dimensional neighborhood which includes the points determined by the ordinary coefficients of the functions $\xi_1(x \mid r, s), \cdots, \xi_n^0(x \mid r, s)$, functions which are analytic in x, in the neighborhood of $x = 0$, with coefficients that in T are continuous functions of r, s. We have then the following theorem:

THEOREM 2. *There is one and only one system of solutions $\xi_1, \cdots, \xi_n$ of (4) under the given conditions,* which are analytic at the origin in x and y, and which, for $y = 0$, take the given values $\xi_1^0, \cdots, \xi_n^0$.*

By means of the process of analytic extension we may extend our theorem to the case where the given functions are analytic along a certain given segment of the x-axis.

7. Two of Volterra's integro-differential equations, viz.,

$$\nabla^2 u(x, y \mid r, s) + \int_y^x \left\{ A_1(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial x^2} + A_2(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial y^2} \right\} d\tau = 0 \tag{5}$$

and

$$\nabla^2 u(x, y \mid r, s) + \lambda \int_a^b \left\{ A_1(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial x^2} + A_2(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial y^2} \right\} d\tau = 0 \tag{6}$$

have the common symbolic form

$$\alpha_1 \frac{\partial^2 \xi}{\partial x^2} + \alpha_2 \frac{\partial^2 \xi}{\partial y^2} = 0, \tag{7}$$

in which are to be considered only the solutions of nullity.

For small values of λ, the equation (7) may be written in the form

$$\frac{\partial^2 \xi}{\partial y^2} = \alpha \frac{\partial^2 \xi}{\partial x^2}, \tag{8}$$

in which

$$\alpha = -\frac{1}{\alpha_2} \alpha_1.$$

In close relation to (8) stands the system of equations

* As in § 5, parameters $\lambda_1, \cdots, \lambda_n$ must be introduced when the limits of integration are constant.

$$(9) \qquad \frac{\partial \xi_2}{\partial y} = \alpha \frac{\partial \xi_1}{\partial x}, \qquad \frac{\partial \xi}{\partial y} = \xi_2, \qquad \frac{\partial \xi_1}{\partial y} = \frac{\partial \xi_2}{\partial x}.$$

Under the boundary conditions

$$(9') \quad (\xi)_{y=0} = \xi^0(x, r, s), \quad (\xi_2)_{y=0} = \xi_2^0(x, r, s), \quad (\xi_1)_{y=0} = \frac{\partial}{\partial x}\xi^0(x, r, s),$$

the system (9) has one and only one analytic solution.*

But the system (9) under the conditions (9′) is equivalent to the system (8) under the conditions

$$(8') \qquad (\xi)_{y=0} = \xi^0(x, r, s), \qquad (\partial\xi/\partial y)_{y=0} = \xi_2^0(x, r, s),$$

and (8), then, under these conditions, has one and only one analytic solution. If ξ^0 and ξ_2^0 are functions of nullity, the solution is a function of nullity, and (8) is equivalent to (5) or (6).

Equation (5) needs no further discussion. In equation (6), however, we made the assumption that λ was small. If λ is not small (6) is equivalent to (8) under the condition that λ is not a special parameter value for $A_2(r, s)$. *Equation* (6) *has therefore a unique analytic solution under the given conditions, provided that* λ *is not a special parameter value for* $A_2(r, s)$.

The correspondence between symbolic differential and integro-differential equations

8. Let us consider the equation

$$(10) \qquad \Phi\left(U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial^2 U}{\partial x^2}, \cdots; y, x \mid r, s\right) = 0,$$

of which the left hand member is an integral rational expression in U and its derivatives up to the nth order, with coefficients that are analytic in x and y throughout a neighborhood σ, independent of r, s, of a given analytic curve σ. The coefficients in the analytic developments of these coefficients are continuous functions of r, s in T. Multiplication of U by itself or by any of its derivatives, or of any of these by a function of r, s is to be interpreted as combination according to the formula (1) [§ 1]. Multiplication of U or any of its derivatives by a function not involving r, s may be interpreted in the ordinary way; or, on the other hand, that function may be regarded as implicitly a function of r, s, according as we care to consider one equation or another. In this way we generate an integro-differential equation which includes that discussed by Volterra in the Theorem V already referred to.

Equation (10) may be rewritten as a symbolic differential equation. In

* It is not necessary to introduce parameters λ_i into these boundary conditions (see note to § 5).

fact if we replace in it U and its derivatives by $jU = \xi$ and its derivatives, and replace every function involving r, s, explicitly, or by definition, by j times that function, leaving the functions not involving r, s as they are, and if we then collect terms, we shall have a symbolic differential equation whose solutions of nullity are the solutions of (10), and vice-versa. Let us write this equation as

$$\Psi\left(\xi, \frac{\partial \xi}{dx}, \frac{\partial \xi}{dy}, \frac{\partial^2 \xi}{\partial x^2}, \cdots; y, x \mid r, s\right) = 0. \tag{11}$$

9. If the curve σ is the x-axis, and (11) can be solved for $\partial^n \xi / \partial y^n$, the equation (11) may be replaced by a system of equations of the type of (4), and Theorem 1 or Theorem 2 may be applied. We thus are able to restate for integro-differential equations of the form (10) the ordinary Cauchy existence theorems for differential equations.

If the coefficients in the analytic development of (10) are permutable among themselves, the solution of (10) is permutably analytic, and analytically permutable with the coefficients, provided that the same statement is true of the initial conditions.

Characteristics of integro-differential equations*

10. If the curve σ is not the x-axis but is some analytic curve $y = \varphi(x)$, we may still determine a solution of the equation (10), or, what is the same thing, a solution of nullity of equation (11), by assigning proper values for the solution and its derivatives up to and including those of order $n - 1$ along the curve. For, by a transformation of variable, this case may be reduced to that already discussed. For certain families of curves, however, as with differential equations, this transformation is impossible; the solution is not uniquely determined by means of the given conditions. These families of curves, to which correspond the characteristics of differential equations, are the subject of this last section.

Without any essential loss of generality, for our treatment, we can assume that equations (10) and (11) are of the second order. We shall suppose them, for the present, to be linear in the derivatives of highest order. Equation (11) may then be rewritten as

$$\alpha_{11} \frac{\partial^2 \xi}{\partial x^2} + \alpha_{12} \frac{\partial^2 \xi}{\partial x \partial y} + \alpha_{22} \frac{\partial^2 \xi}{\partial y^2} + \lambda = 0. \tag{12}$$

In this equation α_{11}, α_{12}, α_{22} and λ are functions of ξ, $\partial \xi / \partial x$, $\partial \xi / \partial y$, x, y, r,

* This treatment may be compared with that in Hadamard's *Leçons sur la propogation des ondes*, chap. 7, Paris, 1913. The equations (12), (13), (14), (15), (17), (18′), below, correspond to the equations (1), (7), (8), (8′) and (12), (11), (13) respectively in the chapter cited.

s of the type specified in § 8. We assume that on the curve $y = \varphi(x)$ we are given ξ and $\partial\xi / \partial y$, or what amounts to the same thing, ξ, $\partial\xi / \partial x$ and $\partial\xi / \partial y$ as analytic functions of x in the region σ, with coefficients continuous functions of r, s in T.*

Precisely as in the theory of differential equations, we have from (12), by expressing $\partial^2 \xi / \partial x^2$ and $\partial^2 \xi / \partial x \partial y$ for points on $y = \varphi(x)$ in terms of $\partial\xi / \partial y$ and $\partial^2 \xi / \partial y^2$, as the equation for the determination of $\partial^2 \xi / \partial y^2 = \xi_{22}$

$$\Gamma \xi_{22} + \Pi = 0. \tag{13}$$

In this equation

$$\begin{aligned} \Gamma &= \alpha_{11}\left(\frac{dy}{dx}\right)^2 - \alpha_{12}\left(\frac{dy}{dx}\right) + \alpha_{22}, \\ \Pi &= \alpha_{11}\left(\frac{d\xi_1}{dx} - \frac{dy}{dx}\frac{d\xi_2}{dx}\right) + \alpha_{12}\frac{d\xi_2}{dx} + \lambda, \end{aligned} \tag{14}$$

where d / dx refers to differentiation along the curve, and ξ_1 and ξ_2 denote $\partial\xi / \partial x$ and $\partial\xi / \partial y$ respectively. In these formulæ it is important to preserve the order of all quantities that involve the j, so as not to necessitate the introduction of the hypothesis of permutability.

The equation (13) *enables us to determine* ξ_{22}, *unless* Γ *is a function of nullity at some point of the curve* $y = \varphi(x)$; *i. e., unless*

$$\operatorname{Ord} \Gamma = 0. \tag{15}$$

The curves defined by the differential equation (15) may be called the *ordinary characteristics* of the integro-differential equation (10). On account of the way (11) is formed from (10), equation (15), which involves no j-coefficients, must be independent of the solution $\xi = jU$. *The ordinary characteristics are independent of the solution of the equation* (10).

11. If we make an analytic transformation of the independent variables x, y that reduces the curve $y = \varphi(x)$ to the x-axis, it is immediately verifiable that a sufficient condition that the transformation of the equation (11) be solvable for $\partial^n \xi / \partial y^n$ is that the curve $y = \varphi(x)$ be nowhere tangent to an ordinary characteristic. That it is not also a necessary condition depends on the fact that it is often possible to divide by a function of nullity.

THEOREM 3. *If* $y = \varphi(x)$ *is not tangent to any ordinary characteristic, the given values of* U *and* $\partial U / \partial y$ *uniquely determine an analytic solution of* (10).

12. It may happen that all the curves in the plane are ordinary characteristics of the equation (10). In other words Ord Γ may vanish identically. A

* The j-coefficients of these functions of r, s and of the functions and values arbitrarily assigned are supposed to contain parameters λ, if we are dealing with integration with constant limits.

necessary and sufficient condition that Ord Γ be identically zero is that we have identically

$$\text{Ord } \alpha_{11} = \text{Ord } \alpha_{12} = \text{Ord } \alpha_{22} = 0 . \tag{16}$$

We may still, however, be able to determine solutions of the equation by giving arbitrary values of U and $\partial U / \partial y$ along $y = \varphi(x)$.

As an illustration of this fact, let us restrict ourselves to an integro-differential equation in which the limits of integration are s, r, and let

$$\alpha_{ij} = jA_{ij}(x, y \mid r, s) + jB_{ij}(x, y \mid r, s)\beta_{ij}, \quad \lambda = jC(x, y \mid r, s)\gamma,$$

where β_{ij}, γ are functions of ξ, $\partial\xi / \partial x$, $\partial\xi / \partial y$, x, y, r, s of the kind already described, and A_{ij}, B_{ij}, C and their first derivatives in regard to r are continuous functions of r, s and analytic functions of x, y, which for $s = r$ become merely functions a_{ij}, b_{ij}, c, of x, y alone. We assume, moreover, that the β_{ij} are functions of nullity.

To investigate this equation, differentiate it with regard to r. It will then, as is directly verifiable, take the form

$$\left\{ a_{11} + j\frac{\partial A_{11}}{\partial r} + \left(b_{11} + j\frac{\partial B_{11}}{\partial r}\right)\beta_{11} \right\} \frac{\partial^2 \xi}{\partial x^2}$$
$$+ \left\{ a_{12} + j\frac{\partial A_{12}}{\partial r} + \left(b_{12} + j\frac{\partial B_{12}}{\partial r}\right)\beta_{12} \right\} \frac{\partial^2 \xi}{\partial x \partial y}$$
$$+ \left\{ a_{22} + j\frac{\partial A_{22}}{\partial r} + \left(b_{22} + j\frac{\partial B_{22}}{\partial r}\right)\beta_{22} \right\} \frac{\partial^2 \xi}{\partial y^2} + \lambda' = 0 .$$

The solutions of nullity of this last equation are solutions of nullity of the original one, and vice-versa. But this last equation has definite ordinary characteristics unless a_{11}, a_{12}, a_{22} are all zero.

13. Let us now suppose that we have an integro-differential equation (12) for which the ordinary characteristics are defined, and that $y = \varphi(x)$ is one of them. Let us consider how much is arbitrary in the solution.

The equation (13) shows that it is not generally possible to assign values of ξ and ξ_2 arbitrarily along an ordinary characteristic. For if, for instance, we restrict ourselves to variable limits of integration r, s and take account of the fact that since $y = \varphi(x)$ is an ordinary characteristic Γ is a function of nullity, it follows that in general $\Gamma\xi_{22}$ vanishes to a higher order than Π along $r = s$, and equation (13) is not satisfied. By properly choosing the values of ξ and ξ_2, however, we may still have solutions.

By successive differentiation of (12) and elimination of all partial derivatives except those with regard to y, by means of the relation $y = \varphi(x)$, we have the equation

$$\Gamma\xi_{222} + \Pi_1 = 0, \qquad \Gamma\xi_{2222} + \Pi_2 = 0, \qquad \cdots, \tag{17}$$

in which

$$\Pi_i = \left(\alpha_{12} - 2\alpha_{11}\frac{dy}{dx}\right)\frac{d}{dx}\left(\frac{\partial^{i+1}\xi}{\partial y^{i+1}}\right) + \mu_i, \tag{18}$$

where μ_i does not involve differentiation along $y = \varphi(x)$ of $\partial^{i+1}\xi / \partial y^{i+1}$. Hence, if all the derivatives of ξ of order up to i are known, Π_i becomes an integro-differential expression of the first order in $\partial^{i+1}\xi / \partial y^{i+1}$. We may also write

$$\Pi = \alpha_{11}\left(\frac{d^2\xi}{dx^2} - 2\frac{dy}{dx}\frac{d\xi_2}{dx} - \frac{d^2y}{dx^2}\xi_2\right) + \alpha_{12}\frac{d\xi_2}{dx} + \lambda. \tag{18'}$$

Let us assume that we are able to choose our arbitrary values so that

$$j\text{-coefficient } \Gamma = 0. \tag{19}$$

From (13) it follows that necessarily

$$\Pi = 0, \tag{20}$$

and from (17),

$$\Pi_1 = 0, \qquad \Pi_2 = 0, \qquad \cdots. \tag{21}$$

We must then, if we choose values to satisfy (19), so choose them that they also satisfy (20) and (21).

We may regard (19) and (20) as simultaneous equations that hold along $y = \varphi(x)$ for ξ and ξ_2. In fact, if in the expression for j-coefficient Γ we substitute for the ξ_1 that appears in α_{11}, α_{12}, α_{22}, by means of the formula

$$\xi_1 = \frac{d\xi}{dx} - \frac{dy}{dx}\xi_2,$$

we get for (19) an integro-differential equation of order zero in ξ_2 and of the first order in ξ. Considered as an equation to determine ξ_2, it is an integral equation, and may generally be solved for ξ_2. On substituting this value in (20), we have an integro-differential equation of the second order to determine ξ, simpler than those discussed here, since it involves differentiation with regard to but one variable. In general, then, *if* (19) *holds we may assign at one point of the characteristic arbitrary values of* ξ *and* $d\xi / dx$, *whence will be determined the values of* ξ *and* ξ_2 *all along the curves.* The equations $\Pi_1 = 0$, $\Pi_2 = 0$, $\cdots$ now become integro-differential equations of the first order to determine ξ_{22}, ξ_{222}, $\cdots$ respectively. Therefore *we may assign arbitrarily the values of* ξ_{22}, ξ_{222}, $\cdots$ *at one point of the characteristic, whence their values will be determined all along the curve.*

14. These results may be generalized in various ways. The extension is immediate to the case where instead of x, y we have n variables x_1, x_2, $\cdots$, x_n. The characteristics become spaces of $n - 1$ dimensions,

$$x_n = f_i(x_1, x_2, \cdots, x_{n-1}),$$

and provided that the given $(n-1)$-space

$$x_n = \varphi(x_1, x_2, \cdots, x_{n-1})$$

contains no $(n-1)$-dimensional element of a characteristic, arbitrarily assigned values of ξ and $\partial\xi / \partial x_n = \xi_n$ determine a solution of the integro-differential equation throughout the n-space. In a characteristic space the values of ξ and $\partial\xi / \partial x_n$ may no longer be chosen arbitrarily. If they are taken so that (19) is satisfied, then along an $(n-2)$-space

$$x_{n-1} = \Psi(x_1, x_2, \cdots, x_{n-2})$$

values of ξ, $\partial\xi / \partial x_{n-1}$, ξ_{nn}, $\cdots$ may be chosen arbitrarily, their values being then determined by the integro-differential equation throughout the given $(n-1)$-space. Here by $\partial\xi / \partial x_{n-1}$ we mean the total derivative in the $(n-1)$-space. In fact, the equation (19) may be rewritten as an integral equation in ξ_n, as before it was rewritten for ξ_2, and the result of substituting in (20) is an integro-differential equation of the second order in ξ. This equation and the equations (21) are integro-differential equations of the kind treated in this article, referred to $n-1$ variables of differentiation. Thus a formal development for a solution is obtained.

In the extension of the results for equations of the second order to those of higher order, or from equations linear in the derivatives of highest order to those that are not, there is no difference in the case that we are now treating from the similar extensions in the case of differential equations.

BOSTON, *July* 26, 1913.

UNIVERSITY OF THE

A NEW PRINCIPLE IN THE GEOMETRY OF NUMBERS, WITH SOME APPLICATIONS

BY

H. F. BLICHFELDT

1. Minkowski has discovered a geometrical principle which he applied with success to certain important problems in the theory of numbers. If we define *lattice-points* as those points in space of n dimensions whose (rectangular) coördinates are positive or negative integers or zero, his principle may be stated as follows:*

A surface in n-dimensional space, nowhere concave, possessing a center which coincides with one of the lattice-points of this n-space, and having a volume $\geqq 2^n$, will contain at least two more lattice-points, either inside the surface or upon its boundary.

He gave this theorem the following analytic form: Let $f(x_1, \cdots, x_n)$ be any function of $x_1, \cdots, x_n$, which vanishes when $x_1 = 0, \cdots, x_n = 0$, but possesses a definite positive value for any other system of values assigned to $x_1, \cdots, x_n$; let, moreover, the following functional equations be fulfilled:

$$\text{(I)} \qquad f(tx_1, \cdots, tx_n) = tf(x_1, \cdots, x_n), \quad \text{when} \quad t > 0,$$

$$\text{(II)} \qquad f(y_1 + z_1, \cdots, y_n + z_n) \leqq f(y_1, \cdots, y_n) + f(z_1, \cdots, z_n),$$

$$\text{(III)} \qquad f(-x_1, \cdots, -x_n) = f(x_1, \cdots, x_n).$$

Then the n-tuple integral $\int dx_1 \cdots dx_n$, taken over the region

$$f(x_1, \cdots, x_n) \leqq 1$$

in positive directions along the paths of integration, will possess a definite value J, and there exists at least one system of integers $l_1, \cdots, l_n$, such that

$$0 < f(l_1, \cdots, l_n) \leqq \frac{2}{\sqrt[n]{J}}.$$

2. Among the applications given by Minkowski, two will be mentioned here, and one later on (§ 12).

* *Geometrie der Zahlen*, Leipzig und Berlin, 1910, p. 76.

(α) Let

$$[f(x_1, \cdots, x_n)]^2 \equiv \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

be a positive, definite quadratic form in n variables, having a determinant D. Then the variables may be replaced by such integers $l_1, \cdots, l_n$ that we have*

$$0 < f^2 \leqq \frac{4}{\pi}\left[\Gamma\left(1+\frac{n}{2}\right)\right]^{2/n} D^{1/n}.$$

(β) Let

$$f(x_1, \cdots, x_n) \equiv |v_1| + \cdots + |v_n|,$$

where $v_1, \cdots, v_n$ are linear homogeneous forms in $x_1, \cdots, x_n$, with a non-vanishing determinant Δ. Let s pairs of these forms have conjugate imaginary coefficients. Then integers $l_1, \cdots, l_n$ exist such that we have, upon substitution,*

$$0 < f \leqq \left[\left(\frac{4}{\pi}\right)^s \Gamma(1+n) \cdot |\Delta|\right]^{1/n}.$$

3. A new geometrical principle will now be stated and proved. To make it somewhat more general in scope, a new definition of lattice-points will be necessary.

Definition. Let the n-space defined by rectangular coördinates $x_1, \cdots, x_n$ be divided into equal rectangular spaces by the n systems of planes

$$x_1 = a_1 + b_1 t, \cdots, x_n = a_n + b_n t \qquad (t = 0, \pm 1, \pm 2, \cdots),$$

where $a_1, \cdots, a_n, b_1, \cdots, b_n$ are given real numbers. We shall call these spaces *fundamental parallelepipeds*. In each of them let there be located, in an arbitrary manner, a given number of points, say k; none of them, however, lying upon the boundaries of the parallelepipeds. *These points shall be designated lattice-points.*

It will be observed that this definition of lattice-points includes the former: set $a_i = \frac{1}{2}$, $b_i = 1$ $(i = 1, \cdots, n)$; and locate one lattice-point $(k = 1)$ at the center of each of the resulting parallelepipeds.

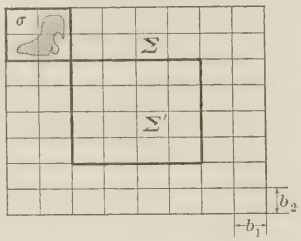
4. **Theorem I.** *Let S represent any limited open n-dimensional continuum in the n-space $x_1, \cdots, x_n$, having the (outer) volume V. By a suitable translation*

$$x_i' = x_i + \delta_i \qquad (i = 1, \cdots, n),$$

this continuum can be placed in such a position with reference to the fundamental parallelepipeds that the number of lattice-points L contained in the continuum or lying as near as we please to its boundary is greater than Vk/W, where W represents the volume, and k the number of lattice-points of a fundamental parallelepiped.

* *Geometrie der Zahlen*, p. 122.

Proof. We construct a parallelepiped σ whose edges parallel to the coördinate axes $X_1, \cdots, X_n$ are of lengths $Ab_1, \cdots, Ab_n$, A being a fixed integer (2 in the figure) taken so large that S can, by a translation, be placed entirely within σ. When so placed, the double surface obtained will be denoted by (S, σ). We then construct a parallelepiped Σ whose edges, parallel to the axes, are of lengths $Bb_1, \cdots, Bb_n$, B being a large (variable) integer (8 in the figure), which exceeds $2A$. We may assume that Σ contains just B^n fundamental parallelepipeds. Inside Σ we have a parallelepiped Σ' whose edges are of lengths $(B - 2A)\, b_1, \cdots, (B - 2A)\, b_n$, its faces being at distances $Ab_1, \cdots, Ab_n$ from those of Σ.



Now place (S, σ) in such a position inside Σ that one of the vertices of σ coincides with one of Σ, so that n of its faces lie upon n of the faces of Σ. From this position we obtain $\{ (B - A)\, C + 1 \}^n$ different positions of (S, σ), all inside Σ, by means of translations of the form

$$x_i' = x_i + t_i \frac{b_i}{C} \qquad (i = 1, \cdots, n), \tag{1}$$

where t_i runs through all integral values from $-\infty$ to $+\infty$, independently for each subscript i, and where C represents a large positive integer.

5. Each of the $k\,(B - 2A)^n$ lattice-points inside Σ' will lie inside or near to several of the surfaces S, if C be chosen large enough. We shall proceed to count these lattice-points in such a way that each is counted as often as it appears inside of, or as near as we please to a surface S. Call this number N.

Consider any one of these lattice-points P. To count the number of surfaces S inside of which it will appear, we may regard S as stationary and P as subjected to the translations (1). Then P will appear as vertices of parallelepipeds whose edges are of lengths $b_1/C, \cdots, b_n/C$. Let the least number of such parallelepipeds which entirely contain the continuum S be M_P. We may take C so large that all the vertices of these M_P parallelepipeds lie either within the continuum S, or as near as we please to its boundary. Summing for all of the lattice-points considered, we get

$$N > k\,(B - 2A)^n\, M,$$

M being the smallest of the numbers M_P.

6. Since there are $\{ (B - A)\, C + 1 \}^n$ different surfaces S inside Σ, it follows that there are

$$L > k \frac{(B - 2A)^n\, M}{\{ (B - A)\, C + 1 \}^n}$$

lattice-points inside or near the surface of at least one of them. If now B and C increase indefinitely, the right-hand member approaches

$$\lim_{C=\infty} k\frac{M}{C^n} = k\frac{V}{b_1 \cdots b_n} = \frac{k}{W}V$$

so that for a given positive ϵ we may take

$$L > \frac{k}{W}V - \epsilon.$$

Finally, since L is an integer, the inequality of the theorem follows unless Vk/W is an integer also. In this case however we consider first a continuum lying arbitrarily near to S and containing S within it. For this continuum of volume greater than V the desired inequality holds, and thus for S also. The theorem is therefore true.

7. When the different fundamental parallelepipeds are congruent with reference to the lattice-points contained, the statement in Theorem I reading "the number of lattice-points L contained in the continuum or lying as near as we please to limiting points of the continuum is greater than kV/W . . ." may be accentuated to read as follows: "*the number of lattice-points L contained in the continuum or belonging to limiting points of the continuum is greater than Vk/W. . . .*" The above proof may be modified to cover this statement. However, instead of doing this, we shall give a very elegant and independent proof of Theorem I, under the restrictions mentioned, furnished the author by Professor Birkhoff, which proof throws into evidence the sharper wording of the theorem. We shall limit ourselves to one lattice-point in each fundamental parallelepiped, although the proof, as given by Professor Birkhoff, is valid for any number. The spirit of the proof is shown equally well by limiting ourselves to space of two dimensions only.

Let us therefore assume the plane divided into a network R of equal rectangles, a single lattice-point A in each, similarly placed. Let a closed curve C be drawn, having an (outer) area V, the area of a rectangle being W.

We will superpose, by a translation, upon a single rectangle R_1 all those which are covered, in whole or in part, by V (its boundary included). It is then evident that a point P can be located in R_1 at which the superposed or adjoining portions of V are in number $L > V/W$, unless V/W = an integer and no part of the boundary of V appears inside R_1. But such a case could be avoided by subjecting C to a proper translation at the outset. In the reconstructed area V, P will appear as L points, congruent with reference to the rectangles R. Applying a translation such that these points coincide with L points A, the translated area V will contain in its interior or upon its boundary more than V/W lattice-points A.

8. **Extension of Theorem I.** *Let there be given a finite number of continua $S_1, \cdots, S_m$, overlapping or not, forming a network $\bar{S}$, and let $\alpha_1, \cdots, \alpha_m$ be arbitrary positive constants. Then by a suitable translation, $\bar{S}$ may be brought into such a position that, if L_i designates the number of lattice-points inside of or as near as we please to the surface of S_i, and V_i the outer volume of S_i, then* $\alpha_1 L_1 + \cdots + \alpha_m L_m > (\alpha_1 V_1 + \cdots + \alpha_m V_m) k / W$. The demonstration follows that of Theorem I, after S in the symbol (S, σ) is replaced by $\bar{S}$. Considering each continuum S_i in turn, we prove (using corresponding notation N_i, M_i):

$$\Sigma \alpha_i N_i > k (B - 2A)^n \Sigma \alpha_i M_i,$$

and then divide by $\{(B - 2A) C + 1\}^n$. Finally, $\Sigma \alpha_i L_i$ not being a continuous magnitude, the argument at the end of § 6 is valid here.

9. Minkowski's general analytical theorem (§ 1), so far as the last inequalities are concerned, follows immediately from Theorem I. Let the symbols $f(x_1, \cdots, x_n)$, J be defined as in Minkowski's theorem, and let S represent the continuum of points $(x_1, \cdots, x_n)$ satisfying the condition

$$f(x_1, \cdots x_n) < J^{-1/n}. \tag{2}$$

Then $V = 1$, and if "lattice-points" are defined as in § 1, we have $k/W = 1$ (cf. § 3). Applying Theorem I, we have $L \geqq 2$. Let $(y_1, \cdots, y_n)$ and $(-z_1, \cdots, -z_n)$ be two lattice-points inside S or upon its boundary (§ 7), in its new position. Set $y_1 + z_1 = l_1, \cdots, y_n + z_n = l_n$.

Now, if $(x_1', \cdots, x_n')$ be a lattice-point which after a translation lies in or on the boundary of the continuum defined by (2) and if $\delta_1, \cdots, \delta_n$ be the components of the translation, we have

$$f(x_1' - \delta_1, \cdots, x_n' - \delta_n) \leqq \frac{1}{J^{1/n}}.$$

Hence, we obtain

$$\begin{aligned} 0 < f(l_1, \cdots, l_n) &= f(y_1 + z_1, \cdots, y_n + z_n) \\ &\leqq f(y_1 - \delta_1, \cdots, y_n - \delta_n) + f(-z_1 - \delta_1, \cdots, -z_n - \delta_n) \leqq 2J^{-1/n}. \end{aligned}$$

10. **Quadratic forms.** Let F designate a positive definite quadratic form in n variables $x_1, \cdots, x_n$. We may write

$$F \equiv v_1^2 + \cdots + v_n^2,$$

where $v_1, \cdots, v_n$ are linear in the n variables and of determinant Δ. Then the determinant of F will be $D = \Delta^2$.

To apply the theorem in § 8, let us construct the network $\bar{S}$, consisting of the points respectively inside the surfaces $S_1, \cdots, S_m$; S_λ having for its

equation

$$F = (\lambda\phi)^{2/n}, \quad \text{where} \quad \phi = \frac{n+2}{2mJ};$$

here m is a large (variable) integer, and

$$J = \frac{\pi^{\frac{1}{2}n}}{\Delta\Gamma\,(1+\frac{1}{2}n)}$$

is the volume of $F = 1$. Let "lattice-points" be those points having integral coördinates, so that $k / W = 1$. Then, after a suitable translation,

$$\begin{aligned} \alpha_1 L_1 + \cdots + \alpha_m L_m &> \alpha_1 V_1 + \cdots + \alpha_m V_m \\ &= \phi J\,(\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m). \end{aligned} \tag{3}$$

We shall put $\alpha_i = (i+1)^{2/n} - i^{2/n}$ when $i < m$,

$$\alpha_m = \frac{(n+2)\,g}{n\,m} - m^{2/n},$$

where $g = 1 + 2^{2/n} + \cdots + m^{2/n}$, and shall denote $L_i - L_{i-1}$ by p_i. Then (3) becomes

$$\frac{(n+2)\,g}{n\,m} L_m - (p_1 + 2^{2/n} p_2 + \cdots + m^{2/n} p_m) > \frac{(n+2)\,g}{n\,m}. \tag{4}$$

Now, if $x_1', \cdots, x_m'$ be a lattice-point inside or near S_λ, and if we indicate the result of substituting $x_1' - \delta_1, \cdots, x_n' - \delta_n$ for $x_1, \cdots, x_n$ in v_i by v_i', then

$$(v_1')^2 + \cdots + (v_n')^2 < (\lambda\phi)^{2/n} + \epsilon,$$

where ϵ is a quantity small at will. If $x_1'', \cdots, x_n''$ be any other lattice-point, then the differences $v_1' - v_1''$, etc., are equal to the results obtained by substituting the integers $x_1' - x_1''$, etc., for x_1, etc., in $v_1, \cdots, v_n$. Hence

$$(v_1' - v_1'')^2 + \cdots + (v_n' - v_n'')^2 \tag{5}$$

will be a numerical value of F, for integral values of the variables involved.

Adding all the expressions (5) for every pair of lattice-points contained in the network $\overline{S}$, and denoting the sum $p_1 + p_2 + \cdots + p_m = L_m$ by P, we get

$$\begin{aligned} P \sum_{j=1}^{P} \{ (v_1^{(j)})^2 + \cdots + (v_n^{(j)})^2 \} - \left(\sum_{j=1}^{P} v_1^{(j)}\right)^2 - \cdots - \left(\sum_{j=1}^{P} v_n^{(j)}\right)^2 \\ < P\,\{ p_1\,\phi^{2/n} + p_2\,(2\phi)^{2/n} + \cdots + p_m\,(m\phi)^{2/n} \} + \epsilon P^2 \\ < P\phi^{2/n}(P-1)\,\frac{(n+2)\,g}{n\,m} + \epsilon P^2, \end{aligned}$$

by (4). If we now divide by $\frac{1}{2}P(P-1)$, substitute the value of ϕ and pass to the limit $m = \infty$, we obtain

THEOREM II. *Let F be a positive definite quadratic form in n variables and of determinant D. Then integers $l_1, \cdots, l_n$, not all zero, may be substituted for the n variables such that the numerical value of F is not greater than*

$$\frac{2}{\pi}\left[\Gamma\left(1+\frac{n+2}{2}\right)\right]^{2/n} D^{1/n}.$$

It will be observed that the asymptotic value $nD^{1/n}/\pi e$ of this expression is one half that of Minkowski's limit (§ 2). Minkowski* has proved that the asymptotic value cannot fall below $nD^{1/n}/2\pi e$. The actual limit for $n = 2$ was first determined by Hermite,† and for $n = 3$, 4 and 5 by Korkine and Zolotareff.‡ The quotients of these limits divided by $D^{1/n}$ are respectively $2/\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt{2}$, $\sqrt[5]{8}$.

11. **Linear forms.** Let $f \equiv |v_1| + \cdots + |v_n|$, where $v_1, \cdots, v_n$ are linear homogeneous forms in $x_1, \cdots, x_n$, of determinant $\Delta \neq 0$. If imaginary coefficients occur we assume that the forms having such coefficients appear in conjugate imaginary pairs. We can then utilize Theorem II. For, from the value of $|v_1|^2 + |v_2|^2$, the maximum value of $|v_1| + |v_2|$ is obtained by putting $|v_1| = |v_2|$. We find the following

THEOREM III. *Integers $l_1, \cdots, l_n$, not all zero, can be substituted for the variables $x_1, \cdots, x_n$ such that*

$$0 < |v_1| + \cdots + |v_n| \leqq \sqrt{\frac{2n}{\pi}}\left[\Gamma\left(1+\frac{n+2}{2}\right)\right]^{1/n} |\Delta|^{1/n}.$$

The asymptotic value

$$\frac{n}{\sqrt{\pi e}}|\Delta|^{1/n}$$

of this limit is smaller than that (§ 2) of Minkowski

$$\frac{n}{e}\left(\frac{4}{\pi}\right)^{s/n}|\Delta|^{1/n},$$

the more so the greater the number of pairs s of conjugate imaginary forms contained among $v_1, \cdots, v_n$. On the other hand, for low values of n, the limit given above is higher than that given by Minkowski.

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*Journal für Mathematik, vol. 129 (1905), pp. 268–9.

†Journal für Mathematik, vol. 40 (1850), p. 263. The limits for $n = 2$ and $n = 3$ were virtually determined by Gauss and Seeber. Cf. Gauss' Werke, Bd. 1, p. 307, ff. and Bd. 2, p. 192, ff. Göttingen, 1876.

‡*Sur les formes quadratiques positives*, Mathematische Annalen, vol. 11 (1877), p. 242, ff.

12. **Approximation of irrational numbers by means of fractions.** Let $\alpha_1, \cdots, \alpha_{n-1}$ represent $n-1$ real positive numbers. The problem of finding $n-1$ rational fractions $x_1/z, \cdots, x_{n-1}/z$, having a common denominator, which are close approximations to the quantities α, has been solved by Hermite,* Kronecker,† and Minkowski,‡ and for $n=2$ by Hurwitz.§ We shall prove a theorem giving slightly closer limits, except for $n=2$, than those obtained by these writers.

Consider the linear forms

$$y_1 \equiv x_1 - \alpha_1 z, \quad y_2 \equiv x_2 - \alpha_2 z, \quad \cdots, \quad y_{n-1} \equiv x_{n-1} - \alpha_{n-1} z.$$

In the n-space $(y_1, \cdots, y_{n-1}, z)$ construct the lattice-points obtained by substituting all possible integers for $x_1, \cdots, x_{n-1}, z$. Construct also the planes

$$y_1 = a_1 + h_1, \quad \cdots, \quad y_{n-1} = a_{n-1} + h_{n-1}, \quad z = \tfrac{1}{2} + kh_n.$$

Here k is a given positive integer; $a_1, \cdots, a_{n-1}$ any set of $n-1$ numbers *not* representable by the forms $y_1, \cdots, y_{n-1}$; while $h_1, \cdots, h_{n-1}$ run, independently of each other, through all integers from $-\infty$ to $+\infty$. These planes divide the n-space into fundamental parallelepipeds, each of volume $W = k$, and each containing just k lattice-points. For there are just k sets of integers $x_1, \cdots, x_{n-1}, z$ which satisfy the inequalities

$$a_i + h_i < x_i - \alpha_i z < a_i + h_i + 1 \qquad (i = 1, \cdots, n-1),$$
$$\tfrac{1}{2} + kh_n < z < \tfrac{1}{2} + k(h_n + 1).$$

We now apply Theorem I, where S represents the continuum of points satisfying the conditions

$$|z| < a;$$

$$(6) \qquad \left|\frac{y_i}{b}\right| + \left|\frac{2^n (n-1)^{n-1} z}{n^n a}\right| < 1 \quad \text{when} \quad \left|\frac{z}{a}\right| \leqq \left(\frac{n}{2(n-1)}\right)^{n-1},$$
$$\left|\frac{z}{a}\right| \left(\left|\frac{y_i}{b}\right| + 1\right)^{n-1} < 1 \quad \text{when} \quad 1 > \left|\frac{z}{a}\right| > \left(\frac{n}{2(n-1)}\right)^{n-1}$$
$$(i = 1, \cdots, n-1).$$

Here a and b are two positive real numbers subject to the restrictions

$$(7) \qquad ab^{n-1} \left\{ \frac{n^{n-1}}{(n-1)^{n-1}} - \frac{(n-2)^n}{(n-1)^{n-1} n} + 2^n (n-1) \int_0^{1-2/n} \frac{v^{n-1}\, dv}{(1+v)^n} \right\} = 1, \quad b < \frac{1}{2}.$$

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* Journal für Mathematik, vol. 40 (1850), p. 266.

† *Werke*, Bd. I, p. 636.

‡ *Geometrie der Zahlen*, p. 108 ff.

§ Mathematische Annalen, vol. 39 (1891), p. 279.

The volume of S is $V = 1$ by (7). We can therefore apply a translation

$$y_1' = y_1 + \delta_1, \quad \cdots, \quad y_{n-1}' = y_{n-1} + \delta_{n-1}, \quad z' = z + \delta_n$$

to S which will bring it into such a position that two lattice-points, $(y_{11}, \cdots, y_{1\,n-1}, z_1)$, $(y_{21}, \cdots, y_{2\,n-1}, z_2)$ are contained in S, or lie as near as we wish to its boundary.

The two integers z_1, z_2 cannot be equal. For otherwise

$$|x_{1i} - x_{2i}| = |y_{1i} - y_{2i}| = |(y_{1i} - \delta_i) - (y_{2i} - \delta_i)| < 2b + \epsilon < 1,$$

by (6) and (7), i. e., $x_{1i} - x_{2i} = 0$ for every subscript i, and hence the two lattice-points would not be distinct. If we therefore set

$$X_1 = x_{11} - x_{21}, \quad \cdots, \quad X_{n-1} = x_{1\,n-1} - x_{2\,n-1}, \quad Z = z_1 - z_2,$$

we have $Z \geqq 1$, and we can prove that

$$|X_i - \alpha_i Z| < 2b + \epsilon_1, \qquad |X_i - \alpha_i Z|\,|Z|^{1/n-1} < a^{1/n-1} b + \epsilon_2,$$

where ϵ_1 and ϵ_2 are quantities as small as we please. If now we notice that from (7)

$$ab^{n-1}\left(\frac{n}{n-1}\right)^{n-1}\left[1 + \left(\frac{n-2}{n}\right)^{n-2}\right] \leqq 1,$$

and that z may be assumed positive, these results give at once:

THEOREM IV. *Given $n - 1$ positive quantities $\alpha_1, \cdots, \alpha_{n-1}$, and an arbitrarily small quantity $b < \frac{1}{2}$, we can find n integers $X_1, \cdots, X_{n-1}, Z$ such that the $n - 1$ differences*

$$\left|\frac{X_i}{Z} - \alpha_i\right|$$

are not greater than $2b$, and at the same time are all not greater than

$$\frac{\gamma}{Z^{n/n-1}} = \frac{(n-1)\,Z^{-(n/n-1)}}{n\left[1 + \left(\frac{n-2}{n}\right)^{n+2}\right]^{1/n-1}}.$$

The common denominator Z need not be taken greater than $2a \leqq 2\,(\gamma / b)^{n-1}$.

For $n = 2$, Hurwitz proved (l. c.) that $\gamma = 1 / \sqrt{5}$, and Minkowski* proved that, for $n \geqq 2$, $\gamma < (n-1)/n$. For large values of n, this may be written $\gamma = e^{-1/n}$, while the expression for γ in Theorem IV becomes $(e + 1/e)^{-1/n}$.

* *Geometrie der Zahlen*, p. 112.

AN APPLICATION OF SEVERI'S THEORY OF A BASIS TO THE KUMMER AND WEDDLE SURFACES*

BY

F. R. SHARPE AND C. F. CRAIG

In a series of papers F. Severi† has developed a theory by means of which any linear system of algebraic curves on an algebraic surface can be expressed linearly in terms of a finite number of such systems. He has also applied this theory to the study of the curves on the Fano quartic surface and to the determination of the group of birational transformations which leave this surface invariant.

In the following paper this method is applied to the study of certain of the birational transformations which leave the Kummer and Weddle surfaces invariant. In particular the question of the periodicity of the product of two such transformations is studied. Some new relations among the different types of transformation are obtained.

1. The Kummer Surface

Let A, B, E, F be any Göpel even tetrad of nodes on the general Kummer surface K_4. The notation‡

$$\begin{matrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{matrix} \tag{1}$$

will be used to denote the complete configuration of the sixteen nodes on K_4. Variables x, y, z, w will be so chosen that $A \equiv (1,0,0,0)$, $B \equiv (0,1,0,0)$, $E \equiv (0,0,1,0)$ and $F \equiv (0,0,0,1)$.

* Presented to the Society February 28, 1914.

† *Sulla totalità delle curve algebriche tracciate sopra una superficie algebrica*, Mathematische Annalen, vol. 62 (1906), pp. 194–225; *La base minima pour la totalité des courbes tracées sur une surface algébrique*, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), pp. 449–468; *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

‡ See Hudson, *Kummer's Quartic Surface*, Chapters I, II, and VII.

The surface K_4, referred to this system of coördinates, is invariant under the cubic inversion*

$$x' : y' : z' : w' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{w}. \tag{2}$$

Under this transformation the planes of each pencil through an edge of the tetraedron of inversion are interchanged in pairs. Through each pair of vertices are two six-point conics, namely:

$$\begin{cases} ABDGKO, \\ ABCHLP, \end{cases} \quad \begin{cases} AEJKLM, \\ AEIONP, \end{cases} \quad \begin{cases} AFCDJN, \\ AFGHIM, \end{cases}$$

$$\begin{cases} EFCHKO, \\ EFDGLP, \end{cases} \quad \begin{cases} BFIKLN, \\ BFJMOP, \end{cases} \quad \begin{cases} BECDIM, \\ BEGHJN. \end{cases}$$

Hence these pairs of conics are interchanged under (2). This requires that twelve nodes interchange in pairs as follows

$$(CG)\,(DH)\,(IJ)\,(KP)\,(LO)\,(MN), \tag{3}$$

where (CG) indicates the interchange of the points C and G. It follows that each of the four remaining conics $CGIJLO$, $DHLOMN$, $DHIJKP$, and $CGKPMN$, which are plane sections of K_4 by the same quadric, is invariant under (2). Since K_4 is invariant and each of the vertices A, B, E, F is transformed into the opposite face of the tetraedron of inversion by (2), the image on K_4 of each vertex is the rational plane section of the surface by the opposite face. Denote a general plane section of K_4 by C_4 and use the symbol $\backsim$ to denote "is transformed into." The transformations of the vertices are

$$\begin{aligned} A &\backsim C_4 - B - E - F, \qquad B \backsim C_4 - A - E - F, \\ E &\backsim C_4 - A - B - E, \qquad F \backsim C_4 - A - B - E, \end{aligned} \tag{4}$$

respectively.† By (2) any plane is transformed into a cubic surface which contains the edges of the tetraedron $ABEF$. This surface intersects K_4 in C_{12}, a curve of order 12. Since genus is invariant under birational transformation and C_{12} is the image of any plane section of K_4, the genus of this curve is 3. The inversion is also symmetric with respect to all the vertices: hence

$$C_4 \backsim 3C_4 - 2(A + B + E + F). \tag{5}$$

* See J. I. Hutchinson, *On some birational transformations of the Kummer surface into itself*, Bulletin of the American Mathematical Society, ser. 2, vol. 7 (1901), pp. 211–217; see p. 212.

† The symbol $\lambda C_4 - \alpha A - \beta B - \cdots$ denotes the complete curve of intersection of the Kummer surface with a surface of order λ. This curve of intersection passes through the points A, B, $\cdots$, 2α, 2β, $\cdots$, times respectively, is of order 4λ, and is of genus $1 + 2\lambda^2 - \alpha^2 - \beta^2 - \cdots$. For details as to the meaning of symbolic addition and subtraction see Picard et Simart, *Fonctions algébriques de deux variables*, vol. 2, pp. 104–116 or the papers of Severi already cited.

The partial transformations (3), (4), and (5) completely* express the transformation of the linear systems of the curves on K_4 under the inversion (2). This complete transformation will be denoted by $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]$. The inversion as thus expressed is geometric in character and independent of the coördinate system.

The inversion with respect to any other Göpel even tetrad of nodes can readily be written. The transformation $\left[\begin{smallmatrix}AB\\IJ\end{smallmatrix}\right]$ is

$$(CK)\,(DL)\,(EF)\,(GP)\,(HO)\,(MN),$$

$$A \sim C_4 - B - I - J, \quad B \sim C_4 - A - I - J, \quad I \sim C_4 - A - B - J,$$

$$J \sim C_4 - A - B - I, \quad C_4 \sim 3C_4 - 2(A + B + I + J);$$

and $\left[\begin{smallmatrix}AB\\MN\end{smallmatrix}\right]$ is

$$(CO)\,(DP)\,(EF)\,(GL)\,(HK)\,(IJ),$$

$$A \sim C_4 - B - M - N, \quad B \sim C_4 - A - M - N,$$

$$M \sim C_4 - A - B - N, \quad N \sim C_4 - A - B - M,$$

$$C_4 \sim 3C_4 - 2(A + B + M + N).$$

By the symbol $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}AB\\IJ\end{smallmatrix}\right]$, referred to as a product, is meant that a basis (α) of elements is transformed into a configuration (β) by $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]$ and that (β) is transformed into (γ) by $\left[\begin{smallmatrix}AB\\IJ\end{smallmatrix}\right]$. The product transforms (α) into (γ). Thus $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}AB\\IJ\end{smallmatrix}\right]$ is

$$A \sim 2C_4 - A - 2B - E - F - I - J,$$

$$B \sim 2C_4 - 2A - B - E - F - I - J,$$

$$E \sim C_4 - A - B - E, \quad F \sim C_4 - A - B - F, \quad I \sim C_4 - A - B - I,$$

$$J \sim C_4 - A - B - J, \quad C_4 \sim 5C_4 - 4(A + B) - 2(E + F + I + J),$$

$$(CP)\,(DO)\,(GK)\,(HL)\,(MM)\,(NN).$$

The symmetry of this result in E, F, I, and J shows that the transformation is of period two.† Similarly the products $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}AB\\MN\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}AB\\IJ\end{smallmatrix}\right]\left[\begin{smallmatrix}AB\\MN\end{smallmatrix}\right]$ and

* See Severi, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), p. 465.

† See H. F. Baker, *Note on some transformations of a general Kummer surface*, Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912–13), pp. 302–312, where this result is obtained by a different method. The operations in Baker's paper may be identified with those of the present paper by the following key:

$$\omega \equiv \begin{bmatrix}AB\\EF\end{bmatrix}, \qquad \omega U_{DA} \equiv \begin{bmatrix}AB\\IJ\end{bmatrix}, \qquad \omega V_{DA} \equiv \begin{bmatrix}AB\\MN\end{bmatrix},$$

and

$$\vartheta_2\,\vartheta_3, \qquad \vartheta_3\,\vartheta_1, \qquad \vartheta_1\,\vartheta_2$$

with

$$(AF)\,(BE)\,(CH)\,(DG)\,(IN)\,(JM)\,(KP)\,(LO),$$
$$(AE)\,(BF)\,(CG)\,(DH)\,(IM)\,(JN)\,(KO)\,(LP),$$
$$(AB)\,(CD)\,(EF)\,(GH)\,(IJ)\,(KL)\,(MN)\,(OP),$$

C_4 remaining invariant under each collineation.

$\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}AB\\IJ\end{smallmatrix}\right]\left[\begin{smallmatrix}AB\\MN\end{smallmatrix}\right]$ are each of period two. Hence *three cubic inversions, taken with respect to Göpel even tetraedra which have an edge in common, generate an abelian group of order eight and type* (1, 1, 1). There are 120 such edges and hence 120 such groups.

The tetraedron $ACIK$ has a vertex in common with the tetraedron $ABEF$. The product $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}AC\\IK\end{smallmatrix}\right]$ is

$$
\begin{aligned}
&A \sim 3C_4 - 2A - 2(C+I+K) - (G+J+P),\\
&B \sim 2C_4 - 2A - (C+I+K) - (G+P),\\
&E \sim 2C_4 - 2A - (C+I+K) - (J+P),\\
&F \sim 2C_4 - 2A - (C+I+K) - (G+J),\\
(6)\qquad &G \sim C_4 - A - (I+K), \qquad C \sim E,\\
&J \sim C_4 - A - (C+K), \qquad I \sim B,\\
&P \sim C_4 - A - (C+I), \qquad K \sim F,\\
&C_4 \sim 7C_4 - 6A - 4(C+I+K) - 2(G+J+P),\\
&\qquad (DNO)\,(HLM),
\end{aligned}
$$

where (DNO) is the cyclic substitution on its elements. If we write

$$\alpha = C+I+K, \qquad \beta = G+J+P, \qquad \gamma = B+E+F,$$

then (6) in part becomes

$$
(7)\qquad \begin{aligned}
&C_4 \sim 7C_4 - 6A - 4\alpha - 2\beta, \qquad A \sim 3C_4 - 2A - 2\alpha - \beta, \qquad \alpha \sim \gamma,\\
&\beta \sim 3C_4 - 3A - 2\alpha, \qquad \gamma \sim 6C_4 - 6A - 3\alpha - 2\beta.
\end{aligned}
$$

If (7) is non-periodic, (6) can not be periodic. Transform (7) through

$$X = C_4 - 2A, \qquad Y = A - \beta, \qquad Z = \alpha - 2\beta + \gamma,$$
$$W = A - \alpha, \qquad T = \alpha,$$

and it becomes

$$X \sim X, \qquad Y \sim Y, \qquad Z \sim Z, \qquad W \sim 3X + 3Y - Z + W,$$
$$T \sim -2Y + Z + 2W + T,$$

which is clearly non-periodic. Hence (6) is non-periodic. By suitable choice of variables the periodicity of the transformation given by the product of any two inversions, taken with respect to Göpel even tetraedra which have one node in common, can always be made to depend on the periodicity of (7). Hence the products of this type are all non-periodic.

Consider next the product $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}IJ\\MN\end{smallmatrix}\right]$ by which

$$
\begin{gathered}
A \backsim 3C_4 - (A+E+F) - 2(I+J+M+N),\\
B \backsim 3C_4 - (B+E+F) - 2(I+J+M+N),\\
E \backsim 3C_4 - (A+B+E) - 2(I+J+M+N),\\
F \backsim 3C_4 - (A+B+F) - 2(I+J+M+N),\\
I \backsim C_4 - (I+M+N), \qquad J \backsim C_4 - (J+M+N),\\
M \backsim C_4 - (I+J+M), \qquad N \backsim C_4 - (I+J+N),\\
C_4 \backsim 9C_4 - 2(A+B+E+F) - 6(I+J+M+N),\\
(CD)\,(GH)\,(KL)\,(OP).
\end{gathered}
\tag{8}
$$

The periodicity of (8) depends on that of

$$C_4 \backsim 9C_4 - 2\alpha - 6\beta, \qquad \alpha \backsim 12C_4 - 3\alpha - 8\beta, \qquad \beta \backsim 4C_4 - 3\beta,$$

where $\alpha = A + B + E + F$ and $\beta = I + J + M + N$. Transform this through

$$X = C_4 - \beta, \qquad Y = 2C_4 - \alpha - \beta, \qquad Z = \beta,$$

and it becomes

$$Y \backsim Y, \qquad X \backsim 2Y + X, \qquad Z \backsim 4X + Z,$$

which is non-periodic. Hence (8) is non-periodic. It is to be noticed that the four vertices of each tetraedron of this product are invariant as a whole under the inversion taken with respect to the other tetraedron. The non-periodicity of the product of two inversions, taken with respect to any similarly chosen pair of Göpel even tetraedra, is an immediate consequence.

In a similar manner the transformation $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]\left[\begin{smallmatrix}JK\\NO\end{smallmatrix}\right]$ may be shown to be non-periodic. The distinction between this case and that immediately preceding is that $J\,K\,N\,O$ are not invariant as a whole under $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right]$ and $A\,B\,E\,F$ are not invariant as a whole under $\left[\begin{smallmatrix}JK\\NO\end{smallmatrix}\right]$. The non-periodicity of all similar products follows.

This completes the analysis of the product of two inversions taken with respect to tetraedra chosen in all possible positions. Hence follows the

THEOREM. *The product of any two cubic inversions of the general Kummer surface is periodic only if the Göpel even tetraedra, with respect to which the respective inversions are taken, have an edge in common.*

An immediate consequence of this is the theorem of Hutchinson* that the sixty inversions generate a group of infinite order.

* Loc. cit., p. 212.

Any line through a node of K_4 meets the surface in two other points. The transformation which interchanges these points is a monoidal cubic involution having the node for fundamental point. Under this nodal projection any plane section of K_4 is transformed into a C_{12} of genus three. Hence for the node A

$$C_4 \sim 3C_4 - 4A. \tag{9}$$

The image of the node is the intersection of the surface K_4 with the tangent cone at the node, namely a C_8 of genus zero. Hence

$$A \sim 2C_4 - 3A. \tag{10}$$

The other nodes remain invariant. Equations (9) and (10) are all that will be written to denote the complete nodal projection and will be indicated by $[A]$.

The product of two nodal projections $[B][A]$ gives

$$C_4 \sim 9C_4 - 12A - 4B, \qquad A \sim 2C_4 - 3A, \qquad B \sim 6C_4 - 8A - 3B,$$

the rest of the nodes being invariant. This transformation may be shown to be non-periodic by the methods employed above.*

If a point be transformed by a cubic inversion and its image transformed by a nodal projection, the resulting transformation is either involutorial or non-periodic. For $\left[\begin{smallmatrix}AB\\EF\end{smallmatrix}\right][A]$ gives

$$C_4 \sim 5C_4 - 6A - 2(B + E + F), \qquad A \sim 3C_4 - 4A - (B + E + F),$$

$$B \sim C_4 - A - (E + F), \qquad E \sim C_4 - A - (B + F),$$

$$F \sim C_4 - A - (B + F), \qquad (CG)(DH)(IJ)(KP)(LO)(MN).$$

This is easily verified to be involutorial. If the node of projection is not a vertex of the tetraedron of inversion, every such product is of the same type as $\left[\begin{smallmatrix}EF\\AB\end{smallmatrix}\right][C]$, under which

$$C_4 \sim 9C_4 - 12C - 2(A + B + E + F),$$

$$A \sim 3C_4 - 4C - (B + E + F), \qquad B \sim 3C_4 - 4C - (A + E + F),$$

$$E \sim 3C_4 - 4C - (A + B + F), \qquad F \sim 3C_4 - 4C - (A + B + E),$$

$$G \sim 2C_4 - 3C, \qquad C \sim G, \qquad (DH)(IJ)(KP)(LO)(MN).$$

The non-periodicity of this transformation may be established as above. Hence *the product of a cubic inversion and a nodal projection is non-periodic except when the fundamental point of the projection is one of the basis points of the inversion.*

* For another proof see V. Snyder, *An application of a* (1, 2) *quaternary correspondence to the Weddle and Kummer surfaces*, these Transactions, vol. 12 (1911), pp. 354–366; in particular, see p. 364.

2. The Weddle Surface

The Weddle surface W_4 can be obtained from the Kummer surface by birational transformation.* Under this transformation one node of the Kummer surface is transformed into a cubic curve on W_4. The six conics of K_4 through this node become the six nodes on the cubic curve on W_4. The remaining ten conics on K_4 are transformed into the ten lines of intersection of the pairs of planes determined by the six nodes on W_4. The other fifteen nodes of K_4 become the fifteen connecting lines of the six nodes of W_4. A plane section of K_4 is transformed into a curve of order eight on W_4.

In the diagram (1), if the cubic curve is taken as A, the residual elements will then be the fifteen lines joining the pairs of nodes on W_4. The six nodes will be denoted by b, c, d, e, i, and m; the remaining ten lines will be $a, f, g, h, j, k, l, n, o$, and p. These may be identified by the conics on K_4 to which they correspond. For example, through the node d pass the lines B, C, H, L, P, and the cubic curve A; the line k meets the lines C, G, I, J, L, and O.

Let variables x, y, z, w be so chosen that

$$e \equiv (1, 0, 0, 0), \qquad i \equiv (0, 1, 0, 0), \qquad m \equiv (0, 0, 1, 0),$$

$$b \equiv (0, 0, 0, 1), \qquad c \equiv (a, b, c, d), \qquad d \equiv (1/a, 1/b, 1/c, 1/d).$$

The equation of the Weddle surface, written in terms of these coördinates is (18) and is invariant under the transformation† (2). Under this transformation the pencils of planes through the edges of the tetraedron and the nodes c and d are interchanged in pairs. Hence the lines joining each vertex to c and d are interchanged. Thus

$$(CD)\,(GH)\,(KL)\,(OP). \tag{12}$$

Under (2) an edge of the tetraedron of inversion is transformed into the opposite edge, or

$$(EF)\,(IJ)\,(MN). \tag{13}$$

Further, the cubic curve A, through the six nodes, has the line B as image and conversely, thus

$$(AB). \tag{14}$$

A quadric surface through the six nodes intersects W_4 in a curve C_8 of order eight which has a double point at each node and the lines $a, f, g, h, j, k, l, n, o$, and p for bisecants. Under (2) this curve is transformed into another

* The transformation here used is employed by Snyder, loc. cit., p. 360 and is the dual of the correspondence given by Hudson, loc. cit., p. 169.

† See Hutchinson, loc. cit., p. 216.

curve of the same system, since the quadric surface is transformed into another one through the six nodes; hence

$$(C_8\, C_8)\,. \tag{15}$$

It follows from (12), (13), (14), and (15) that the complete inversion is

$$(C_8\, C_8)\ (AB)\ (CD)\ (EF)\ (GH)\ (IJ)\ (KL)\ (MN)\ (OP)\,.$$

Since C_8 on W_4 corresponds to C_4 on K_4, it follows that this is a collineation on K_4. Hence *the collineation group on* K_4 *corresponds to the inversion group on* W_4. *On* W_4 *the product of two inversions is an inversion* and the inversion group is of order* 16.

Any line through a node of W_4 meets the surface in two other points. The transformation which interchanges these points is involutorial. Consider the projection N_c from the node c. The tangent cone at the node meets W_4 in a curve of order eight, which consists of the lines B, D, G, K, and O and the cubi A, hence (Ac). A plane through B intersects W_4 in B and a cubic curve. This curve intersects B in c, d, and a residual point R which is the image of d in this plane. As the plane rotates, R describes the line B. Hence (Bd), and similarly for the four remaining lines through c. Under N_c the line M is transformed into the ine of intersection of the planes Mc and Nd, and conversely. Hence (Mo), and similarly for all remaining lines. The collected result is

$$\begin{gathered}(Ac)\ (Bd)\ (Ca)\ (Db)\ (Eg)\ (Fh)\ (Ge)\ (Hf)\\ (Ik)\ (Jl)\ (Ki)\ (Lj)\ (Mo)\ (Np)\ (Om)\ (Pn)\,.\end{gathered} \tag{16}$$

Since $C_8 = 2a + B + C + D + E + I + M$, it follows from (16) that $C_8 \sim 2C + d + a + b + g + k + o$. By writing the symbolic sums for C_8 which contain the elements a, b, d, g, k, o, these may be expressed in terms of C_8 and the elements $A, B, \cdots, P$. Hence

$$C_8 \sim 3C_8 - (A + B + \cdots + P)\,. \tag{17}$$

The transformation N_c is given by (16) and (17). In like manner the transformation N_d is given by (17) and

$$\begin{gathered}(Ad)\ (Bc)\ (Cb)\ (Da)\ (Eh)\ (Fg)\ (Gf)\ (He)\\ (Il)\ (Jk)\ (Kj)\ (Li)\ (Mp)\ (No)\ (On)\ (Pm)\,.\end{gathered}$$

* In Hutchinson's paper, loc. cit., p. 217, the product of two inversions is stated to be of infinite order. The formulas used for changing the fundamental tetraedron from $e\ i\ m\ b$ to $e\ i\ m\ c$ should read, in his notation,

$$w_1 = w\,, \qquad \sqrt{b^2 - a^2}\, x_1 = bw - ax\,, \qquad \sqrt{c^2 - a^2}\, y_1 = cw - ay\,, \qquad \sqrt{d^2 - a^2}\, z_1 = dw - az\,.$$

With this change it is seen that the product of the two is an inversion.

The product of these transformations is the inversion with respect to the remaining nodes. Hence *the cubic inversion with respect to any tetraedron of nodes on the Weddle surface is the product of the nodal projections with respect to the two residual nodes.**

The points (x, y, z, w), $(1/x, 1/y, 1/z, 1/w)$, (a, b, c, d), and $(1/a, 1/b, 1/c, 1/d)$ are coplanar if

$$\begin{vmatrix} x & y & z & w \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} & \frac{1}{w} \\ a & b & c & d \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \end{vmatrix} = 0. \tag{18}$$

This condition is satisfied if (x, y, z, w) is on the Weddle surface. In other words, a point on W_4 and its image under a cubic inversion are coplanar with the residual nodes of the tetraedron of inversion. Thus any plane through the residual nodes is invariant under an inversion, as is well known for the equivalent product of the two nodal projections.

Let the nodal projections be designated by N_i, $i = 1, 2, \cdots, 6$. The inversions will be $N_i N_k = I_{ik}$. Hence $N_i = I_{ik} N_k$. There results immediately the known theorem that the group of the nodal projections is of order 32. For it is the direct product of the inversion group of order 16 by the group generated by any nodal projection.

Cornell University,
January, 1914.

* The algebraical equivalent of this theorem is given by Baker, *Multiply Periodic Functions*, p. 154.

TRANSFORMATIONS OF SURFACES OF VOSS*

BY

LUTHER PFAHLER EISENHART

INTRODUCTION

Voss† was the first to study surfaces which are characterized by the property of containing two families of geodesics which form a conjugate system. These surfaces belong to the class admitting a continuous deformation in which a conjugate system remains conjugate; for a *surface of Voss* it is the geodesic conjugate system which possesses this property. These surfaces play an important rôle also in the determination of congruences whose developables meet the focal surfaces in their lines of curvature. It is the purpose of this paper to establish transformations of surfaces of Voss into surfaces of the same kind, and to study some of the geometrical properties of these transformations.

The geodesic conjugate system of a surface‡ V has the same spherical representation as the asymptotic lines of a pseudospherical surface P. Moreover, the determination of all surfaces V with the spherical representation of a given surface P requires the integration of an equation of Laplace with equal invariants. We shall say that each of these surfaces is *conjugate* to P. In § 1 we recall the formulas defining a pseudospherical surface P, the formulas for a conjugate surface V and the equations of a Bäcklund transformation of P into another pseudospherical surface P_1. Such a transformation involves a constant σ (the angle between the tangent planes to P and P_1) and a function θ whose determination requires the solution of a Riccati equation. When a surface V is given, each pair of quantities θ and σ determine a new surface V_1 such that the developables of the congruence of lines joining corresponding points on V and V_1 meet these surfaces in geodesic conjugate systems. Moreover, V_1 is a conjugate of the surface P_1 which is determined by the Bäcklund transformation of P by means of θ and σ. In order to put in evidence the functions we say that V_1 is obtained from V by a transformation $\Omega(\theta, \sigma)$.

* Presented to the Society, December 31, 1913.

† *Ueber diejenigen Flächen auf denen zwei Scharen geodätischer Linien ein conjugirtes System bilden,* Sitzungsberichte der K. Akademie zu München (1888), pp. 95–102.

‡ Throughout the paper we shall denote by V a surface of Voss.

Each surface V conjugate to P is transformable by the same pair of functions θ and σ. However, for a given pair (θ, σ) there exists a surface V such that the lines joining its points to the corresponding points on the surface V_1 resulting from the transformation $\Omega(\theta, \sigma)$ are concurrent. By means of certain additional functions connected with this special case we are able to show that the transformations $\Omega(\theta, \sigma)$ are of the Moutard type for equations with equal invariants. In a subsequent paper we shall show that there exist transformations of general conjugate systems with equal tangential invariants — transformations which are, in certain respects, generalizations of the transformations $\Omega(\theta, \sigma)$.

The transformations $\Omega(\theta, \sigma)$ admit a "theorem of permutability" (cf. § 6). The remainder of the paper deals with the congruence of lines joining corresponding points on V and V_1, and also the other congruence formed by the lines of intersection of the tangent planes at corresponding points on V and V_1; the latter is a normal congruence.

1. Equations of a Surface V and Preliminary Formulas

We consider a surface V referred to the geodesic conjugate system. Since this system has the same spherical representation as the asymptotic lines on a pseudospherical surface P, the linear element of this spherical representation can be given the form

$$d\sigma^2 = du^2 + 2\cos 2\omega\, du dv + dv^2, \tag{1}$$

where ω is a function of u and v satisfying the equation

$$\frac{\partial^2 \omega}{\partial u \partial v} + \sin\omega\cos\omega = 0.^* \tag{2}$$

If X_1, Y_1, Z_1 and X_2, Y_2, Z_2, denote respectively the direction-cosines of the bisectors of the angles between the parametric curves of the spherical representation, and X, Y, Z the direction-cosines of the normal to a surface V with this representation, we have

$$\begin{aligned} \frac{\partial X_1}{\partial u} &= -\frac{\partial \omega}{\partial u} X_2 - \sin\omega\, X, & \frac{\partial X_1}{\partial v} &= \frac{\partial \omega}{\partial v} X_2 + \sin\omega\, X, \\ \frac{\partial X_2}{\partial u} &= \frac{\partial \omega}{\partial u} X_1 - \cos\omega\, X, & \frac{\partial X_2}{\partial v} &= -\frac{\partial \omega}{\partial v} X_1 - \cos\omega\, X, \\ \frac{\partial X}{\partial u} &= \sin\omega\, X_1 + \cos\omega\, X_2, & \frac{\partial X}{\partial v} &= -\sin\omega\, X_1 + \cos\omega\, X_2.^\dagger \end{aligned} \tag{3}$$

* E., p. 289. A reference of this kind is to the author's *Treatise on the differential geometry of curves and surfaces*, Ginn and Company, Boston (1909).

† Cf. Bianchi, *Lezioni di geometria differenziale*, vol. 1, p. 320, Pisa (1902).

The equation satisfied by the tangential coördinates X, Y, Z, and W of the surface V is

$$\frac{\partial^2 \phi}{\partial u \partial v} + \cos 2\omega\, \phi = 0 .* \tag{4}$$

In terms of the tangential coördinates the rectangular coördinates are of the form

$$\begin{aligned} x = WX + \frac{1}{\sin 2\omega}\Bigg[& X_1 \cos \omega \left(\frac{\partial W}{\partial u} - \frac{\partial W}{\partial v} \right) \\ & + X_2 \sin \omega \left(\frac{\partial W}{\partial u} + \frac{\partial W}{\partial v} \right) \Bigg] .\dagger \end{aligned} \tag{5}$$

From these we obtain

$$\begin{aligned} \frac{\partial x}{\partial u} &= -\frac{D}{\sin 2\omega} (\cos \omega\, X_1 + \sin \omega\, X_2), \\ \frac{\partial x}{\partial v} &= \frac{D''}{\sin 2\omega} (\cos \omega\, X_1 - \sin \omega\, X_2), \end{aligned} \tag{6}$$

where

$$\begin{aligned} D &= -\left(\frac{\partial^2 W}{\partial u^2} - 2 \cot 2\omega \frac{\partial \omega}{\partial u} \frac{\partial W}{\partial u} + \frac{2}{\sin 2\omega} \frac{\partial \omega}{\partial u} \frac{\partial W}{\partial v} + W \right), \\ D'' &= -\left(\frac{\partial^2 W}{\partial v^2} + \frac{2}{\sin 2\omega} \frac{\partial \omega}{\partial v} \frac{\partial W}{\partial u} - 2 \cot 2\omega \frac{\partial \omega}{\partial v} \frac{\partial W}{\partial v} + W \right) .\ddagger \end{aligned} \tag{7}$$

The Codazzi equations for the surface V may be given the form

$$\frac{\partial D}{\partial v} - \frac{2}{\sin 2\omega} \frac{\partial \omega}{\partial u} D'' = 0, \qquad \frac{\partial D''}{\partial u} - \frac{2}{\sin 2\omega} \frac{\partial \omega}{\partial v} D = 0 .\S \tag{8}$$

When the pseudospherical surface P with the representation (1) of its asymptotic lines is subjected to a Bäcklund transformation, the linear element of the spherical representation of the asymptotic lines on the new surface P_1 is given by

$$d\sigma_1^2 = du^2 + 2 \cos 2\theta\, du dv + dv^2, \tag{9}$$

where θ is a solution of equation (2) given by

$$\begin{aligned} \sin \sigma \left(\frac{\partial \theta}{\partial u} - \frac{\partial \omega}{\partial u} \right) + (\cos \sigma + 1) \sin (\theta + \omega) &= 0, \\ \sin \sigma \left(\frac{\partial \theta}{\partial v} + \frac{\partial \omega}{\partial v} \right) + (\cos \sigma - 1) \sin (\theta - \omega) &= 0, \end{aligned} \tag{10}$$

* E., p. 415.

† E., p. 163 with the aid of equations (3).

‡ Cf. E., p. 164.

§ E., p. 200. Equations (36).

σ being the constant angle between the tangent planes to P and P_1 at corresponding points.* The direction-cosines X', Y', and Z' of the normal to P_1 are of the form

$$X' = \cos\sigma\, X + \sin\sigma\,(\sin\theta\, X_1 - \cos\theta\, X_2)\,,\dagger \tag{11}$$

and the direction-cosines $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ of the line joining corresponding points of P and P_1, and tangent to these surfaces at these points, are of the form

$$\bar{X} = \cos\theta\, X_1 + \sin\theta\, X_2\,.\ddagger \tag{12}$$

From (11) we obtain with the aid of equations (3)

$$\frac{\partial X'}{\partial u} = \sin\theta\, X_1' + \cos\theta\, X_2', \qquad \frac{\partial X'}{\partial v} = -\sin\theta\, X_1' + \cos\theta\, X_2', \tag{13}$$

where we have put

$$\begin{aligned} X_1' &= \sin\omega\left(\cos\sigma\,(\sin\theta\, X_1 - \cos\theta\, X_2) - \sin\sigma\, X\right) \\ &\qquad\qquad - \cos\omega\,(\cos\theta\, X_1 + \sin\theta\, X_2)\,, \\ X_2' &= -\cos\omega\left(\cos\sigma\,(\sin\theta\, X_1 - \cos\theta\, X_2) - \sin\sigma\, X\right) \\ &\qquad\qquad - \sin\omega\,(\cos\theta\, X_1 + \sin\theta\, X_2)\,. \end{aligned} \tag{14}$$

From these follow by differentiation

$$\begin{aligned} \frac{\partial X_1'}{\partial u} &= -\frac{\partial\theta}{\partial u} X_2' - \sin\theta\, X', & \frac{\partial X_1'}{\partial v} &= \frac{\partial\theta}{\partial v} X_2' + \sin\theta\, X', \\ \frac{\partial X_2'}{\partial u} &= \frac{\partial\theta}{\partial v} X_1' - \cos\theta\, X', & \frac{\partial X_2'}{\partial v} &= -\frac{\partial\theta}{\partial v} X_1' - \cos\theta\, X', \end{aligned} \tag{15}$$

equations which are analogous to (3). Equations (14) are equivalent to

$$\begin{aligned} \cos\theta\, X_1' + \sin\theta\, X_2' &= -\sin(\theta-\omega)\left[\cos\sigma\,(\sin\theta\, X_1 \right. \\ &\qquad \left. - \cos\theta\, X_2) - \sin\sigma\, X\right] - \cos(\theta-\omega)\,(\cos\theta\, X_1 + \sin\theta\, X_2)\,, \\ \cos\theta\, X'_1 - \sin\theta\, X_2' &= \sin(\theta+\omega)\left[\cos\sigma\,(\sin\theta\, X_1 \right. \\ &\qquad \left. - \cos\theta\, X_2) - \sin\sigma\, X\right] - \cos(\theta+\omega)\,(\cos\theta\, X_1 + \sin\theta\, X_2)\,. \end{aligned} \tag{16}$$

2. The Transformation $\Omega(\theta, \sigma)$ of the Surfaces V

Each solution W_1 of the equation

$$\frac{\partial^2 \phi_1}{\partial u \partial v} + \cos 2\theta\, \phi_1 = 0 \tag{17}$$

* Equations (10) may be obtained from equations (44) E., p. 289, by replacing $\cos\sigma$ by $-\cos\sigma$.

† This follows from E., p. 284.

‡ E., l. c.

determines a surface V with the spherical representation (9) of its geodesic conjugate system. The rectangular coördinates of the corresponding surface V_1 are of the form

$$
(18)\qquad x_1 = W_1 X' + \frac{1}{\sin 2\theta}\left[X_1' \cos\theta \left(\frac{\partial W_1}{\partial u} - \frac{\partial W_1}{\partial v} \right) + X_2' \sin\theta \left(\frac{\partial W_1}{\partial u} + \frac{\partial W_1}{\partial v} \right) \right],
$$

where X', X_1', and X_2' have the significance given by equations (11) and (14).

It is our purpose to show that when a surface V whose coördinates are given by (5) is known, it is possible to find a surface V_1 defined by (18), such that the developables of the congruence formed by the joins of corresponding points of V and V_1 cut these surfaces in their respective geodesic conjugate systems.

If we write the coördinates of any point on a line of this congruence in the form

$$
(19)\qquad x + t(x_1 - x), \qquad y + t(y_1 - y), \qquad z + t(z_1 - z),
$$

it must be possible to find two values of t, say t_1 and t_2, such that

$$
(20)\qquad \begin{aligned} \frac{(1-t_1)\dfrac{\partial x}{\partial u} + t_1 \dfrac{\partial x_1}{\partial u}}{x_1 - x} &= \frac{(1-t_1)\dfrac{\partial y}{\partial u} + t_1 \dfrac{\partial y_1}{\partial u}}{y_1 - y} = \frac{(1-t_1)\dfrac{\partial z}{\partial u} + t_1 \dfrac{\partial z_1}{\partial u}}{z_1 - z}, \\ \frac{(1-t_2)\dfrac{\partial x}{\partial v} + t_2 \dfrac{\partial x_1}{\partial v}}{x_1 - x} &= \frac{(1-t_2)\dfrac{\partial y}{\partial v} + t_2 \dfrac{\partial y_1}{\partial v}}{y_1 - y} = \frac{(1-t_2)\dfrac{\partial z}{\partial v} + t_2 \dfrac{\partial z_1}{\partial v}}{z_1 - z}. \end{aligned}
$$

The necessary and sufficient condition that these equations be consistent is that the following equations be satisfied:

$$
(21)\qquad \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial y_1}{\partial u} & \dfrac{\partial z_1}{\partial u} \\ x_1 - x & y_1 - y & z_1 - z \end{vmatrix} = 0, \qquad \begin{vmatrix} \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x_1}{\partial v} & \dfrac{\partial y_1}{\partial v} & \dfrac{\partial z_1}{\partial v} \\ x_1 - x & y_1 - y & z_1 - z \end{vmatrix} = 0.
$$

The equations for V_1 analogous to (6) are

$$
(22)\qquad \begin{aligned} \frac{\partial x_1}{\partial u} &= -\frac{D_1}{\sin 2\theta}(\cos\theta\, X_1' + \sin\theta\, X_2'), \\ \frac{\partial x_1}{\partial v} &= \frac{D_1''}{\sin 2\theta}(\cos\theta\, X_1' - \sin\theta\, X_2'). \end{aligned}
$$

Again with the aid of equations (11) and (14) we obtain from (5) and (18) the expression

$$x_1 - x = AX + BX_1 + CX_2, \tag{23}$$

where

$$\begin{aligned}
A &= W_1 \cos\sigma - W + \frac{\sin\sigma}{\sin 2\theta}\left(\sin(\theta-\omega)\frac{\partial W_1}{\partial u} + \sin(\theta+\omega)\frac{\partial W_1}{\partial v}\right),\\
B &= W_1 \sin\sigma \sin\theta - \frac{\cos\sigma\sin\theta}{\sin 2\theta}\left(\sin(\theta-\omega)\frac{\partial W_1}{\partial u}\right.\\
&\qquad \left. + \sin(\theta+\omega)\frac{\partial W_1}{\partial v}\right) - \frac{\cos\omega}{\sin 2\omega}\left(\frac{\partial W}{\partial u} - \frac{\partial W}{\partial v}\right)\\
&\qquad + \frac{\cos\theta}{\sin 2\theta}\left(\cos(\theta+\omega)\frac{\partial W_1}{\partial v} - \cos(\theta-\omega)\frac{\partial W_1}{\partial u}\right),\\
C &= -W_1 \sin\sigma\cos\theta + \frac{\cos\sigma\cos\theta}{\sin 2\theta}\left(\sin(\theta-\omega)\frac{\partial W_1}{\partial u}\right.\\
&\qquad \left. + \sin(\theta+\omega)\frac{\partial W_1}{\partial v}\right) - \frac{\sin\omega}{\sin 2\omega}\left(\frac{\partial W}{\partial u} + \frac{\partial W}{\partial v}\right)\\
&\qquad + \frac{\sin\theta}{\sin 2\theta}\left(\cos(\theta+\omega)\frac{\partial W_1}{\partial v} - \cos(\theta-\omega)\frac{\partial W_1}{\partial u}\right).
\end{aligned} \tag{24}$$

When these values are substituted in (21) and use is made of (16) in the reduction, we obtain

$$A\cos(\theta-\omega)(\cos\sigma - 1) + B\sin\omega\sin\sigma - C\cos\omega\sin\sigma = 0,$$

$$A\cos(\theta+\omega)(\cos\sigma + 1) - B\sin\omega\sin\sigma - C\cos\omega\sin\sigma = 0.$$

In consequence of (24) these equations, and consequently equations (21), are equivalent to

$$\begin{aligned}
\frac{\partial W_1}{\partial u} + \frac{\cos\sigma + 1}{\sin\sigma}\cos(\theta+\omega)W_1 &= -\frac{\partial W}{\partial u} + \frac{1+\cos\sigma}{\sin\sigma}\cos(\theta+\omega)W,\\
\frac{\partial W_1}{\partial v} + \frac{\cos\sigma - 1}{\sin\sigma}\cos(\theta-\omega)W_1 &= \frac{\partial W}{\partial v} + \frac{1-\cos\sigma}{\sin\sigma}\cos(\theta-\omega)W.
\end{aligned} \tag{25}$$

Since θ satisfies equations (10), the condition of integrability of equations (25) is satisfied, as is readily shown, and furthermore the function W_1 so defined is a solution of equation (17). We shall find shortly (§ 5) that the integration of equations (25) requires one quadrature only. Hence we have

THEOREM I. *Each solution of equations* (10) *determines a transformation* $\Omega(\theta, \sigma)$ *of a given surface* V *into a surface* V_1 *such that the developables of the congruence formed by the joins of corresponding points on* V *and* V_1 *cut these*

surfaces in the geodesic conjugate systems. Moreover, when θ is known, the further determination of V_1 requires only a quadrature.

The second fundamental coefficients of V_1, namely D_1, D_1'', are given by expressions similar to (7). In consequence of (25) these can be given the form

$$
(26)\qquad
\begin{aligned}
D_1 &= -D - \frac{2A}{\sin\sigma\sin 2\omega}\left(2\cos(\theta-\omega)\frac{\partial\omega}{\partial u} - \frac{\cos\sigma+1}{\sin\sigma}\sin 2\omega\right),\\
D_1'' &= D'' - \frac{2A}{\sin\sigma\sin 2\omega}\left(2\cos(\theta+\omega)\frac{\partial\omega}{\partial v} + \frac{1-\cos\sigma}{\sin\sigma}\sin 2\omega\right).
\end{aligned}
$$

3. The Congruence Associated with a Transformation $\Omega(\theta, \sigma)$

With the aid of (25) the expressions (24) are reducible to

$$
(27)\qquad
\begin{aligned}
A &= \frac{\sin\sigma}{\sin 2\theta}\left(\frac{\sin 2\omega}{\sin\sigma}(W_1 - W\cos\sigma) - \sin(\theta-\omega)\frac{\partial W}{\partial u}\right.\\
&\qquad\qquad \left. + \sin(\theta+\omega)\frac{\partial W}{\partial v}\right),\\
B &= \frac{A}{\sin\sigma\sin\omega}(\cos\theta\cos\omega - \cos\sigma\sin\theta\sin\omega),\\
C &= \frac{A}{\sin\sigma\cos\omega}(\cos\theta\cos\omega\cos\sigma - \sin\theta\sin\omega).
\end{aligned}
$$

From these expressions and (23) it follows that the direction-cosines α, β, and γ of the line joining corresponding points on V and V_1 are of the form

$$
(28)\qquad
\begin{aligned}
\alpha\rho = {} & \cos\omega(\cos\theta\cos\omega - \cos\sigma\sin\omega\sin\theta)X_1\\
& + \sin\omega(\cos\theta\cos\omega\cos\sigma - \sin\theta\sin\omega)X_2\\
& + \sin\sigma\sin\omega\cos\omega X,
\end{aligned}
$$

where

$$
(28)\qquad \rho = \sqrt{2}\sqrt{1 + \cos 2\omega\cos 2\theta - \cos\sigma\sin 2\omega\sin 2\theta}.
$$

Since the expressions for these direction-cosines involve only the functions of the Bäcklund transformation, we have

Theorem II. *If V and V' are any two surfaces conjugate to P, and V_1 and V_1' are obtained by transformations involving the same functions θ and σ, the developables of the congruences of lines joining corresponding points of V and V_1 and of V' and V_1' have the same spherical representation.*

In § 4 we shall find that in certain cases these congruences coincide.

In the tangent plane to V at a point M we draw the line l parallel to the line joining corresponding points of the pseudospherical surfaces P and P_1 determining a transformation of V. The direction-cosines $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ of

this line are of the form

$$\bar{X} = \cos\theta\, X_1 + \sin\theta\, X_2. \tag{29}$$

The direction-cosines $\bar{X}'$, $\bar{Y}'$, and $\bar{Z}'$ of the line in this plane perpendicular to l are of the form

$$\bar{X}' = -\sin\theta\, X_1 + \cos\theta\, X_2. \tag{30}$$

With the aid of these expressions equation (28) may be written

$$\begin{aligned} 2\alpha\rho = \sin\sigma \sin 2\omega\, X + (\cos 2\omega + \cos 2\theta)\,\bar{X} \\ + (-\sin 2\theta + \cos\sigma \sin 2\omega)\, X'. \end{aligned} \tag{31}$$

In view of this expression we are able to give the following construction for the direction of the line through M upon which lies the point M_1 of the transformed surface V_1. In the tangent plane to V at M we lay off a unit vector which makes the angle -2θ with the line l, and in the plane through l and inclined at the angle σ to the tangent plane we lay off a unit vector which makes the angle 2ω with l; the vector which is the sum of these two vectors has the desired direction.

It is our purpose now to determine the abscissas of the focal points of the congruence of joins of corresponding points on V and V_1. To this end we observe that in consequence of (20) we have

$$\frac{(1-t_1)\sum X'\dfrac{\partial x}{\partial u}}{\sum X'(x_1 - x)} = \frac{t_1 \sum X \dfrac{\partial x_1}{\partial u}}{\sum X(x_1 - x)}, \qquad \frac{(1-t_2)\sum X'\dfrac{\partial x}{\partial v}}{\sum X'(x_1 - x)} = \frac{t_2 \sum X \dfrac{\partial x_1}{\partial v}}{\sum X(x_1 - x)}.$$

It is readily shown that

$$\sum X \frac{\partial x_1}{\partial u} = -\frac{D_1}{\sin 2\theta} \sin\sigma \sin(\theta - \omega),$$

$$\sum X \frac{\partial x_1}{\partial v} = -\frac{D_1''}{\sin 2\theta} \sin\sigma \sin(\theta + \omega),$$

$$\sum X' \frac{\partial x}{\partial u} = -\frac{D}{\sin 2\omega} \sin\sigma \sin(\theta - \omega),$$

$$\sum X' \frac{\partial x}{\partial v} = \frac{D''}{\sin 2\omega} \sin\sigma \sin(\theta + \omega).$$

In consequence of the values (27) we have also

$$\sum X'(x_1 - x) = \frac{\sin 2\theta}{\sin 2\omega} A.$$

Substituting these expressions in the above equations, we obtain

$$(1 - t_1) D = t_1 D_1, \qquad (1 - t_2) D'' = -t_2 D_1'',$$

from which we derive the following:

$$
(32)\qquad
\begin{aligned}
t_1 &= \frac{D}{D + D_1}, & 1 - t_1 &= \frac{D_1}{D + D_1}, \\
t_2 &= \frac{D''}{D'' - D_1''}, & 1 - t_2 &= \frac{-D_1''}{D'' - D_1''}.
\end{aligned}
$$

From these values it follows that the cross-ratio of the points M and M_1 on V and V_1 and the foci F_1 and F_2 of their joins is

$$
(33)\qquad R(MM_1, F_1F_2) = -\frac{DD_1''}{D''D_1}.
$$

Substituting the values (32) in (19), we find that the coördinates of the focal points are of the form

$$
(34)\qquad \frac{xD_1 + x_1D}{D_1 + D}, \quad \text{etc.}; \qquad \frac{xD_1'' - x_1D''}{D_1'' - D''}, \quad \text{etc.}
$$

Hence we have

THEOREM III. *The focal points of the lines joining corresponding points M and M_1 on the surfaces V and V_1 divide the segment MM_1 in the ratios D/D_1 and $-D''/D_1''$.*

4. Special Surfaces of Voss

In the preceding discussion we have ignored the possibility that the joins of corresponding points on V and V_1 may pass through a point, in which case any ruled surface of the congruence is developable and Theorem I loses part of its significance. It is our purpose now to consider this possibility. We shall find that every surface V does not admit of such a transformation. We use w to denote the function W for a surface V which possesses this property.

If we take the origin for the point of concurrence of all the rays, we must have

$$
x + t(x_1 - x) = 0, \qquad y + t(y_1 - y) = 0, \qquad z + t(z_1 - z) = 0.
$$

When the expressions from (5) and (23) are substituted in these equations, we obtain three conditions of the form

$$
LX + MX_1 + NX_2 = 0,
$$

which coexist only in case $L = M = N = 0$. This gives the three equations of condition

$$
(35)\qquad
\begin{aligned}
&w + At = 0, \qquad \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} + 2\sin\omega t B = 0, \\
&\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} + 2\cos\omega t C = 0.
\end{aligned}
$$

If we solve the last two equations for $\partial w/\partial u$ and $\partial w/\partial v$ and substitute the values of B and C from (27), we have

$$\frac{\partial w}{\partial u} + tA\,\frac{(1+\cos\sigma)}{\sin\sigma}\cos(\theta+\omega) = 0, \qquad \frac{\partial w}{\partial v} + tA\,\frac{\cos\sigma - 1}{\sin\sigma}\cos(\theta-\omega).$$

Hence equations (35) may be replaced by the first of them and

$$\frac{\partial \log w}{\partial u} = \frac{1+\cos\sigma}{\sin\sigma}\cos(\theta+\omega), \qquad \frac{\partial \log w}{\partial v} = \frac{\cos\sigma - 1}{\sin\sigma}\cos(\theta-\omega). \tag{36}$$

From the equations (25) for the determination of w_1 it follows that

$$ww_1 = \text{const.} \tag{37}$$

Hence each solution θ of equations (10) leads by a quadrature to a surface V, which we call a *special surface of Voss* and denote by $V(\theta, \sigma)$, such that there is another surface of the same kind so related that the lines joining corresponding points of the two surfaces are concurrent. We observe that the relation between these surfaces is reciprocal, and consequently we say that each is the *adjoint* of the other.

It is evident that if there is applied to a special surface $V(\theta, \sigma)$ a transformation involving a different θ, the relation between the two surfaces will be that of Theorem I.

When the values (36) are substituted in equations (7), we find that the expressions for the second fundamental coefficients Δ, Δ'' of $V(\theta, \sigma)$ are reducible to

$$\begin{aligned} \Delta &= \frac{2w}{\sin\sigma}\left(\frac{2\cos(\theta-\omega)}{\sin 2\omega}\frac{\partial\omega}{\partial u} - \frac{1+\cos\sigma}{\sin\sigma}\right), \\ \Delta'' &= -\frac{2w}{\sin\sigma}\left(\frac{2\cos(\theta+\omega)}{\sin 2\omega}\frac{\partial\omega}{\partial v} + \frac{1-\cos\sigma}{\sin\sigma}\right). \end{aligned} \tag{38}$$

From (26) it follows that the coefficients Δ_1, Δ_1'' of the adjoint surface are given by

$$\Delta_1 = -\Delta\left(1+\frac{A}{w}\right), \qquad \Delta_1'' = \Delta''\left(1+\frac{A}{w}\right). \tag{39}$$

Since we have

$$\Delta_1\Delta'' + \Delta_1''\Delta = 0, \tag{40}$$

it follows that the asymptotic lines on a special surface of Voss correspond to a conjugate system on its adjoint.

We consider the converse problem, that is, we suppose that D, D'', D_1, D_1'' satisfy an equation of the form (40). From (26) it follows that either $A = 0$,

or there exists a function λ such that

$$(41)\qquad \begin{aligned} D &= \frac{\lambda}{\sin 2\omega}\left(2\cos(\theta-\omega)\frac{\partial\omega}{\partial u} - \frac{1+\cos\sigma}{\sin\sigma}\sin 2\omega\right), \\ D'' &= -\frac{\lambda}{\sin 2\omega}\left(2\cos(\theta+\omega)\frac{\partial\omega}{\partial v} + \frac{1-\cos\sigma}{\sin\sigma}\sin 2\omega\right). \end{aligned}$$

If we take $A = 0$ and substitute in (25) the value of W_1 so determined, we find that $D = D'' = 0$, which evidently is impossible.

When the values (41) are substituted in the Codazzi equations (8), we find

$$\lambda = cw.$$

Hence V is a special surface of Voss, and we have

THEOREM IV. *A necessary and sufficient condition that a surface of Voss be "special" is that it admit a transformation Ω such that the asymptotic lines of the given surface V correspond to a conjugate system on the surface V_1.*

5. EQUATIONS OF A TRANSFORMATION $\Omega(\theta, \sigma)$ IN ANOTHER FORM

In consequence of equations (36) we may put the fundamental equations (25) in the form

$$(42)\qquad \begin{aligned} \frac{\partial}{\partial u}(W_1 w) &= -w\frac{\partial W}{\partial u} + \frac{\partial w}{\partial u}W, \\ \frac{\partial}{\partial v}(W_1 w) &= w\frac{\partial W}{\partial v} - \frac{\partial w}{\partial v}W. \end{aligned}$$

This shows that the transformation $\Omega(\theta, \sigma)$ is of the general class of transformations of Moutard of equations with equal invariants.* It is a well-known fact that three linearly independent solutions ν_1, ν_2, ν_3 of an equation of the form

$$(43)\qquad \frac{\partial^2\theta}{\partial u\partial v} = M\theta$$

determine a surface Σ referred to its asymptotic lines; its coördinates ξ, η, ζ are given by the formulas of Lelieuvre†

$$(44)\qquad \frac{\partial\xi}{\partial u} = \begin{vmatrix} \nu_2 & \nu_3 \\ \dfrac{\partial\nu_2}{\partial u} & \dfrac{\partial\nu_3}{\partial v} \end{vmatrix}, \qquad \frac{\partial\xi}{\partial v} = -\begin{vmatrix} \nu_2 & \nu_3 \\ \dfrac{\partial\nu_2}{\partial v} & \dfrac{\partial\nu_3}{\partial v} \end{vmatrix}.$$

If a fourth solution of equation (43) is known, say θ_1, the functions $\bar{\nu}_i$, defined

* Bianchi, *Lezioni di geometria differenziale*, vol. 2, p. 47; also Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, p. 145.

† E., p. 193.

by the quadratures

$$(45)\qquad \frac{\partial}{\partial u}(\theta_1 \bar{\nu}_i) = -\begin{vmatrix} \theta_1 & \nu_i \\ \dfrac{\partial \theta_1}{\partial u} & \dfrac{\partial \nu_i}{\partial u} \end{vmatrix}, \qquad \frac{\partial}{\partial v}(\theta_1 \bar{\nu}_i) = \begin{vmatrix} \theta_1 & \nu_i \\ \dfrac{\partial \theta_1}{\partial v} & \dfrac{\partial \nu_i}{\partial v} \end{vmatrix} \quad (i = 1, 2, 3)$$

satisfy the differential equation

$$(43')\qquad \frac{\partial^2 \theta}{\partial u \partial v} = \theta_1 \frac{\partial^2}{\partial u \partial v} \frac{1}{\theta_1} \cdot \theta .$$

Moreover, the surface Σ_1, determined by equations in $\bar{\nu}_i$ analogous to (44), can be so placed in space that Σ and Σ_1 are the focal surfaces of a W-congruence.*

Since the total curvature of Σ is given by†

$$K = \frac{-1}{(\nu_1^2 + \nu_2^2 + \nu_3^2)^2},$$

if Σ and Σ_1 are to be pseudospherical surfaces of curvature $-(1/a^2)$ in the relation of a transformation of Bäcklund,

$$\nu_1 = \sqrt{a}X, \qquad \nu_2 = \sqrt{a}Y, \qquad \nu_3 = \sqrt{a}Z,$$

$$\bar{\nu}_1 = \sqrt{a}X', \qquad \bar{\nu}_2 = \sqrt{a}Y', \qquad \bar{\nu}_3 = \sqrt{a}Z',$$

where X', Y', Z' are given by (11). When these values are substituted in (45), we find that θ_1 equals w, given by (36), to within a constant factor. Now, equations (43) and (43′) are respectively (4) and (17). Hence we have

THEOREM V. *A transformation $\Omega(\theta, \sigma)$ is a transformation of Moutard in which the transforming function is the tangential coördinate w of the corresponding special surface $V(\theta, \sigma)$.*

It is a fundamental property of W-congruences that each focal surface admits an infinitesimal deformation parallel to the normal to the other surface. The surface corresponding with orthogonality of linear elements to Σ, defined by (44), in this case has for coördinates ‡

$$\theta_1 \bar{\nu}_1, \qquad \theta_1 \bar{\nu}_2, \qquad \theta_1 \nu_3 .$$

Moreover, in the general case $\theta_1 \sqrt[4]{-K}$ is the tangential coördinate W of the corresponding associate surface. Hence we have

THEOREM VI. *A special surface $V(\theta, \sigma)$ associated with a pseudospherical surface P is that associate surface in the infinitesimal deformation of P whose directrices are parallel to the corresponding normals to the Bäcklund transform of P by means of θ and σ.*

* E., pp. 418, 419.
† E., p. 194.
‡ E., p. 420.

Since $W_1 w$ is determined by (42) only to within an additive constant, and since also $1/w$ is a solution of equation (17), if V_1 is a transform of V by a transformation $\Omega(\theta, \sigma)$, so also is the suite of surfaces whose tangential coördinates are

$$X', \quad Y', \quad Z', \quad W_1 + \frac{a}{w},$$

where a is an arbitrary constant.

In like manner it may be observed that V_1 is obtained by a transformation $\Omega(\theta, \sigma)$ from any one of the surfaces whose tangential coördinates are

$$X, \quad Y, \quad Z, \quad W + aw,$$

provided it is true when $a = 0$.

We have seen that when a surface $V(\theta, \sigma)$ is transformed by $\Omega(\theta, \sigma)$, the tangential coördinates w and w_1 are such that their product is constant. We shall show that this property is characteristic of special surfaces. In fact, if we differentiate the equation

$$WW_1 = \text{const.}$$

with respect to u and v respectively and make use of equations (42), we get

$$(W_1 - W)\frac{\partial}{\partial u}\log\frac{W}{w} = 0, \qquad (W_1 - W)\frac{\partial}{\partial v}\log\frac{W}{w} = 0.$$

Hence V is a special surface $V(\theta, \sigma)$ and the transformation is $\Omega(\theta, \sigma)$. We shall say that in this case the two special surfaces are in perspective correspondence.

6. Theorem of Permutability of the Transformations Ω

It is our purpose to show that the transformations which have been established in §2 admit a "theorem of permutability" as follows:

Theorem VII. *If two surfaces V_1 and V_2 are obtained from a surface V by means of transformations $\Omega(\theta_1, \sigma_1)$ and $\Omega(\theta_2, \sigma_2)$ respectively, there exists a surface V' which may be obtained from V_1 by a transformation $\Omega(\phi, \sigma_2)$ and from V_2 by a transformation $\Omega(\phi, \sigma_1)$; moreover, the determination of V' does not involve integration.*

If W_1 and W_2 are tangential coördinates of V_1 and V_2 respectively, we must have in accordance with (42)

$$(46) \qquad \begin{aligned} \frac{\partial W_i}{\partial u} + \frac{W_i}{w_i}\frac{\partial w_i}{\partial u} &= -\frac{\partial W}{\partial u} + \frac{W}{w_i}\frac{\partial w_i}{\partial u}, \\ \frac{\partial W_i}{\partial v} + \frac{W_i}{w_i}\frac{\partial w_i}{\partial v} &= \frac{\partial W}{\partial v} - \frac{W}{w_i}\frac{\partial w_i}{\partial v} \end{aligned} \qquad (i = 1, 2),$$

where the functions w_1 and w_2 are given by

$$
(47)\qquad \begin{aligned} \frac{\partial \log w_i}{\partial u} &= \frac{1+\cos\sigma_i}{\sin\sigma_i}\cos(\theta_i+\omega), \\ \frac{\partial \log w_i}{\partial v} &= \frac{\cos\sigma_i - 1}{\sin\sigma_i}\cos(\theta_i-\omega) \end{aligned} \qquad (i=1,2),
$$

and

$$
(48)\qquad \begin{aligned} \sin\sigma_i\left(\frac{\partial\theta_i}{\partial u}-\frac{\partial\omega}{\partial u}\right)+(\cos\sigma_i+1)\sin(\theta_i+\omega)&=0, \\ \sin\sigma_i\left(\frac{\partial\theta_i}{\partial v}+\frac{\partial\omega}{\partial v}\right)+(\cos\sigma_i-1)\sin(\theta_i-\omega)&=0 \end{aligned} \qquad (i=1,2).
$$

On the assumption that a surface V' exists satisfying the requirements of the above theorem, there must exist functions W', w_1', and w_2' satisfying equations analogous to (46), namely

$$
(49)\qquad \begin{aligned} \frac{\partial W'}{\partial u}+\frac{W'}{w_i'}\frac{\partial w_i'}{\partial u} &= -\frac{\partial W_i}{\partial u}+\frac{W_i}{w_i'}\frac{\partial w_i'}{\partial u}, \\ \frac{\partial W'}{\partial v}+\frac{W'}{w_i'}\frac{\partial w_i'}{\partial v} &= \frac{\partial W_i}{\partial v}-\frac{W_i}{w_i'}\frac{\partial w_i'}{\partial v} \end{aligned} \qquad (i=1,2).
$$

By means of (46) these are reducible to

$$
(50)\qquad \begin{aligned} \frac{\partial W'}{\partial u}+W'\frac{\partial \log w_i'}{\partial u} &= W_i\cdot\frac{\partial}{\partial u}\log(w_i'w_i)+\frac{\partial W}{\partial u}-W\frac{\partial \log w_i}{\partial u}, \\ \frac{\partial W'}{\partial v}+W'\frac{\partial \log w_i'}{\partial v} &= -W_i\cdot\frac{\partial}{\partial v}\log(w_i'w_i)+\frac{\partial W}{\partial v}-W\frac{\partial \log w_i}{\partial v} \end{aligned} \qquad (i=1,2).
$$

The consistency of the first equations for $i=1$ and $i=2$, and likewise for the second equations, requires that

$$
(51)\qquad \begin{aligned} W'\frac{\partial}{\partial u}\log\frac{w_1'}{w_2'} - W_1\frac{\partial}{\partial u}\log(w_1'w_1) \\ + W_2\frac{\partial}{\partial u}\log(w_2'w_2)+W\frac{\partial}{\partial u}\log\frac{w_1}{w_2}&=0, \\ W'\frac{\partial}{\partial v}\log\frac{w_1'}{w_2'} + W_1\frac{\partial}{\partial v}\log(w_1'w_1) \\ - W_2\frac{\partial}{\partial v}\log(w_2'w_2)+W\frac{\partial}{\partial v}\log\frac{w_1}{w_2}&=0. \end{aligned}
$$

Since V_1 and V_2 are general transforms of V, the coexistence of equations

(51) necessitates the conditions

$$
(52)\qquad
\begin{aligned}
&\frac{\partial}{\partial v}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial u}\log w_1' w_1 + \frac{\partial}{\partial u}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial v}\log w_1' w_1 = 0,\\
&\frac{\partial}{\partial v}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial u}\log w_2' w_2 + \frac{\partial}{\partial u}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial v}\log w_2' w_2 = 0,\\
&\frac{\partial}{\partial v}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial u}\log\frac{w_2}{w_1} - \frac{\partial}{\partial u}\log\frac{w_1'}{w_2'}\frac{\partial}{\partial v}\log\frac{w_2}{w_1} = 0.
\end{aligned}
$$

This system may be replaced by the first and the following equations

$$
(53)\qquad
\begin{aligned}
\frac{\partial}{\partial u}(w_1' w_1) &= -\mu\left(w_1\frac{\partial w_2}{\partial u} - w_2\frac{\partial w_1}{\partial u}\right),\\
\frac{\partial}{\partial u}(w_2' w_2) &= +\lambda\left(w_1\frac{\partial w_2}{\partial u} - w_2\frac{\partial w_1}{\partial u}\right),\\
\frac{\partial}{\partial v}(w_1' w_1) &= \mu\left(w_1\frac{\partial w_2}{\partial v} - w_2\frac{\partial w_1}{\partial v}\right),\\
\frac{\partial}{\partial v}(w_2' w_2) &= -\lambda\left(w_1\frac{\partial w_2}{\partial v} - w_2\frac{\partial w_1}{\partial v}\right),
\end{aligned}
$$

where λ and μ are functions to be determined. When these values are substituted in the first of equations (52) we find that it may be replaced by

$$
(54)\qquad \mu w_2 w_2' + \lambda w_1 w_1' = 0.
$$

In consequence of (53) and (54) equations (51) reduce to

$$
(55)\qquad W' = (W_1 - W_2)\lambda\frac{w_1}{w_2'} + W = (W_2 - W_1)\mu\frac{w_2}{w_1'} + W.
$$

If we substitute these expressions in (50), we find that λ and μ must be constant.

It is necessary now that we determine whether functions w_2' and w_1' exist which satisfy equations (53) and also equations analogous to (47), namely

$$
(56)\qquad
\begin{aligned}
\frac{\partial \log w_i'}{\partial u} &= \frac{1+\cos\sigma_k}{\sin\sigma_k}\cos(\phi+\theta_i),\\
\frac{\partial \log w_i'}{\partial v} &= \frac{\cos\sigma_k - 1}{\sin\sigma_k}\cos(\phi-\theta_i)
\end{aligned}
\qquad (i,\ k = 1,\ 2;\ i \neq k),
$$

where the function ϕ must satisfy the equations

$$
(57)\qquad
\begin{aligned}
&\sin\sigma_i\left(\frac{\partial\phi}{\partial u} - \frac{\partial\theta_k}{\partial u}\right) + (\cos\sigma_i + 1)\sin(\phi+\theta_k) = 0,\\
&\sin\sigma_i\left(\frac{\partial\phi}{\partial v} + \frac{\partial\theta_k}{\partial v}\right) + (\cos\sigma_i - 1)\sin(\phi-\theta_k) = 0
\end{aligned}
\qquad (i,\ k = 1,\ 2;\ i \neq k).
$$

From the "theorem of permutability" of transformations of Bäcklund of pseudospherical surfaces it follows that there exists a function ϕ satisfying the four equations (57); it is given by *

$$
(58)\qquad
\begin{aligned}
\sin(\phi-\omega) &= \frac{(\cos\sigma_1-\cos\sigma_2)\sin(\theta_2-\theta_1)}{\sin\sigma_1\sin\sigma_2\cos(\theta_2-\theta_1)+\cos\sigma_1\cos\sigma_2-1},\\
\cos(\phi-\omega) &= \frac{\sin\sigma_1\sin\sigma_2+(\cos\sigma_1\cos\sigma_2-1)\cos(\theta_2-\theta_1)}{\sin\sigma_1\sin\sigma_2\cos(\theta_2-\theta_1)+\cos\sigma_1\cos\sigma_2-1}.
\end{aligned}
$$

Since w_1' and w_2' are given by (56) only to within a constant factor, it follows from (53) that the constants μ and λ may be taken equal to unity, and consequently equations (53) may be written

$$
(59)\qquad
\begin{aligned}
\frac{\partial}{\partial u}(w_i' w_i) &= w_k\frac{\partial w_i}{\partial u}-w_i\frac{\partial w_k}{\partial u},\\
\frac{\partial}{\partial v}(w_i' w_i) &= -w_k\frac{\partial w_i}{\partial v}+w_i\frac{\partial w_k}{\partial v}
\end{aligned}
\qquad (i,\ k=1,\ 2;\ i\neq k).
$$

If we reduce these equations for $i=1$, $k=2$ by means of (56) and (47), we obtain

$$
(60)\qquad
\begin{aligned}
&w_1'\left(\frac{1+\cos\sigma_2}{\sin\sigma_2}\cos(\phi+\theta_1)+\frac{1+\cos\sigma_1}{\sin\sigma_1}\cos(\theta_1+\omega)\right)\\
&\qquad = -w_2\left(\frac{1+\cos\sigma_2}{\sin\sigma_2}\cos(\theta_2+\omega)-\frac{1+\cos\sigma_1}{\sin\sigma_1}\cos(\theta_1+\omega)\right),\\
&w_1'\left(\frac{\cos\sigma_2-1}{\sin\sigma_2}\cos(\phi-\theta_1)+\frac{\cos\sigma_1-1}{\sin\sigma_1}\cos(\theta_1-\omega)\right)\\
&\qquad = w_2\left(\frac{\cos\sigma_2-1}{\sin\sigma_2}\cos(\theta_2-\omega)-\frac{\cos\sigma_1-1}{\sin\sigma_1}\cos(\theta_1-\omega)\right).
\end{aligned}
$$

If we add these equations, the result is reducible to

$$
(61)\qquad
\begin{aligned}
&\frac{w_1'}{w_2}\Big[(\cot\sigma_2\cos\phi+\cot\sigma_1\cos\omega)\cos\theta_1\\
&\qquad -\left(\frac{1}{\sin\sigma_2}\sin\phi+\frac{1}{\sin\sigma_1}\sin\omega\right)\sin\theta_1\Big]\\
&\quad =\left(-\frac{\cos\theta_2}{\sin\sigma_2}+\frac{\cos\theta_1}{\sin\sigma_1}\right)\cos\omega+(\cot\sigma_2\sin\theta_2-\cot\sigma_1\sin\theta_1)\sin\omega.
\end{aligned}
$$

By means of the values of ϕ given by (58) this equation becomes

$$
(62)\qquad \frac{w_1'}{w_2}=\frac{\sin\sigma_1\sin\sigma_2\cos(\theta_2-\theta_1)+\cos\sigma_1\cos\sigma_2-1}{\cos\sigma_2-\cos\sigma_1}.
$$

* E., p. 287.

The same result is obtained if equations (60) be subtracted from each other and in the resulting equation the values of the functions of ϕ given by (58) be substituted. It is readily shown with the aid of (58) that this expression for w_1' satisfies equations (56) with $i = 1, k = 2$.

If we proceed in a similar manner with equations (59) with $i = 2, k = 1$, we obtain

$$\frac{w_2'}{w_1} = -\frac{\sin\sigma_1 \sin\sigma_2 \cos(\theta_2 - \theta_1) + \cos\sigma_1 \cos\sigma_2 - 1}{\cos\sigma_2 - \cos\sigma_1}. \tag{63}$$

These values of w' and w_2' are in agreement with (54) which now is

$$w_2' w_2 + w_1' w_1 = 0. \tag{64}$$

From (55) it follows that the tangential coördinate W' of the surface V is

$$W' = W + \frac{(W_2 - W_1)(\cos\sigma_2 - \cos\sigma_1)}{\sin\sigma_1 \sin\sigma_2 \cos(\theta_2 - \theta_1) + \cos\sigma_1 \cos\sigma_2 - 1}. \tag{65}$$

Hence the "theorem of permutability," as stated at the beginning of this section, has been completely established.

Suppose that $W = w_1$ and $W_1 = 1/w_1$; it is evident that the character of W_2 and W' depend upon the choice of the function w_2. If we require that V_2 and V' shall be special surfaces of Voss such that the joins of corresponding points are concurrent, in all generality we can put $W' W_2 = 1$. It follows at once from (65) and (63) that $W_2 = w_2'$. Now V_1 and V' are not in perspective correspondence. Otherwise we should have $w_1' w_1 =$ const., which is inconsistent with (59).

If four surfaces are related in accordance with Theorem VII, we say that they form a *quatern*. In view of the above results we have

THEOREM VIII. *If V, V_1, V_2, V' form a quatern of surfaces of Voss determined by a set of functions w_1, w_2, w_1', w_2', the special surfaces of Voss determined by the functions w_1, $1/w_1$, w_2', $1/w_2'$ form a quatern, and so likewise do the special surfaces determined by the functions w_2, $1/w_2$, w_1', $1/w_1'$.*

Conversely,

THEOREM IX. *If two surfaces V and V_1 of a quatern (V, V_1, V_2, V') are special surfaces in perspective correspondence, the necessary and sufficient condition that V_2 and V' be special surfaces in perspective correspondence is that these surfaces be determined by w_2' and $1/w_2'$, where w_2' is given by* (63), *σ_2 being any constant and θ_2 a solution of the corresponding equations* (10).

7. Normal Congruences Determined by Transformations Ω

We turn now to the consideration of the congruence of the lines of intersection of the tangent planes at the corresponding points M and M_1 on two

surfaces V and V_1, of which the latter arises from V by a transformation Ω. From the general theory of congruences* we know that the tangents at M and M_1 to the geodesics v = const. through these respective points meet in a point F_1; likewise, the tangents at M and M_1 to the geodesics u = const. meet in a point F_2; furthermore, these points F_1 and F_2 are the focal points of their join in the congruence formed by these lines of intersection of the tangent planes to V and V_1 at M and M_1.

If the coördinates of F_1 and F_2 be denoted by $a_1, b_1, c_1; a_2, b_2, c_2$ respectively,

$$(66)\qquad \begin{aligned} a_1 &= x + \lambda_1(\cos\omega\, X_1 + \sin\omega\, X_2) = x_1 + \mu_1(\cos\theta\, X_1' + \sin\theta\, X_2'),\\ a_2 &= x + \lambda_2(\cos\omega\, X_1 - \sin\omega\, X_2) = x_1 + \mu_2(\cos\theta\, X_1' - \sin\theta\, X_2'), \end{aligned}$$

where $\lambda_1, \mu_1, \lambda_2, \mu_2$ are quantities to be determined. From (16) and (23) it follows that the equivalence of the two expressions for a_1, b_1, and c_1 requires that λ_1 and μ_1 satisfy

$$A + \mu_1 \sin\sigma \sin(\theta - \omega) = 0,$$

$$B - \mu_1[\sin(\theta - \omega)\cos\sigma \sin\theta + \cos(\theta - \omega)\cos\theta] - \lambda_1 \cos\omega = 0,$$

$$C + \mu_1[\sin(\theta - \omega)\cos\sigma \cos\theta - \cos(\theta - \omega)\sin\theta] - \lambda_1 \sin\omega = 0.$$

It is found that these equations are consistent in view of the values (27) of A, B, and C.

From the above equations we find

$$(67)\qquad \lambda_1 = \frac{A \sin 2\theta}{\sin\sigma \sin 2\omega \sin(\theta - \omega)}, \qquad \mu_1 = \frac{-A}{\sin\sigma \sin(\theta - \omega)}.$$

In like manner it can be shown that

$$(68)\qquad \lambda_2 = \frac{A \sin 2\theta}{\sin\sigma \sin 2\omega \sin(\theta + \omega)}, \qquad \mu_2 = \frac{A}{\sin\sigma \sin(\theta + \omega)}.$$

We shall need the expressions for $\partial A/\partial u$ and $\partial A/\partial v$. They are most easily obtained indirectly by differentiating equation (23) with respect to u and v separately and equating to zero the coefficients of X in each case. These expressions are reducible to

$$(69)\qquad \begin{aligned} \frac{\partial A}{\partial u} &= -\frac{D_1}{\sin 2\theta}\sin\sigma \sin(\theta - \omega) + \frac{A(1 + \cos\sigma)\cos(\theta + \omega)}{\sin\sigma},\\ \frac{\partial A}{\partial v} &= \frac{-D_1''}{\sin 2\theta}\sin\sigma \sin(\theta + \omega) + \frac{A(\cos\sigma - 1)\cos(\theta - \omega)}{\sin\sigma}. \end{aligned}$$

If we substitute in (66) the values of λ_1 and λ_2 given above, we derive by

* Cf. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, pp. 230, 231.

means of (69) the following:

$$
(70)\quad
\begin{aligned}
\frac{\partial a_1}{\partial u} &= \frac{\lambda_1 \sin 2\omega}{\sin(\theta-\omega)}\left(\frac{\cos\sigma+1}{\sin\sigma}(\cos\omega\, X_1+\sin\omega\, X_2)-\sin(\theta-\omega)\,X\right),\\
\frac{\partial a_1}{\partial v} &= \left(2\lambda_1\frac{\partial\omega}{\partial v}-D''\right)\frac{(\cos\theta\, X_1+\sin\theta\, X_2)}{\sin(\theta-\omega)},
\end{aligned}
$$

$$
(71)\quad
\begin{aligned}
\frac{\partial a_2}{\partial u} &= -\left(2\lambda_2\frac{\partial\omega}{\partial u}+D\right)\frac{(\cos\theta\, X_1+\sin\theta\, X_2)}{\sin(\theta+\omega)},\\
\frac{\partial a_2}{\partial v} &= \frac{\lambda_2 \sin 2\omega}{\sin(\theta+\omega)}\left(\frac{1-\cos\sigma}{\sin\sigma}(\cos\omega\, X_1-\sin\omega\, X_2)+\sin(\theta+\omega)\,X\right).
\end{aligned}
$$

These equations show incidentally that F_1 and F_2 are the focal points of the congruence.

We denote by Σ_1 and Σ_2 the focal surfaces of the congruence, that is the loci of the points F_1 and F_2, and by A_1, B_1, C_1, A_2, B_2, and C_2 the direction-cosines of the normals to Σ_1 and Σ_2. From (70) and (71) we find

$$
(72)\quad
\begin{aligned}
A_1 &= \left(\sin\theta\, X_1-\cos\theta\, X_2+\frac{1+\cos\sigma}{\sin\sigma}X\right)\frac{\sin\sigma}{\sqrt{2(1+\cos\sigma)}},\\
A_2 &= \left(\sin\theta\, X_1-\cos\theta\, X_2+\frac{\cos\sigma-1}{\sin\sigma}X\right)\frac{\sin\sigma}{\sqrt{2(1-\cos\sigma)}},
\end{aligned}
$$

and similar expressions for the B's and C's.

Hence we have

$$
(73)\quad
\begin{aligned}
\sum XA_1 &= \cos\frac{\sigma}{2}, & \sum X'A_1 &= \cos\frac{\sigma}{2},\\
\sum XA_2 &= -\sin\frac{\sigma}{2}, & \sum X'A_2 &= \sin\frac{\sigma}{2}.
\end{aligned}
$$

Consequently we have

THEOREM X. *For the congruence of the lines of intersection of the tangent planes at corresponding points on V_1 and V the focal planes bisect the angles between the tangent planes, and consequently the congruence is normal.*

From (66) it follows that

$$
(74)\quad a_1-a_2=\frac{A\sin 2\theta}{\sin\sigma\sin(\theta-\omega)\sin(\theta+\omega)}(X_1\cos\theta+X_2\sin\theta),
$$

consequently the focal distance 2ρ is given by

$$
(75)\quad 2\rho=\frac{A\sin 2\theta}{\sin\sigma\sin(\theta-\omega)\sin(\theta+\omega)}.
$$

7. Converse Problem

Suppose that we have a pseudospherical congruence, that is the congruence of lines joining points of a pseudospherical surface P to the corresponding points of a Bäcklund transform P_1. Using the notation of the preceding sections, we note that the direction-cosines of this congruence, namely $\bar{X}$, $\bar{Y}$, and $\bar{Z}$, are of the form

$$\bar{X} = \cos\theta\, X_1 + \sin\theta\, X_2. \tag{76}$$

By differentiation we obtain

$$\begin{aligned} \frac{\partial \bar{X}}{\partial u} &= \sin(\theta+\omega)\left[(\sin\theta\, X_1 - \cos\theta\, X_2)\frac{\cos\sigma+1}{\sin\sigma} - X\right], \\ \frac{\partial \bar{X}}{\partial v} &= \sin(\theta-\omega)\left[(\sin\theta\, X_1 - \cos\theta\, X_2)\frac{\cos\sigma-1}{\sin\sigma} - X\right]. \end{aligned} \tag{77}$$

From these expressions we derive the following expressions for the coefficients of the linear element of the spherical representation of this congruence:

$$\sqrt{\mathcal{E}} = \frac{\sin(\theta+\omega)}{\sin\frac{1}{2}\sigma}, \qquad \mathcal{F} = 0, \qquad \sqrt{\mathcal{G}} = \frac{\sin(\theta-\omega)}{\cos\frac{1}{2}\sigma}. \tag{78}$$

It must be remembered that the parametric curves on the unit sphere correspond to the asymptotic lines on the focal surfaces of the congruence. Since this system on the sphere is orthogonal, it represents the lines of curvature of a group of surfaces $\bar{S}$ whose fundamental functions $\bar{D}$ and $\bar{D}''$ satisfy the Codazzi equations

$$\frac{\partial}{\partial v}\left(\frac{\bar{D}}{\sqrt{\mathcal{E}}}\right) = \frac{\bar{D}''}{\mathcal{G}}\frac{\partial\sqrt{\mathcal{E}}}{\partial v}, \qquad \frac{\partial}{\partial u}\left(\frac{\bar{D}''}{\sqrt{\mathcal{G}}}\right) = \frac{\bar{D}}{\mathcal{E}}\frac{\partial\sqrt{\mathcal{G}}}{\partial u}. \tag{79}$$

Each set of functions $\bar{D}$, $\bar{D}''$ satisfying these equations gives a surface $\bar{S}$ by means of the quadratures *

$$\frac{\partial \bar{x}}{\partial u} = -\frac{\bar{D}}{\mathcal{E}}\frac{\partial \bar{X}}{\partial u}, \qquad \frac{\partial \bar{x}}{\partial v} = -\frac{\bar{D}''}{\mathcal{G}}\frac{\partial \bar{X}}{\partial v}. \tag{80}$$

Suppose we have such a surface $\bar{S}$. Through each normal we draw planes parallel to the corresponding tangent planes to the pseudospherical surfaces P and P_1. The tangential coördinates W and W_1 of the envelopes of these planes are given by

$$W = \sum \bar{x}X, \qquad W_1 = \sum \bar{x}X'.$$

* E., p. 161.

With the aid of (79) and (80) we show that

$$\frac{\partial^2 W}{\partial u \partial v} + \cos 2\omega\, W = 0\,, \qquad \frac{\partial^2 W_1}{\partial u \partial v} + \cos 2\theta\, W_1 = 0\,.$$

Hence the envelopes of these planes are surfaces of Voss, and we have

THEOREM XI. *If P and P_1 are any two pseudospherical surfaces in the relation of a Bäcklund transformation, there exists a family of surfaces $\bar{S}$ each of which has its normals parallel to lines joining corresponding points of P and P_1, and, in this correspondence between the surfaces $\bar{S}$, P, and P_1, the lines of curvature on the first correspond to the asymptotic lines on the last two surfaces. Moreover, the two planes through each normal to $\bar{S}$ and parallel to the tangent planes to P and P_1 at the corresponding points envelope two surfaces of Voss which are in the relation of a transformation Ω.*

PRINCETON,
January 12, 1914.

BIRATIONAL TRANSFORMATIONS OF CERTAIN QUARTIC SURFACES*

BY

F. R. SHARPE AND VIRGIL SNYDER

1. Statement of the problem. It has been shown by Enriques† that if an algebraic surface contains a pencil of elliptic curves and also two curves cutting each curve of the pencil in two groups of points which are not equivalent and which do not possess equivalent multiples, the surface is invariant under a discontinuous infinite series of birational transformations. In a more recent memoir‡ the same author states that, with the exception of the case in which

$$p_g = p_a = P_2 = p^{(1)} = 1,$$

the converse is also true, since transversal curves of each curve of the pencil can always be found. Since the publication of the second theorem the author has discovered that the conclusion must be modified.§ An interesting example of a quartic surface having an infinite discontinuous group of birational transformations but not containing any elliptic curves has been found by Fano.|| In the following paper it is proposed to study the possibility and the nature of birational transformations of quartic surfaces which pass through one or more given curves. The method of the basis of a system of curves on the surface will be employed, as developed by Severi.¶

* Presented to the Society, February 28, 1914.

† F. Enriques, *Sulle superficie algebriche, che ammettono una serie discontinua di trasformazioni birazionali*, Rendiconti della Reale Accademia dei Lincei, ser. 5, vol. 15 (1906), pp. 665–669.

‡ F. Enriques, *Sulle superficie algebriche con un fascio di curve ellittiche*, Rendiconti della Reale Accademia dei Lincei, ser. 5, vol. 21 (1912), pp. 14–17.

§ A. Rosenblatt, *Sur certaines classes des surfaces algébriques irregulières et sur les transformations birationnelles de ces surfaces en ellesmêmes*, Bulletin de l'académie des sciences de Cracovie, Juillet, 1912, pp. 761–810. See footnote 1, page 802.

|| G. Fano, *Sopra alcune superficie del 4. ordine rappresentabili sul piano doppio*, Rendiconti del Reale Istituto Lombardo, ser. 2, vol. 39 (1906), pp. 1071–1086.

¶ F. Severi, *Sulla totalità delle curve algebriche tracciate sopra una superficie algebrica*, Mathematische Annalen, vol. 62 (1906), pp. 194–225.

1. Quartic Surfaces Containing a Straight Line

2. **General case.** If a quartic surface contains a straight line it contains a pencil of plane elliptic cubics. If it is otherwise unrestricted, the surface contains no other rational curves, and consequently no birational transformations into itself.

3. **Rational curve on the surface.** Let the quartic surface contain the straight line C_1 and a rational curve C_n of order n meeting C_1 in $n-1$ points. Any plane through C_1 contains a plane cubic curve C_3 on F_4 that meets C_n in one point K. The involution defined by projecting C_3 upon itself from K leaves C_3 invariant. Associated with this involution is a Cremona transformation of the Jonquières type, having the polar conic of K with respect to C_3 for invariant curve. This Jonquières transformation will be understood when we speak of the involution. As K describes C_n the involution defines a Cremona transformation which leaves F_4 invariant and transforms every C_3 of the pencil into itself. When the equations of the surface and of C_1 and C_n are known, the equations of the transformation can be readily derived.

The system of curves C_1, $|C_3|$, and C_n constitute a basis, since we are concerned with the most general surface which satisfies the conditions.* The values of $[C_i, C_k]$ are

$$[C_1, C_1] = -2, \qquad [C_3, C_3] = 0, \qquad [C_n, C_n] = -2,$$

$$[C_1, C_3] = 3, \qquad [C_1, C_n] = n-1, \qquad [C_3, C_n] = 1.$$

In terms of the above curves the transformation may be expressed symbolically as follows.

Let C_1, C_3, C_n go into $\overline{C}_1$, $\overline{C}_3$, $\overline{C}_n$ respectively. The equations connecting the old and the new curves are found to be

$$T. \qquad \begin{aligned} \overline{C}_1 &= 2C_1 + 2(n+1)C_3 - 3C_n, \\ \overline{C}_3 &= C_3, \\ \overline{C}_n &= C_1 + 2(n+1)C_3 - 2C_n, \end{aligned}$$

by the following considerations. To obtain $\overline{C}_1$ we determine the image of C_1 as K describes C_n. Draw the lines from K to the three points of intersection of C_1 and the plane cubic curve defined by C_1 and K. As K describes C_n these joining lines generate a ruled surface R containing $\overline{C}_1$. On this ruled surface C_n is a threefold curve, and C_1 is an $(r-3)$-fold straight line, r being the order of R. The intersection of R with F_4 is a curve of order $4r$, consisting of C_n counted three times, of C_1 counted $r-2$ times, since the surfaces touch each other at every point of C_1, and of a residual curve, the

* Severi, l. c.

image of C_1. The curve C_n meets C_1 in $n-1$ points, corresponding to each one of which C_1 counts as a double generator of R, hence $r-4=2n-2$. The residual curve therefore has the symbol

$$2(n+1)(C_3+C_1)-2nC_1-3C_n$$

and is of order $2(n+1)4-2n-3n$.

It is evident that each plane cubic curve C_3 goes into itself.

The image of C_n is the locus of the first tangential of K as K describes C_n. The tangents to $|C_3|$ at the points of C_n describe a ruled surface of order $2(n+1)$ passing through C_n and touching F_4 at each point of C_n, and having C_1 as a $(2n+1)$-fold line. The residual intersection is the required image of C_n.

From K four tangents can be drawn to C_3 besides the one at K. As K describes C_n the four tangents generate a ruled surface R_{2n+6} of order $2n+6$, having C_n for a fourfold curve and C_1 for a multiple line of order $2n+2$, as may be seen as follows: The tangent cone to F_4 from any point P on C_1 consists of the tangent plane at P counted twice and of a cone of order 10 on which C_1 is fourfold. This cone intersects the projecting cone of C_n from P in C_1 counted $4(n-1)$ times and in $6n+4$ other lines. These $6n+4$ lines are composed of two groups; those of the first are the $2n+1$ tangents to C_3 at points of C_n, each counted twice, and the remaining lines belong to the second group. The lines of the second group are generators of the ruled surface in question. The residual intersection of R_{2n+6} with F_4 is a curve C counted twice that remains invariant under the transformation T. It is defined symbolically by the equation

$$2C=(2n+6)(C_1+C_3)-4C_n-(2n+2)C_1,$$

or

$$C=2C_1+(n+3)C_3-2C_n.$$

As K describes C_n its first tangential K' describes a rational curve. By projecting each C_3 from the corresponding K' upon itself, another Cremona transformation T' is defined. Since by T' both $|C_3|$ and C_n remain invariant, and since

$$[\bar{C}_1,\bar{C}_1]=-2,\qquad [\bar{C}_1,\bar{C}_3]=3,\qquad [\bar{C}_1,\bar{C}_n]=n-1,$$

we easily obtain the equations

$$T'.\qquad \bar{C}_1=-C_1+2(n+5)C_3+6C_n,\qquad \bar{C}_3=C_3,\qquad \bar{C}_n=C_n.$$

The determinants of T, T' are each -1, and their product is 1; the operation (TT') is not periodic. Thus the surface is invariant under an infinite number of birational transformations. In any plane the operation (TT')

can be replaced by the product of two projections of the curve upon itself, the center of one of which may be chosen anywhere on the curve. But the operation does not define a continuous group, since the C_3 associated with K has not a constant modulus as K describes C_n.*

Moreover, it can be shown by the same method as that employed by Severi† that all the effective transformations that leave F_4 invariant can be expressed in terms of T and T'.

4. **One double point on the line.** Let all the curves C_3 pass through one fixed point P on C_1, the other points of intersection of C_1, C_3 being variable.

The surface is evidently invariant under the projection of the surface upon itself from the point P; the projection is an involutional Cremona transformation of the Jonquières type. The surface is also invariant under the transformation defined by the first tangential of P as in the preceding case. Surfaces of this type necessarily contain rational curves which cut every plane C_3 of the pencil in one point apart from the point P.

Symbolically, the transformation is expressed as follows. For basis we may choose C_1, C_3 and the point P with the following characteristic‡

$$[C_1, C_1] = -2, \qquad [C_1, C_3] = 2, \qquad [C_3, C_3] = 0,$$
$$[C_1, P] = 1, \qquad [C_3, P] = 1, \qquad [P, P] = -2.$$

The point P is a double point on F_4. The tangent cone to F_4 at P is a quadric passing through C_1; the residual intersection is a rational curve of order 7 which passes through P five times. We have therefore

$$[C_7, P] = 5, \qquad [C_7, C_1] = 1, \qquad [C_7, C_3] = 1, \qquad [C_7, C_7] = -2,$$

so that the equations of the projection from P are easily found to be

$$S. \qquad \begin{aligned} \bar{C}_1 &= C_1, \\ \bar{C}_3 &= C_3, \\ \bar{P} &= C_1 + 2C_3 - P. \end{aligned}$$

The equations of the transformation defined by the first tangential are similarly found to be

$$S'. \qquad \begin{aligned} \bar{C}_1 &= -C_1 + 10C_3 + 4P, \\ \bar{C}_3 &= C_3, \\ \bar{P} &= P. \end{aligned}$$

* Picard et Simart, *Fonctions algébriques de deux variables indépendentes*, vol. 2, pp. 517–518.

† Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

‡ For the idea of using a point as part of a basis, see F. Severi, *La base minima pour la totalité des courbes tracés sur une surface algébrique*, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), pp. 449–468.

The product (SS') is not periodic, hence the surface is invariant under a group of infinite order.

5. **Line an inflexional tangent.** The cases in which two distinct or coincident points of intersection of C_3 and C_1 are fixed offer no new peculiarities; similarly for the cases of three distinct fixed points or of two coincident and one distinct fixed points. But the case in which all three points are coincident merits special attention. If C_1 is an inflexional tangent for every C_3 passing through it, the point of inflexion must be the same point for every curve of the pencil, as is easily seen by considering the binary cubic whose roots define the point of contact.

An equation of the form

$$\phi(x_2 : x_3) + mf(x_2 : x_3) = 0$$

must be a cube in $x_2 : x_3$ for every value of m; this is possible only when $\phi \equiv kf$, k being a constant.

The equation of the most general quartic surface Φ_4 passing through a line $(x_1 = 0, x_4 = 0)$ and having the property that every plane section (cubic curve) through the line will have a fixed point of inflexion $(0, 0, 1, 0)$ on the line and the line itself for inflexional tangent, is of the form

$$\Phi_4 \equiv x_1 F_3(x_1, x_2, x_3, x_4) + x_4 \phi_3(x_2, x_3, x_4) = 0,$$

where

$$F_3 = \sum_{i=1}^{4} \sum_{k-1}^{4} \sum_{l=1}^{4} a_{ikl} x_i x_k x_l, \qquad \phi_3 = \sum_{i=2}^{4} \sum_{k=2}^{4} \sum_{l=2}^{4} b_{ikl} x_i x_k x_l,$$

and

$$a_{223} = a_{233} = a_{333} = b_{223} = b_{233} = b_{333} = 0.$$

The tangent plane to F_4 at any point of the line $x_1 = 0$, $x_4 = 0$, except the point $(0, 0, 1, 0)$, is $a_{222} x_1 + b_{222} x_4 = 0$, and is a fixed plane throughout the line. It is no restriction to call this plane $x_1 = 0$, so that $b_{222} = 0$.

The equation of the surface now has the simpler form

$$x_1 F_3 + x_4^2 \phi_2 = 0,$$

where

$$\phi_2 \equiv \sum_{i=2}^{4} \sum_{k-2}^{4} b_{ik} x_i x_k.$$

The point $(0, 0, 1, 0)$ is a biplanar point, the equations of the two tangent planes being

$$a_{133} x_1^2 + a_{334} x_1 x_4 + b_{334} x_4^2 = 0.$$

One of these planes may be taken for $x_4 = 0$, by putting $a_{133} = 0$. The surface may be mapped on a double plane $(x_3 = 0)$ by projecting from the double point $(0, 0, 1, 0)$.

The curve of branch points is the intersection of the cone of tangents from the double point. By arranging the equation of the surface in the form

$$u_2 x_3^2 + 2u_3 x_3 + u_4 = 0,$$

where $u_2 = x_4(a_{334} x_1 + b_{334} x_4)$, $u_3 =$ quadratic in x_1, x_4,

$$u_4 = \text{quadratic in } x_1, x_4 + a_{222} x_1 x_2,$$

the equation of the cone may be written in the form

$$u_3^2 - u_2 u_4 \equiv K_6 = p_3 x_2^3 + p_4 x_2^3 + p_5 x_2 + p_6 = 0,$$

where

$$p_3 = a_{222} x_1 x_4 (a_{334} x_1 + b_{33} x_4).$$

If now a curve exists on F_4 which meets every plane cubic through the line $x_1 = 0$, $x_4 = 0$, in one point, it must be rational and meet the line in $n - 1$ points.

Projected from $(0, 0, 1, 0)$, the curve lies on the cone $qx_2 - q' = 0$, in which q, q' are binary in x_1, x_4 of orders $n - 1$, n respectively.

Since every generator of the cone meets Φ_4 in one other point, the locus of this residual point is another rational curve lying on Φ_4. The first necessary condition that $qx_2 - q' = 0$ shall meet Φ_4 in two distinct curves is that Q shall be a contact curve of K_6. If this condition is satisfied, the equation obtained by eliminating x_2 between K_6 and Q has only double roots, so that

$$\frac{p_3 q'^3 + p_4 \cdot q'^2 q + p_5 q' \cdot q^2 + p_6 q^3}{q^3}$$

is of the form L^2/q^3.

These conditions can be satisfied by properly choosing the constants in q' and q. The second condition is that the cone from $(0, 1, 0, 0)$ through the curve of intersection of Φ_4 and Q shall be composite.

By writing the equation of Φ_4 in the form

$$(u_2 x_2 + u_3)^2 = K_6$$

and making it simultaneous with Q, we have

$$q^3 (u_2 x_2 + u_3)^2 = L^2.$$

This will be composite when and only when q is a square, hence a further condition is stated in the following

THEOREM. *If a rational curve exists on Φ_4 cutting every cubic curve through $x_1 = 0$, $x_4 = 0$ in one point, it must be of odd order.*

If q is a square, it can contain only $n/2 - 1$ constants, while q' contains $n + 2$ constants, and $\frac{3}{2}(n + 2)$ relations among the coefficients must be satisfied.

Since

$$n + 2 + \frac{n}{2} - 1 = \frac{3n + 2}{2},$$

it follows that there are not enough constants to satisfy the conditions, hence: *The general quartic surface through the line* $x_1 = 0$, $x_4 = 0$, *having a point of inflexion at* $(0, 1, 0, 0)$ *and* $x_1 = 0$, $x_4 = 0$ *for inflexional tangent on the cubic curve in every plane through the line, does not contain any rational curve which cuts every cubic curve once.*

The surface does not have an infinite number of birational transformations under which it is invariant, but only the involution defined by projecting Φ_4 upon itself from the point P.

6. **Inflexional tangent obtained by transformation.** More generally, given a surface F_m of order m, having the line $x_1 = 0$, $x_4 = 0$ for an $(m - 3)$-fold line and a rational curve meeting every plane section through the line once, it is possible to transform the surface into a surface of order m' having an $(m' - 3)$-fold straight line on it, and having the further property that every plane cubic section through the line will have the line for inflexional tangent.

At any point P of the rational curve the tangent t and any other line n through P can be chosen in terms of which the equation of the cubic may be expressed in the form

$$lt - n^2 + \cdots = 0, \qquad l(p) \neq 0.$$

The conic $lt - n^2 = 0$ has three point contact with the cubic at P, and meets it in three other points P_1, P_2, P_3. By using the triangle $P_1 P_2 P_3$ for triangle of quadratic inversion, the cubic will go over into a cubic and the conic into a straight line having at the point P_1' (the image of P) three point contact with the image cubic. Since a net of conics can be drawn having three point contact with the cubic at P, there are two independent parameters, enough to make the image line any line in the plane. The coördinates of P_1, P_2, P_3 enter the equations of transformation symmetrically, and all the operations can be expressed rationally.

But it has been seen that, if every C_3 has C_1 for inflexional tangent, the point of contact of C_3 with C_1 is fixed.

It is thus shown that a surface containing a pencil of plane cubic curves and a rational curve cutting each curve of the pencil once can be transformed into another having the given line for inflexional tangent of every plane cubic section through it. The converse, however, is not true. In the transformed surface sufficient restrictions are imposed on the coefficients to permit the existence of a rational curve, and hence of an infinite number. These curves are the transforms of the locus of the successive tangentials of P on the original surface.

2. Quartic Surfaces Containing a Conic

7. F_4 **with two coplanar conics.** If one plane section of a quartic surface is composed of two conics, the number of linearly independent curves on the surface is two.*

Since neither conic is a multiple of the other, the two conics C_2 and C_2' form a basis. The surface is regular and of geometric genus 1, and hence the grade n and the genus π of any curve C on the surface satisfy the relation

$$n = 2\pi - 2 .$$

The curves C are self-adjoint and of dimensions† π. In the preceding cases the question of the existence of a birational transformation did not need to be considered, as each transformation could be determined by geometric considerations. In the present case the existence is not so evident, and we shall proceed algebraically.

Since the conics C_2, C_2' are rational and meet in four points, any two curves $\lambda C_2 + \mu C_2'$, $\lambda' C_2 + \mu' C_2'$ meet in i points, defined by

$$i = -2\lambda\lambda' + 4(\lambda\mu' + \lambda'\mu) - 2\mu\mu',$$

and the grade n of $\lambda C_2 + \mu C_2'$ is given by

$$n = -2\lambda^2 + 8\lambda\mu - 2\mu^2 .‡$$

Hence $\lambda C_2 + \mu C_2'$, which is of order $2\lambda + 2\mu$, meets C_2 in $4\mu - 2\lambda$ points and meets C_2' in $4\lambda - 2\mu$ points. It follows that λ and μ are positive integers and that neither can exceed twice the other except the cases $\lambda = 1$, $\mu = 0$ and $\lambda = 0$, $\mu = 1$. We shall now show that C_2, C_2' form a minimum basis§ so that all curves on F_4 are expressible in the form $\lambda C_2 + \mu C_2'$.

Suppose Γ and Γ' constitute a minimum basis. Then

$$C_2 = \alpha\Gamma + \beta\Gamma',$$

$$C_2' = \gamma\Gamma + \delta\Gamma',$$

in which α, β, γ, δ are integers. If $|\Gamma|$ is of grade a, $|\Gamma'|$ of grade b, and any curve $|\Gamma|$ meets $|\Gamma'|$ in h points, the grade of any curve $l\Gamma + m\Gamma'$ is

$$al^2 + 2hlm + bm^2 .$$

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* Bagnera and de Franchis, *Le nombre ρ de M. Picard pour les surfaces hyperelliptiques*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 185–238. See page 187.

† F. Severi, *Complementi alla theoria delle base per la totalità delle curve di una superfice algebrica*, Rendiconti del circolo matematico di Palermo, vol. 30 (1911), pp. 265–288, p. 276.

‡ F. Severi, Mathematische Annalen, loc. cit., see p. 222.

§ F. Severi, Rendiconti di Palermo, loc. cit., p. 277.

Since $\lambda C_2 + \mu C_2'$ may be written in the form

$$(\lambda\alpha + \mu\gamma)\Gamma + (\lambda\beta + \mu\delta)\Gamma',$$

we have, by making this identical with $l\Gamma + m\Gamma'$,

$$a(\alpha\lambda + \gamma\mu)^2 + 2h(\alpha\lambda + \gamma\mu)(\beta\lambda + \delta\mu) + b(\beta\lambda + \delta\mu)^2$$
$$= -2\lambda^2 + 8\lambda\mu - 2\mu^2.$$

Hence

$$(ab - h^2)(\alpha\delta - \beta\gamma)^2 = -12.$$

Since a, b, and h are integers, it follows that $\alpha\delta - \beta\gamma = \pm 1$ or ± 2. In the former case C_2 and C_2' also form a minimum basis. In any case a and b are even, hence the value ± 2 is impossible, since it requires h^2 to be of the form $4p + 3$, in which p is an integer. We have therefore proved that the conics C_2 and C_2' form a minimum basis. The following table of possible curves on F_4 can now be found.

λ or μ	μ or λ	Order	Genus
1	0	2	0
1	1	4	3
2	1	6	4
2	2	8	9

The curves $\lambda C_2 + \mu C_2'$, $\lambda \geqq \mu$, are the residual intersections of F_4 with surfaces of order λ having C_2' for a $(\lambda - \mu)$-fold curve. All the formulas can be readily verified by the Cayley-Noether equivalence and postulation theorems.* Hence the conics C_2 and C_2' are the only rational curves on the surface; the curves of genus two are composed of the partial series of tangent planes; the only curves of genus three are the plane sections $|C_2 + C_2'|$, etc.

If a birational transformation exists that leaves F_4 invariant, every linear system of curves is transformed into another linear system of the same grade, genus, and dimension. Hence the plane sections must be transformed into plane sections and the two conics either interchanged or left invariant. This is possible only under a linear transformation, and the general quartic surface containing a conic does not admit such a transformation. We may therefore conclude: *The most general quartic surface subjected only to the condition that it must contain a conic is not invariant under any birational transformation.*†

* M. Noether, *Sulle curve multiple di superficie algebriche*, Annali di matematica, ser. 2, vol. 5 (1871), pp. 163–177. See p. 164 and 168.

† L. Godeaux, *Sur la surface du quatrième ordre contenant une conique*, Société des sciences du Hainaut, vol. 62 (1912), pp. 1–8, has considered the problem of possible birational transformations of the quartic surface containing a conic. In consequence of the restriction $\lambda \not> 2\mu$ (in our notation) which he overlooks, the system of rational curves there found are all composite. The proof of the non-existence of involutional transformations is also incomplete because not all possible cases are considered.

3. Quartics Containing One or Two Space Cubic Curves

8. F_4 **through a space cubic curve.** Let F_4 contain a space cubic curve C_3. From any point P on F_4 can be drawn one bisecant to C_3 which meets F_4 in a fourth point P'. Since the bisecant to C_3 from P' meets F_4 in P, we see that F_4 admits an involutional birational transformation. Through C_3 pass a bundle of quadric surfaces, each meeting F_4 in a residual quintic of genus 2. All the quadrics of the bundle which pass through a fixed point P belong to a pencil whose basis curve consists of C_3 and the bisecant through P. Hence all the quintic curves through P on F_4 also pass through P'.

A basis is composed of C_3 and a plane section C_4. By proceeding as in the preceding cases we obtain the transformation

$$U. \qquad \begin{aligned} \overline{C}_4 &= 9C_4 - 5C_3, \\ \overline{C}_3 &= 16C_4 - 9C_3, \end{aligned}$$

which can be verified as follows:

The image of a plane section C_4 is obtained by constructing the ruled surface of bisecants to C_3 from each point of C_4. The curve C_4 is a simple curve on the surface, and the lines joining the three points of intersection of C_3 and C_4 in pairs are double generators. Hence the surface is of order 10. From any point of C_3 a quadric cone of lines can be drawn meeting C_3 again. This cone meets the quartic cone to C_4 in eight lines, but three of them pass through the points of intersection of C_3 and C_4; hence C_3 is a five-fold curve on the ruled surface. Since $\overline{C}_4$ is the residual intersection of F_4 and the ruled surface, we have

$$\overline{C}_4 = 10C_4 - (C_4 + 5C_3).$$

The image of C_3 is obtained as the residual intersection with F_4 of the ruled surface obtained by connecting each point K of C_3 with the point K' in which the tangent plane to F_4 at K meets C_3 again.

From K can be drawn one generator to K'; from K' can be drawn six generators to points of tangency K. Moreover, the ruled surface and F_4 touch each other at every point K of C_3; hence each generator meets 14 others, and the surface is of order 16. The curve C_3 counts for nine intersections, and hence the residual intersection C_3 is defined by the formula

$$\overline{C}_3 = 16C_4 - 9C_3.$$

The other possible values of λ and μ which satisfy the congruences analogous to those of § 2 lead to non-effective curves. Hence the surface does not admit an infinite number of birational transformations.

The transformation is a Cremona transformation, the invariant surface

being the locus of the double points of the involution determined on C_3 and F_4 by the bisecants of C_3.

9. F_4 **through two space cubic curves.** Consider the general quartic surface containing the two space cubic curves C_3 and C_3' which intersect in k points $0 \leqq k \leqq 5$. A basis may be made of C_3, C_3' and a plane section C_4. The involution defined by transforming P into the residual point P' of intersection with the surface of the bisecant of C_3 through P may be expressed by the equations

$$V. \qquad \begin{aligned} \bar{C}_3 &= 16C_4 - 9C_3, \\ \bar{C}_3' &= (12 - 2k)C_4 - (6 - k)C_3 - C_3', \\ \bar{C}_4 &= 9C_4 - 5C_3. \end{aligned}$$

The similar involution as to C_3' has the equations

$$V'. \qquad \begin{aligned} \bar{C}_3 &= (12 - 2k)C_4 - C_3 - (6 - k)C_3', \\ \bar{C}_3' &= 16C_4 - 9C_3', \\ \bar{C}_4 &= 9C_4 - 5C_3'. \end{aligned}$$

The product of these two involutions is a discontinuous birational transformation that is not periodic.

When $k = 5$, the curves C_3 and C_3' lie on a quadric which meets F_4 in a conic C_2. By the transformation V the curves C_3' and C_2 are interchanged.

Similarly, the curves C_3 and C_2 are interchanged by the transformation V'.

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ONE-PARAMETER FAMILIES OF CURVES IN THE PLANE*

BY

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Introduction

Nets of curves in the plane were studied from a projective point of view for the first time by Wilczynski.† As he himself points out,‡ the theory as set up by him is that of the "two component one-parameter families, rather than a theory of the net itself." In the present paper, we shall show how a single one-parameter family of plane curves may be studied by Wilczynski's methods. Although many of the projective properties of a single one-parameter family may be investigated by considering it as a component family of a particular net,—such properties are in fact noted in Wilczynski's memoir—still the theory of one of the families of the net is closely bound up with the properties of the other, if the study be made as in the paper already cited. To make this clearer, we must outline Wilczynski's procedure.

If $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ be interpreted as the homogeneous coördinates of a point in the plane, the equations

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3) \tag{1}$$

define for u = const. and v = const. two one-parameter families of curves. Equations (1) are a fundamental system of solutions of the completely integrable system of partial differential equations

$$\begin{aligned} y_{uu} &= a y_u + b y_v + c y, \\ y_{uv} &= a' y_u + b' y_v + c' y, \\ y_{vv} &= a'' y_u + b'' y_v + c'' y, \end{aligned} \tag{2}$$

any fundamental system of solutions of which, say $\bar{y}^{(1)}$, $\bar{y}^{(2)}$, $\bar{y}^{(3)}$, will be of the form

$$\bar{y}^{(k)} = c_{k1} y^{(1)} + c_{k2} y^{(2)} + c_{k3} y^{(3)}, \qquad |c_{kl}| \neq 0 \quad (k = 1, 2, 3)$$

where the c's are constants. The differential equations therefore define the

* Presented to the Society October 25, 1913.

† *One-parameter families and nets of plane curves.* These Transactions, vol. 12 (1911), pp. 473–510.

‡ Loc. cit., p. 478.

configuration (1) and all its projective transformations. The most general point transformations which leave unchanged the configuration (1) are composed of transformations of the form

$$\bar{y} = \lambda(u, v)\, y,$$

which leave every point of the plane fixed, and

$$\bar{u} = U(u), \qquad \bar{v} = V(v),$$

which leave invariant each of the families u = const., v = const. Suppose these transformations to be made on the differential equations (2). Those properties of (2) which remain unchanged under these transformations will be characteristic of the net (1) and of all its projective transformations, i. e., they will be projective properties. *Invariants* are those functions of the coefficients of (2), and of their derivatives, which remain unchanged under these transformations, except possibly for a factor depending only on the transformations. Covariants have the same nature and properties, but contain also the variable y and its derivatives.

Now evidently the projective properties of the net, as expressed by the vanishing of invariants thus found, can not all be characteristic of a single family of curves of the net, say the family u = const. For instance, the condition that the curves v = const. be straight lines is obviously from the first of equations (2) the vanishing of the invariant b; this has, however, nothing whatever to do with the properties of the curves u = const. But the invariant equation $a'' = 0$ is the condition that the curves u = const. be straight lines, and is therefore characteristic for that family.

Again, the covariants of the net have the following geometric interpretation. The tangents to the curves of one family, constructed at the points where these curves meet a fixed curve of the other family, have a covariant curve as envelope. Evidently neither of the two covariants thus obtained is intrinsically connected with the single family of curves u = const.

The theory of the net of curves, as developed by Wilczynski, is therefore not applicable in its present form to the study of a single family of plane curves, say the curves u = const. defined by (1). We shall in the sequel modify his theory so that it may be used for the theory of a one-parameter family of curves, by making this family one of the components of a net, in which the second family bears an intrinsic geometric relation to the given family.

1. The One-parameter Family of Curves as Part of a Canonical Net

The most general transformation of the independent variables which leaves fixed the family of curves u = const. defined by equations (1) is

$$\bar{u} = U(u), \qquad \bar{v} = V(u, v). \tag{3}$$

By means of this transformation we can change in the most arbitrary way the system of curves v = const. which cut the curves u = const. We seek a particular transformation of form (3) which will associate with the family u = const. a family v = const. intrinsically related to it. For a family of space curves which do not lie on a developable surface and which are not asymptotic curves on the surface determined by them, there is a very evident choice of a second family, namely the conjugate system of curves.* For the system of plane curves there are of course isogonal trajectories, but isogonality is not a projective property. There is in fact no obvious projective relation, analogous say to the relation of conjugacy of curves on a surface, of which we can make use here. We proceed to determine a family of curves v = const. which have a projectively intrinsic relation to the given family u = const., but the relation is by no means so simple as that of conjugacy on a curved surface.

Let us carry out the transformation

$$\bar{u} = u, \qquad \bar{v} = \phi(u, v), \tag{4}$$

which is included in (3), on the system of differential equations. We have

$$\begin{aligned} y_u &= \bar{y}_u + \bar{y}_v \phi_u, \qquad y_v = \bar{y}_v \phi_v, \\ y_{uu} &= \bar{y}_{uu} + 2\bar{y}_{uv} \phi_u + \bar{y}_{vv} \phi_u^2 + \bar{y}_u + \bar{y}_v \phi_{uu}, \\ y_{uv} &= \bar{y}_{uv} \phi_v + \bar{y}_{vv} \phi_u \phi_v + \bar{y}_v \phi_{uv}, \\ y_{vv} &= \bar{y}_{vv} \phi_v^2 + \bar{y}_v \phi_{vv}, \end{aligned} \tag{5}$$

where we have written $\bar{y}_u$ for $\partial y/\partial \bar{u}$, $\bar{y}_v$ for $\partial y/\partial \bar{v}$, etc. Substituting these equations in (2), and solving for the new second derivatives $\bar{y}_{uu}$, $\bar{y}_{uv}$, $\bar{y}_{vv}$, we obtain a system of differential equations of the same form as (2), with coefficients $\bar{a}$, $\bar{b}$, etc., given by

$$\begin{aligned} \bar{a} &= a - 2a' \frac{\phi_u}{\phi_v} + a'' \frac{\phi_u^2}{\phi_v^2}, \\ \bar{b} &= (a - 2b')\phi_u + b\phi_v + (b'' - 2a') \frac{\phi_u^2}{\phi_v} + a'' \frac{\phi_u^3}{\phi_v^2} \\ &\qquad - \phi_{uu} + 2\frac{\phi_u}{\phi_v}\phi_{uv} - \frac{\phi_u^2}{\phi_v^2}\phi_{vv}, \\ \bar{c} &= c - 2c' \frac{\phi_u}{\phi_v} + c'' \frac{\phi_u^2}{\phi_v^2}, \\ \phi_v \bar{a}' &= a' - a'' \frac{\phi_u}{\phi_v}, \qquad \phi_v \bar{c}' = c' - c'' \frac{\phi_u}{\phi_v}, \end{aligned} \tag{6}$$

* The author has already carried out this idea in a paper entitled *Projective Differential Geometry of One-parameter Families of Space Curves, and Conjugate Nets on a Curved Surface*, to appear in the American Journal of Mathematics.

$$\bar{b}' = b' + (a' - b'')\frac{\phi_u}{\phi_v} - a''\frac{\phi_u^2}{\phi_v^2} - \frac{\phi_{uv}}{\phi_v} + \frac{\phi_u}{\phi_v}\frac{\phi_{vv}}{\phi_v},$$

$$\phi_v^2\,\bar{a}'' = a'', \qquad \phi_v\,\bar{b}'' = b'' + a''\frac{\phi_u}{\phi_v} - \frac{\phi_{vv}}{\phi_v}, \qquad \phi_v^2\,\bar{c}'' = c''.$$

Let us put

$$\mathfrak{A}' = a' - \tfrac{1}{3}(a' + b'') + \frac{1}{6}\frac{a_v''}{a''},$$

and calculate the corresponding quantity $\overline{\mathfrak{A}}'$. We have

$$\frac{\bar{a}_v''}{\bar{a}''} = \frac{\partial}{\partial \bar{v}}\log \bar{a}'' = \frac{1}{\phi_v}\frac{\partial}{\partial v}\log \bar{a}'' = \frac{1}{\phi_v}\left(\frac{a_v''}{a''} - 2\frac{\phi_{vv}}{\phi_v}\right),$$

and

$$\begin{aligned}\phi_v\,\overline{\mathfrak{A}}' &= \phi_v\left[\bar{a}' - \tfrac{1}{3}(\bar{a}' + \bar{b}'') + \frac{1}{6}\frac{\bar{a}_v''}{\bar{a}''}\right]\\ &= a' - \tfrac{1}{3}(a' + b'') + \frac{1}{6}\frac{a_v''}{a''} - a''\frac{\phi_u}{\phi_v} \qquad (7)\\ &= \mathfrak{A}' - a''\frac{\phi_u}{\phi_v}.\end{aligned}$$

We may therefore make $\overline{\mathfrak{A}}'$ vanish by taking for ϕ a solution of the partial differential equation of the first order

$$a''\,\phi_u - \mathfrak{A}'\,\phi_v = 0. \qquad (8)$$

This choice of $\phi(u, v)$ gives a uniquely determined family $\bar{v} = \text{const.}$ For, from (7) we see that the most general transformation of the independent variable v which leaves the condition $\mathfrak{A}' = 0$ unchanged is independent of u; also, the condition will be found to remain invariant under any transformation $\bar{u} = U(u)$. Consequently the family $\bar{v} = \text{const.}$ bears a characteristic relation to the family $\bar{u} = \text{const.}$, of such nature, that it remains undisturbed only by those transformations of the independent variables which are of the form

$$\bar{\bar{u}} = U(\bar{u}), \qquad \bar{\bar{v}} = V(\bar{v}).$$

We can make this clearer by inquiring into the geometric meaning of the relation $\mathfrak{A}' = 0$. This condition has been interpreted by Wilczynski.* We shall give here an adequate description of the geometric nature of the net, but must refer to Wilczynski's memoir for the demonstrations.

Consider any net $u = \text{const.}$, $v = \text{const.}$ At each point of a fixed curve $u = \text{const.}$ construct the tangent to the curve $v = \text{const.}$ which passes through that point. The ∞^1 tangents so drawn will envelop a curve. The family of

* Loc. cit., p. 504.

curves corresponding in this way to the family u = const. is given by the equations

$$\rho^{(k)} = y_u^{(k)} - b' y^{(k)} \qquad (k = 1, 2, 3),$$

or, as we may say, by the covariant (of the net)

$$\rho = y_u - b' y,$$

for u = const.* Thus to every point P_y given by the equations

$$y^{(k)} = f^{(k)}(u, v)$$

corresponds a covariant point P_ρ. In the same way, a covariant

$$\sigma = y_v - a' y$$

gives points P_σ, which for v = const. trace the envelope of the tangents to the curves u = const. at the points where these curves meet the fixed curve v = const. given by y. We have then at any point P_y two lines, tangent respectively to the curves u = const. and v = const. which pass through P_y, and on these two lines the respective covariant points P_σ, P_ρ.

Now construct the conic which osculates the curve u = const. at P_y. *The line $P_\sigma P_\rho$ cuts it in a pair of points, which are harmonic to the pair P_ρ, P_σ if and only if $\mathfrak{A}' = 0$.*†

Let us call the net so determined the *canonical net.* The covariants

$$\bar{\rho} = \bar{y}_u - \bar{b}' \bar{y}, \qquad \bar{\sigma} = \bar{y}_v - \bar{a}' \bar{y} \tag{9}$$

will give for each point P_y two points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$, and the line $P_y P_{\bar{\rho}}$ will be the polar of $P_{\bar{\sigma}}$ with respect to the conic which osculates the curve u = const. at P_y. We proceed now to show geometrically how the family u = const. determines the second family, $\bar{v}$ = const., of the canonical net. Of course, the best we can do is to find at each point P_y the direction of the line $P_y P_{\bar{\rho}}$, since the determination of the family $\bar{v}$ = const. itself requires the integration of equation (8).

Suppose known only the family of curves u = const. Draw the tangent t and the osculating conic C at P_y. Consider any line l through P_y. Its pole with respect to the conic C is a point Q on t. The tangents to the curves u = const. at the points where these curves meet l envelop a curve; let R be the point of this curve which lies on t. Consider now the pencil of lines $[l]$ through P_y. The corresponding poles will be a range $[Q]$ on t; corresponding to each Q will be a point R given by the construction above. The ranges $[Q]$ and $[R]$ are in one to one correspondence. There are two double-points $Q \equiv R$; one of them is the point P_y, the other a definite point which is in

* E. J. Wilczynski, loc. cit., p. 486.

† E. J. Wilczynski, loc. cit., p. 504.

fact the covariant point $P_{\bar\sigma}$. The polar of $P_{\bar\sigma}$ with respect to C gives a certain direction passing through P_y. This will give the direction of $P_y P_{\bar\rho}$, although the point $P_{\bar\rho}$ is not thereby determined. It is possible, therefore, to construct at each point P_y the direction $P_y P_{\bar\rho}$; the ∞^2 directions so constructed give on integration the second family, $\bar v$ = const., of the canonical net.

Since the canonical net is completely determined by the given family of curves u = const., it follows that the invariants and covariants of the net are invariants and covariants of the given family of curves. Thus the quantities $\bar\rho$, $\bar\sigma$ given by (9) are covariants of the family u = const. The invariants may be calculated as in Wilczynski's memoir. Put

$$\bar f_u = \bar a + \bar b', \qquad \bar f_v = \bar a' + \bar b'', \tag{10}$$

$$\begin{aligned}
&\bar A = \bar a - \tfrac{2}{3}\bar f_u, \qquad \bar B = \bar b, \qquad \bar C = \bar c + \tfrac{1}{3}\bar a \bar f_u + \tfrac{1}{3}\bar b \bar f_v - \tfrac{1}{3}\bar f_{uu} - \tfrac{1}{9}\bar f_u^2,\\
&\bar A' = \bar a' - \tfrac{1}{3}\bar f_v, \qquad \bar B' = \bar b' - \tfrac{1}{3}\bar f_u,\\
&\qquad\qquad \bar C' = \bar c' + \tfrac{1}{3}\bar a' \bar f_u + \tfrac{1}{3}\bar b' \bar f_v - \tfrac{1}{3}\bar f_{uv} - \tfrac{1}{9}\bar f_u \bar f_v,\\
&\bar A'' = \bar a'', \qquad \bar B'' = \bar b'' - \tfrac{2}{3}\bar f_v,\\
&\qquad\qquad \bar C'' = \bar c'' + \tfrac{1}{3}\bar a'' \bar f_u + \tfrac{1}{3}\bar b'' \bar f_v - \tfrac{1}{3}\bar f_{vv} - \tfrac{1}{9}\bar f_v^2.
\end{aligned} \tag{11}$$

Then every invariant of the canonical net, and hence of the family of curves u = const., is a function of the following fundamental system of invariants and their derivatives:*

$$\begin{aligned}
&\bar{\mathfrak{B}} = \bar B, \qquad \bar{\mathfrak{C}} = \bar C - \bar B'_u - 2\bar B'^2 + \bar A' \bar B,\\
&\bar{\mathfrak{A}}' = \bar A' + \frac{1}{6}\frac{\bar A''_v}{\bar A''}, \qquad \bar{\mathfrak{B}}' = \bar B' + \frac{1}{6}\frac{\bar B_u}{\bar B}, \qquad \bar{\mathfrak{C}}' = \bar C' + \bar A' \bar B',\\
&\bar{\mathfrak{A}}'' = \bar A'', \qquad \bar{\mathfrak{C}}'' = \bar C'' - \bar A'_v - 2\bar A'^2 + \bar A'' \bar B'.
\end{aligned} \tag{12}$$

The last statement is evidently true from a geometric point of view; analytically, it may be verified as follows. In equations (6) appear the first and second derivatives of ϕ. From (8) we obtain by differentiation the equations

$$\begin{aligned}
\phi_{uu} - \frac{\phi_u}{\phi_v}\phi_{uv} &= \frac{\mathfrak{A}'_u}{a''}\phi_v - \frac{a''_u}{a''}\phi_u,\\
\phi_{uv} - \frac{\phi_u}{\phi_v}\phi_{vv} &= \frac{\mathfrak{A}'_v}{a''}\phi_v - \frac{a''_v}{a''}\phi_u.
\end{aligned} \tag{13}$$

By means of these and (8) we may eliminate from expressions (6) all derivatives of ϕ but one, say ϕ_v, except in the expression for $\bar b''$, from which ϕ_v and ϕ_{vv} cannot be removed. In all of the expressions thus obtained ϕ_v comes

* The invariant $\bar{\mathfrak{A}}' = \bar A' + \frac{1}{6}\frac{\bar A''_v}{\bar A''}$ is of course identically equal to zero.

out as a factor; moreover, in the invariants (12) $\bar{b}''$ occurs only in combinations from which ϕ_{vv} disappears. It follows upon calculation that *except for extraneous factors (powers of ϕ_v), the invariants* (12) *are expressible rationally in terms of the coefficients of the original system of differential equations* (2), *and the derivatives of these coefficients.*

The covariants (9) are expressible in the same way; we have, in fact,

$$\bar{y}_u = y_u - \frac{\phi_u}{\phi_v} y_v, \qquad \bar{y}_v = \frac{1}{\phi_v} y_v,$$

so that on calculating the values of $\bar{a}'$ and $\bar{b}'$ we obtain

$$(14) \qquad \begin{aligned} \bar{\rho} &= y_u - \frac{\mathfrak{A}'}{a''} y_v - \left[b' + \frac{\mathfrak{A}'}{a''}\left(a' - b'' - \mathfrak{A}' + \frac{a''_v}{a''} - \frac{\mathfrak{A}'_v}{\mathfrak{A}'} \right) \right] y, \\ \phi_v \bar{\sigma} &= y_v - (a' - \mathfrak{A}') y. \end{aligned}$$

All covariants of the canonical net are functions of these, of y, and of invariants; it may be shown, moreover, that the complete system of invariants, as set up by Wilczynski, is expressible entirely in terms of the coefficients of (2). We may state our results as follows: *the projective geometry of a one-parameter family of plane curves may be investigated by the study of a canonical net whose determination requires the integration of a partial differential equation of the first order. This integration is however unnecessary for the formulation of the projective properties of the one-parameter family of curves, since the invariants and covariants of this geometric configuration are expressible entirely in terms of the coefficients and variables of the original system of differential equations which define the one-parameter family as one component family of a net in which the second component family is arbitrary.*

2. Canonical Development for a One-parameter Family of Plane Curves

Suppose the family $v =$ const. has been determined so that it forms with the given system $u =$ const. a canonical net. We have then

$$(15a) \qquad \bar{y}_u = \bar{\rho} + \bar{b}' \bar{y}, \qquad \bar{y}_v = \bar{\sigma} + \bar{a}' \bar{y},$$

$$(15b) \qquad \begin{aligned} \bar{y}_{uu} &= \bar{a}\bar{y}_u + \bar{b}\bar{y}_v + \bar{c}\bar{y}, \\ \bar{y}_{uv} &= \bar{a}' \bar{y}_u + \bar{b}' \bar{y}_v + \bar{c}' \bar{y}, \\ \bar{y}_{vv} &= \bar{a}'' \bar{y}_u + \bar{b}'' \bar{y}_v + \bar{c}'' \bar{y}. \end{aligned}$$

The second derivatives of $\bar{y}$ are therefore expressible linearly in terms of $\bar{\rho}, \bar{\sigma}, \bar{y}$; the same is easily seen to be true of all higher derivatives of $\bar{y}$. We have, then, in general,

$$\frac{\partial^{i+j}}{\partial \bar{u}^i \, \partial \bar{v}^j} \bar{y} = \alpha^{(ij)} \bar{\rho} + \beta^{(ij)} \bar{\sigma} + \gamma^{(ij)} \bar{y}, \tag{15c}$$

where the α's, β's, and γ's are functions of the coefficients in (15b) and of their derivatives. Thus

$$\alpha^{(10)} = 1, \qquad \beta^{(10)} = 0, \qquad \gamma^{(10)} = \bar{b}',$$
$$\alpha^{(01)} = 0, \qquad \beta^{(01)} = 1, \qquad \gamma^{(01)} = \bar{a}',$$

from (15a). We may call these "coefficients of the first order." The coefficients of the second order are

$$\begin{aligned} &\alpha^{(20)} = \bar{a}, && \beta^{(20)} = \bar{b}, && \gamma^{(20)} = \bar{c} + \bar{a}\bar{b}' + \bar{b}\bar{a}', \\ &\alpha^{(11)} = \bar{a}', && \beta^{(11)} = \bar{b}', && \gamma^{(11)} = \bar{c}' + 2\bar{a}'\bar{b}', \\ &\alpha^{(02)} = \bar{a}'', && \beta^{(02)} = \bar{b}'', && \gamma^{(02)} = \bar{c}'' + \bar{a}''\bar{b}' + \bar{b}''\bar{a}'. \end{aligned} \tag{16}$$

Some of the coefficients of the third and higher orders are expressible in more than one way; this results from the fact that an expression like

$$\alpha^{(21)} \bar{\rho} + \beta^{(21)} \bar{\sigma} + \gamma^{(21)} \bar{y}$$

may be found either from $\partial \bar{y}_{uu}/\partial \bar{v}$ or from $\partial \bar{y}_{uv}/\partial \bar{u}$. We suppose that in every case two or more expressions for the same coefficient are identically equal* (for all values of $\bar{u}$ and $\bar{v}$). Thus, we have

$$\alpha^{(21)} = \bar{a}'_u + \bar{a}\bar{a}' + \bar{a}'\bar{b}' + \bar{c}' = \bar{a}_v + \bar{a}\bar{a}' + \bar{b}\bar{a}'',$$

etc. There is no theoretical difficulty in the calculation of these coefficients, which we leave for the present.

Let $P_{\bar{y}}$ be any point of the plane. We may suppose without loss of generality that for $P_{\bar{y}}$ the parameters $\bar{u}$, $\bar{v}$ have the values 0, 0. Assume that about $P_{\bar{y}}$ the geometric configuration is regular, so that for a sufficiently small neighborhood about $P_{\bar{y}}$ we have the expansion

$$\mathrm{Y} = \bar{y} + \bar{y}_u \bar{u} + \bar{y}_v \bar{v} + \tfrac{1}{2}(\bar{y}_{uu} \bar{u}^2 + 2\bar{y}_{uv} \bar{u}\bar{v} + \bar{y}_{vv} \bar{v}^2) + \cdots,$$

where Y corresponds to general values of the parameters $\bar{u}$, $\bar{v}$. By using equations (15) we find, therefore,

$$\mathrm{Y} = y_1 \bar{\rho} + y_2 \bar{\sigma} + y_3 \bar{y}, \tag{17}$$

where y_1, y_2, y_3 are the following power series in $\bar{u}$, $\bar{v}$:

* The equality of the pairs of expressions which represent the six coefficients $\alpha^{(21)}$, $\beta^{(21)}$, $\gamma^{(21)}$, $\alpha^{(12)}$, $\beta^{(12)}$, $\gamma^{(12)}$ is sufficient to ensure the equality of ambiguous expressions for all other coefficients. In fact, the six equations expressing the uniqueness in value of $\alpha^{(21)}$, $\beta^{(21)}$, etc., are the conditions of complete integrability for the system of differential equations (15b).

$$
\begin{aligned}
y_1 &= \bar{u} + \tfrac{1}{2}\bar{a}\bar{u}^2 + \bar{a}'\,\bar{u}\bar{v} + \tfrac{1}{2}\bar{a}''\,\bar{v}^2 \\
&\quad + \tfrac{1}{6}\alpha^{(30)}\,\bar{u}^3 + \tfrac{1}{2}\alpha^{(21)}\,\bar{u}^2\,\bar{v} + \tfrac{1}{2}\alpha^{(12)}\,\bar{u}\bar{v}^2 + \tfrac{1}{6}\alpha^{(03)}\,\bar{v}^{(3)} \\
&\quad + \tfrac{1}{24}\alpha^{(40)}\,\bar{u}^4 + \tfrac{1}{6}\alpha^{(31)}\,\bar{u}^3\,\bar{v} + \tfrac{1}{4}\alpha^{(22)}\,\bar{u}^2\,\bar{v}^2 + \tfrac{1}{6}\alpha^{(13)}\,\bar{u}\bar{v}^3 + \tfrac{1}{24}\alpha^{(04)}\,\bar{v}^4 \\
&\quad + \cdots, \\
y_2 &= \bar{v} + \tfrac{1}{2}\bar{b}\bar{u}^2 + \bar{b}'\,\bar{u}\bar{v} + \tfrac{1}{2}\bar{b}''\,\bar{v}^2 \\
&\quad + \tfrac{1}{6}\beta^{(30)}\,\bar{u}^3 + \tfrac{1}{2}\beta^{(21)}\,\bar{u}^2\,\bar{v} + \tfrac{1}{2}\beta^{(12)}\,\bar{u}\bar{v}^2 + \tfrac{1}{6}\beta^{(03)}\,\bar{v}^3 \\
&\quad + \tfrac{1}{24}\beta^{(40)}\,\bar{u}^4 + \tfrac{1}{6}\beta^{(31)}\,\bar{u}^3\,\bar{v} + \tfrac{1}{4}\beta^{(22)}\,\bar{u}^2\,\bar{v}^2 + \tfrac{1}{6}\beta^{(13)}\,\bar{u}\bar{v}^3 \\
&\quad + \tfrac{1}{24}\beta^{(04)}\,\bar{v}^4 + \cdots, \\
y_3 &= 1 + \bar{b}'\,\bar{u} + \bar{a}'\,\bar{v} + \tfrac{1}{2}\gamma^{(20)}\,\bar{u}^2 + \gamma^{(11)}\,\bar{u}\bar{v} + \tfrac{1}{2}\gamma^{(02)}\,\bar{v}^2 \\
&\quad + \tfrac{1}{6}\gamma^{(30)}\,\bar{u}^3 + \tfrac{1}{2}\gamma^{(21)}\,\bar{u}^2\,\bar{v} + \tfrac{1}{2}\gamma^{(12)}\,\bar{u}\bar{v}^2 + \tfrac{1}{6}\gamma^{(03)}\,\bar{v}^3 \\
&\quad + \cdots,
\end{aligned}
\tag{18}
$$

in which terms omitted are in each case of higher order than the last term written.

Except for isolated points $\bar{y}$ of the plane, the covariant points $\bar{\rho}$, $\bar{\sigma}$ which correspond to the point $\bar{y}$ form with $\bar{y}$ a non-degenerate triangle, which we shall take as a moving triangle of reference. A point whose three coördinates are

$$x^{(k)} = x_1\,\bar{\rho}^{(k)} + x_2\,\bar{\sigma}^{(k)} + x_3\,\bar{y}^{(k)} \qquad (k = 1, 2, 3),$$

or, as we may say, a point given by

$$x = x_1\,\bar{\rho} + x_2\,\bar{\sigma} + x_3\,\bar{y},$$

will have coördinates x_1, x_2, x_3 when referred to the covariant triangle $P_{\bar{\rho}}$, $P_{\bar{\sigma}}$, $P_{\bar{y}}$, if the unit point of the new coördinate system be properly chosen. Hence the point Y, given by (17), has coördinates y_1, y_2, y_3 when referred to the triangle $P_{\bar{\rho}}$, $P_{\bar{\sigma}}$, $P_{\bar{y}}$.

We now introduce non-homogeneous coördinates ξ, η, by putting

$$\xi = \frac{y_1}{y_3}, \qquad \eta = \frac{y_2}{y_3}.$$

We have, to terms of the third order,

$$
\begin{aligned}
\frac{1}{y_3} = 1 - \bar{b}'\,\bar{u} - \bar{a}'\,\bar{v} + C^{(20)}\,\bar{u}^2 + C^{(11)}\,\bar{u}\bar{v} + C^{(02)}\,\bar{v}^2 \\
+ C^{(30)}\,\bar{u}^3 + C^{(21)}\,\bar{u}^2\,\bar{v} + C^{(12)}\,\bar{u}\bar{v}^2 + C^{(03)}\,\bar{v}^3 + \cdots,
\end{aligned}
$$

where

$$
\begin{aligned}
&C^{(20)} = \bar{b}'^2 - \tfrac{1}{2}\gamma^{(20)}, \qquad C^{(11)} = 2\bar{a}'\,\bar{b}' - \gamma^{(11)}, \qquad C^{(02)} = \bar{a}'^2 - \tfrac{1}{2}\gamma^{(02)},\\
&C^{(30)} = -\,\bar{b}'^3 + \bar{b}'\,\gamma^{(20)} - \tfrac{1}{6}\gamma^{(30)},\\
(19)\quad &C^{(21)} = -\,3\bar{a}'\,\bar{b}'^2 + \bar{a}'\,\gamma^{(20)} + 2\bar{b}'\,\gamma^{(11)} - \tfrac{1}{2}\gamma^{(21)},\\
&C^{(12)} = -\,3\bar{a}'^2\,\bar{b}' + \bar{b}'\,\gamma^{(02)} + 2\bar{a}'\,\gamma^{(11)} - \tfrac{1}{2}\gamma^{(12)},\\
&C^{(03)} = -\,\bar{a}'^3 + \bar{a}'\,\gamma^{(02)} - \tfrac{1}{6}\gamma^{(03)}.
\end{aligned}
$$

We therefore obtain for ξ and η the following expressions, the one for ξ being exact to terms of the *fourth* order, and the one for η being calculated only to terms of the third order:

$$
\begin{aligned}
\xi = {} & \bar{u} + (\tfrac{1}{2}\bar{a} - \bar{b}')\,\bar{u}^2 + \tfrac{1}{2}\bar{a}''\,\bar{v}^2\\
& + A^{(30)}\,\bar{u}^3 + A^{(21)}\,\bar{u}^2\,\bar{v} + A^{(12)}\,\bar{u}\bar{v}^2 + A^{(03)}\,\bar{v}^3\\
(20)\qquad & + A^{(40)}\,\bar{u}^4 + A^{(31)}\,\bar{u}^3\,\bar{v} + A^{(22)}\,\bar{u}^2\,\bar{v}^2 + A^{(13)}\,\bar{u}\bar{v}^3 + A^{(04)}\,\bar{v}^4 + \cdots,\\
\eta = {} & \bar{v} + \tfrac{1}{2}\bar{b}\bar{u}^2 + (\tfrac{1}{2}\bar{b}'' - \bar{a}')\,\bar{v}^2\\
& + B^{(30)}\,\bar{u}^3 + B^{(21)}\,\bar{u}^2\,\bar{v} + B^{(12)}\,\bar{u}\bar{v}^2 + B^{(03)}\,\bar{v}^3 + \cdots,
\end{aligned}
$$

where

$$
\begin{aligned}
&A^{(30)} = \tfrac{1}{6}\alpha^{(30)} - \tfrac{1}{2}\bar{a}\bar{b}' + C^{(20)}, \qquad A^{(21)} = \tfrac{1}{2}\alpha^{(21)} - \bar{a}'\,\bar{b}' - \tfrac{1}{2}\bar{a}\bar{a}' + C^{(11)},\\
&A^{(12)} = \tfrac{1}{2}\alpha^{(12)} - \tfrac{1}{2}\bar{a}''\,\bar{b}' - \bar{a}'^2 + C^{(02)}, \qquad A^{(03)} = \tfrac{1}{6}\alpha^{(03)} - \tfrac{1}{2}\bar{a}'\,\bar{a}'',\\
&A^{(40)} = \tfrac{1}{24}\alpha^{(40)} - \tfrac{1}{6}\bar{b}'\,\alpha^{(30)} + \tfrac{1}{2}\bar{a}C^{(20)} + C^{(30)},\\
&A^{(31)} = \tfrac{1}{6}\alpha^{(31)} - \tfrac{1}{2}\bar{b}'\,\alpha^{(21)} - \tfrac{1}{6}\bar{a}'\,\alpha^{(30)} + \bar{a}'\,C^{(20)} + \tfrac{1}{2}\bar{a}C^{(11)} + C^{(21)},\\
&A^{(22)} = \tfrac{1}{4}\alpha^{(22)} - \tfrac{1}{2}\bar{b}'\,\alpha^{(12)} - \tfrac{1}{2}\bar{a}'\,\alpha^{(21)} + \tfrac{1}{2}\bar{a}''\,C^{(20)} + \bar{a}'\,C^{(11)}\\
(21)\qquad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \tfrac{1}{2}\bar{a}C^{(02)} + C^{(12)},\\
&A^{(13)} = \tfrac{1}{6}\alpha^{(13)} - \tfrac{1}{6}\bar{b}'\,\alpha^{(03)} - \tfrac{1}{2}\bar{a}'\,\alpha^{(12)} + \tfrac{1}{2}\bar{a}''\,C^{(11)} + \bar{a}'\,C^{(02)} + C^{(03)},\\
&A^{(04)} = \tfrac{1}{24}\alpha^{(04)} - \tfrac{1}{6}\bar{a}'\,\alpha^{(03)} + \tfrac{1}{2}\bar{a}''\,C^{(02)},\\
&B^{(30)} = \tfrac{1}{6}\beta^{(30)} - \tfrac{1}{2}\bar{b}\bar{b}', \qquad B^{(21)} = \tfrac{1}{2}\beta^{(21)} - \bar{b}'^2 - \tfrac{1}{2}\bar{a}'\,\bar{b} + C^{(20)},\\
&B^{(12)} = \tfrac{1}{2}\beta^{(12)} - \tfrac{1}{2}\bar{b}'\,\bar{b}'' - \bar{a}'\,\bar{b}' + C^{(11)},\\
&B^{(03)} = \tfrac{1}{6}\beta^{(03)} - \tfrac{1}{2}\bar{a}'\,\bar{b}'' + C^{(02)}.
\end{aligned}
$$

We wish to express ξ as a power series in $\bar{u}$ and η. We have from the second of (20), to terms of the fourth order,*

* Our subsequent developments are valid even though the last term in the second of equations (20) is of the third order.

$$\eta^4 = \bar{v}^4, \qquad \eta^3 = \bar{v}^3 + \tfrac{3}{2}\bar{b}\bar{u}^2\,\bar{v}^2 + \tfrac{3}{2}(\bar{b}'' - 2\bar{a}')\,\bar{v}^4,$$

$$\begin{aligned}\eta^2 = \bar{v}^2 &+ \bar{b}\bar{u}^2\,\bar{v} + (\bar{b}'' - 2\bar{a}')\,\bar{v}^3 + \tfrac{1}{4}\bar{b}^2\,\bar{u}^4 + 2B^{(30)}\,\bar{u}^3\,\bar{v} \\ &+ [2B^{(21)} + \tfrac{1}{2}\bar{b}(\bar{b}'' - 2\bar{a}')]\,\bar{u}^2\,\bar{v}^2 + 2B^{(12)}\,\bar{u}\bar{v}^3 \\ &+ [2B^{(03)} + \tfrac{1}{4}(\bar{b}'' - 2\bar{a}')^2]\,\bar{v}^4.\end{aligned}$$

We have therefore, exact to terms of the fourth order,

$$\bar{v}^4 = \eta^4, \qquad \bar{u}\bar{v}^3 = \bar{u}\eta^3, \qquad \bar{u}^2\,\bar{v}^2 = \bar{u}^2\,\eta^2, \qquad \bar{u}^3\,\bar{v} = \bar{u}^3\,\eta,$$

$$\bar{v}^3 = \eta^3 - \tfrac{3}{2}\bar{b}\bar{u}^2\,\eta^2 - \tfrac{3}{2}(\bar{b}'' - 2\bar{a}')\,\eta^4,$$

$$\bar{u}\bar{v}^2 = \bar{u}\eta^2 - \bar{b}\bar{u}^3\,\eta - (\bar{b}'' - 2\bar{a}')\,\bar{u}\eta^3,$$

$$\bar{u}^2\,\bar{v} = \bar{u}^2\,\eta - \tfrac{1}{2}\bar{b}\bar{u}^4 - \tfrac{1}{2}(\bar{b}'' - 2\bar{a}')\,\bar{u}^2\,\eta^2,$$

$$\begin{aligned}\bar{v}^2 = \eta^2 &- \bar{b}\bar{u}^2\,\eta - (\bar{b}'' - 2\bar{a}')\,\eta^3 + \tfrac{1}{4}\bar{b}^2\,\bar{u}^4 - 2B^{(30)}\,\bar{u}^3\,\eta \\ &+ [\tfrac{3}{2}\bar{b}(\bar{b}'' - 2\bar{a}') - 2B^{(21)}]\,\bar{u}^2\,\eta^2 - 2B^{(12)}\,\bar{u}\eta^3 \\ &+ [\tfrac{5}{4}(\bar{b}'' - 2\bar{a}')^2 - 2B^{(03)}]\,\eta^4.\end{aligned}$$

Substituting these in the first of equations (20), we obtain

$$\begin{aligned}\xi = \bar{u} &+ \tfrac{1}{2}(\bar{a} - 2\bar{b}')\,\bar{u}^2 + A^{(30)}\,\bar{u}^3 + [A^{(40)} - \tfrac{1}{2}\bar{b}A^{(21)} + \tfrac{1}{8}\bar{a}''\,\bar{b}^2]\,\bar{u}^4 \\ &+ \cdots + \tfrac{1}{2}\bar{a}''\,\eta^2 + [A^{(03)} - \tfrac{1}{2}\bar{a}''(\bar{b}'' - 2\bar{a}')]\,\eta^3 \\ &+ [A^{(04)} - \tfrac{3}{2}(\bar{b}'' - 2\bar{a}')\,A^{(03)} + \tfrac{5}{8}\bar{a}''(\bar{b}'' - 2\bar{a}')^2 - \bar{a}''\,B^{(03)}]\,\eta^4 \\ &+ \cdots + \{(A^{(21)} - \tfrac{1}{2}\bar{a}''\,\bar{b})\,\bar{u}^2 + (A^{(31)} - \bar{b}A^{(12)} - \bar{a}''\,B^{(30)})\,\bar{u}^3 \\ &+ \cdots\}\eta + \{A^{(12)}\,\bar{u} + [A^{(22)} - \tfrac{3}{2}\bar{b}A^{(03)} - \tfrac{1}{2}(\bar{b}'' - 2\bar{a}')\,A^{(21)} \\ &\qquad - \bar{a}''\,B^{(21)} + \tfrac{3}{4}\bar{a}''\,\bar{b}(\bar{b}'' - 2\bar{a}')]\,\bar{u}^2 + \cdots\}\eta^2 \\ &+ \{[A^{(13)} - (\bar{b}'' - 2\bar{a}')\,A^{(12)} - \bar{a}''\,B^{(12)}]\,\bar{u} + \cdots\}\eta^3 \\ &+ \cdots + \cdots.\end{aligned} \tag{22}$$

We have here arranged the expression for ξ in three parts. The first part is a power series in $\bar{u}$ alone, the second part is a power series in η alone, and the third part is a power series in η whose coefficients are power series in $\bar{u}$.

Since the triangle of reference is a covariant triangle, we might expect the coefficients in the development to be invariants; but not all of them are, because the parameter $\bar{u}$ is not chosen in any unique way. In fact, in the part containing powers of $\bar{u}$ alone, none of the coefficients is an invariant, while in the part containing powers of η alone all of the coefficients are invariants. In particular, the coefficient of η^3 vanishes identically:

$$A^{(03)} - \tfrac{1}{2}\bar{a}''(\bar{b}'' - 2\bar{a}') = 0. \tag{23}$$

Again, in the part which contains powers of both $\bar{u}$ and η, we have enclosed the coefficient of each power of η in a brace $\{\ \}$. Only the first coefficient in each brace is an invariant. Thus, in the term

$$(24)\qquad \{(A^{(21)} - \tfrac{1}{2}\bar{a}''\,\bar{b})\,\bar{u}^2 + (A^{(31)} - \bar{b}A^{(12)} - \bar{a}''\,B^{(30)})\,\bar{u}^3 + \cdots\}\eta,$$

the quantity $A^{(21)} - \frac{1}{2}\bar{a}''\,\bar{b}$ is an invariant, while $A^{(31)} - \bar{b}A^{(12)} - \bar{a}''\,B^{(30)}$ is not. In fact, following out the notation of Wilczynski, we may verify that

$$A^{(21)} - \tfrac{1}{2}\bar{a}''\,\bar{b} = -\tfrac{1}{2}(\bar{H} + \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}}),$$

$$(25)\qquad A^{(31)} - \bar{b}A^{(12)} - \bar{a}''\,B^{(30)} = -\tfrac{1}{2}\bar{\mathfrak{B}}(\bar{\mathfrak{A}}'', \bar{B}') - \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}}\bar{\mathfrak{B}}' - \tfrac{1}{6}(\bar{H}, \bar{B}')$$
$$- \tfrac{1}{2}(\bar{a} - 2\bar{b}')(\bar{H} + \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}}),$$

where

$$\bar{H} = \bar{c}' + \bar{a}'\,\bar{b}' - \bar{a}'_u, \qquad \bar{\mathfrak{A}}'' = \bar{a}'', \qquad \bar{\mathfrak{B}} = \bar{b}, \qquad \bar{\mathfrak{B}}' = \tfrac{1}{3}(2\bar{b}' - \bar{a}) + \frac{1}{6}\frac{\bar{b}_u}{\bar{b}},$$
$$(\bar{\mathfrak{A}}'', \bar{B}') = \bar{\mathfrak{A}}''_u - \bar{B}'\,\bar{\mathfrak{A}}'' = \bar{a}''_u + (\bar{a} - 2\bar{b}')\,\bar{a}'',$$
$$(\bar{H}, \bar{B}') = \bar{H}_u + \bar{B}'\,\bar{H} = \bar{H}_u - (\bar{a} - 2\bar{b}')\,\bar{H}$$

are invariants of the canonical net. Then the second of (25) is not invariant, since the quantity $\bar{a} - 2\bar{b}'$ is not.

If instead of the parameter $\bar{u}$ we choose a uniquely determined parameter τ, it will be possible to transform (22) into a new expansion in which all the coefficients are relative invariants. This may be done by taking for τ the part of (22) containing powers of $\bar{u}$ alone, i. e.,

$$(26)\qquad \tau = \bar{u} + \tfrac{1}{2}(\bar{a} - 2\bar{b}')\,\bar{u}^2 + A^{(30)}\,\bar{u}^3 + \cdots,$$

the inversion of which gives

$$\bar{u} = \tau - \tfrac{1}{2}(\bar{a} - 2\bar{b}')\,\tau^2 + \cdots,$$
$$\bar{u}^2 = \tau^2 - (\bar{a} - 2\bar{b}')\,\tau^3 + \cdots.$$

Each brace in (22) becomes a new power series in τ, with invariants as coefficients; thus, in particular, the term (24) becomes

$$\{-\tfrac{1}{2}(\bar{H} + \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}})\,\tau^2 - [\tfrac{1}{2}\bar{\mathfrak{B}}(\bar{\mathfrak{A}}'', \bar{B}') + \bar{\mathfrak{A}}''\,\bar{\mathfrak{B}}\bar{\mathfrak{B}}' - \tfrac{1}{6}(\bar{H}, \bar{B}')]\,\tau^3 + \cdots\}\eta.$$

The coefficient of η^4 in (22) is found without much difficulty to have the value

$$-\frac{\bar{\mathfrak{A}}''}{8}(\bar{\mathfrak{A}}'', \bar{B}'),$$

so that we have the development

$$
\begin{aligned}
\xi = \tau &+ \tfrac{1}{2}\bar{a}''\,\eta^2 - \tfrac{1}{8}\bar{a}''\,(\bar{\mathfrak{A}}'', \bar{B}')\,\eta^4 + \cdots \\
&+ \{\theta^{(21)}\,\tau^2 + \theta^{(31)}\,\tau^3 + \cdots\}\eta \\
&+ \{\theta^{(12)}\,\tau + \theta^{(22)}\,\tau^2 + \cdots\}\eta^2 \\
&+ \{\theta^{(13)}\,\tau + \cdots\}\eta^3 + \cdots,
\end{aligned}
\tag{27}
$$

where $\theta^{(ij)}$, the coefficient of $\tau^i\,\eta^j$, is a relative invariant of the canonical net.

We may make some further transformations, which will yield a development with absolute invariants as coefficients. The coefficients in (26) are of course constants, being the values of the corresponding invariants for the values 0, 0 of the arguments $\bar{u}$, $\bar{v}$. Consider the invariant $\theta^{(ij)}$; it is easily verified that the point transformation

$$\bar{\bar{u}} = U(\bar{u}), \qquad \bar{\bar{v}} = V(\bar{v})$$

changes it into

$$\bar{\bar{\theta}}^{(ij)} = \frac{1}{(U')^{i-1}} \cdot \frac{1}{(V')^j}\,\theta^{(ij)}, \tag{28}$$

where U' and V' are the first derivatives of U and V. The same transformation changes the invariants $\bar{a}''$ and $\bar{b}$ into

$$\bar{\bar{a}}'' = \frac{U'}{V'^2}\,\bar{a}'', \qquad \bar{\bar{b}} = \frac{V'}{U'^2}\,\bar{b}.$$

Therefore

$$\bar{\bar{a}}''\,\bar{\bar{b}}^2 = \frac{1}{U'^3}\,\bar{a}''\,\bar{b}^2, \qquad \bar{\bar{a}}''^2\,\bar{\bar{b}} = \frac{1}{V'^3}\,\bar{a}''^2\,\bar{b}. \tag{29}$$

Let us put

$$\mu = \sqrt[3]{\bar{a}''\,\bar{b}^2}, \qquad \nu = \sqrt[3]{\bar{a}''^2\,\bar{b}}. \tag{30}$$

Then the quantity

$$I^{(ij)} = \mu \cdot \left(\frac{1}{\mu}\right)^i \left(\frac{1}{\nu}\right)^j \theta^{(ij)}$$

is an *absolute invariant.* In fact, the transformed quantity is seen from (27) and (28) to satisfy the equation

$$\bar{\bar{I}}^{(ij)} = I^{(ij)}.$$

If, then, in the expansion (26) we put*

$$x = \mu\xi, \qquad y = \nu\eta, \qquad t = \mu\tau, \tag{31}$$

we obtain the final canonical development,

$$
\begin{aligned}
x = t &+ \tfrac{1}{2}y^2 + I^{(04)}\,y^4 + \cdots \\
&+ \{I^{(21)}\,t^2 + I^{(31)}\,t^3 + \cdots\}y \\
&+ \{I^{(12)}\,t + I^{(22)}\,t^2 + \cdots\}y^2 \\
&+ \{I^{(13)}\,t + \cdots\}y^3 + \cdots,
\end{aligned}
\tag{32}
$$

the coefficients of which are all absolute invariants.

* The transformation (31) leaves fixed the covariant triangle $P_{\bar{y}}$, $P_{\bar{\rho}}$, $P_{\bar{\sigma}}$.

In recapitulation, we state the result:

A one-parameter family of analytic plane curves, given by the equations

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3)$$

for u = const., may be represented in the neighborhood of a regular point by the canonical development (31) *for t = const. In this development all the coefficients are absolute projective invariants. The triangle of reference which gives rise to this development is the covariant triangle $P_{\bar{y}}$, $P_{\bar{\rho}}$, $P_{\bar{\sigma}}$, of which a purely geometric interpretation has been given. The parameter t is uniquely determined as an analytic function of the old parameter u.*

In the derivation of this canonical development, the essential steps to be noted are first, the determination of a triangle of reference which bears a covariant relation to the one-parameter family of curves (considered apart from any net of which it may be a component family), and secondly the characteristic choice of the parameter t. From the results stated in § 1, we see that the absolute invariants which enter into the canonical development as coefficients are all expressible rationally in terms of the coefficients of the original differential equations which define the family of curves as a component of a net in which the second family bears only an accidental relation to the given family. Our canonical development is therefore independent of the integration which is necessary to determine the canonical net.

College of the City of New York,
October, 1913.

THE MINIMUM OF A DEFINITE INTEGRAL FOR UNILATERAL VARIATIONS IN SPACE*

BY

G. A. BLISS AND A. L. UNDERHILL

For the problem of the calculus of variations with fixed end points, the curves which minimize an integral of the form

$$J = \int F(x, y, z, x', y', z')\, ds$$

have previously been studied, provided that the minimizing arc along which the integral J is taken is free to traverse any portion of space or restricted to lie entirely within a given region R. It may happen, however, that while no extremal joining the two fixed end points exists, which is entirely interior to a region R, yet there may be a minimizing curve consisting partly of arcs interior to the region and partly of arcs along its boundary. This gives rise to the problem of investigating the properties which characterize a minimizing

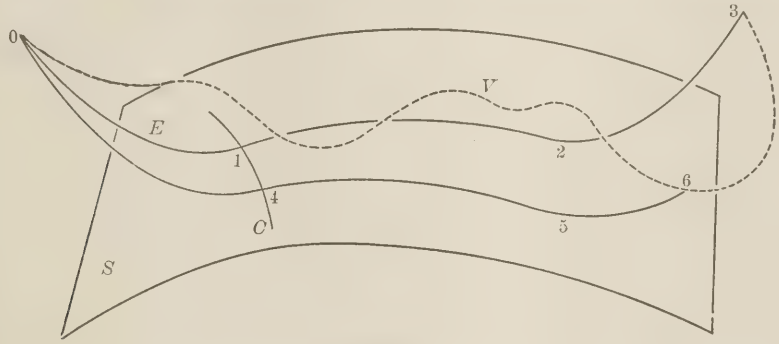

FIG. 1.

curve lying partly on a given surface S, while the variations with which it is compared are restricted to lie on one side of the surface, a problem analogous to one in the plane which has been discussed by Bliss.† For this problem in space a number of necessary conditions have been derived by other writers, but the theory of the analogue of Jacobi's condition is still incomplete.

* Presented to the Society, December 27, 1913.

† These Transactions, vol. 5 (1904), pp. 477–492.

Suppose that the arc E, whose minimizing properties are to be investigated, consists of three parts E_{01}, E_{12}, and E_{23}, as shown in the accompanying figure (Fig. 1). The arc E_{12} must be an extremal of the integral J on the surface S, while the other two, lying on the same side of S and joining the end points of E_{12} to the two fixed points 0 and 3, are necessarily extremals in space and tangent to the arc E_{12}. For the corresponding situation in the plane, the analogue of Jacobi's condition requires only that Jacobi's condition itself be satisfied by each of the arcs E_{01} and E_{23}. For the problem here considered, however, there is in general more required than this. It is not sufficient, in order that E shall minimize the integral, that no conjugate point to 0 lies on the extremal arc E_{01}, and no conjugate point to 2 on the arc E_{23}.*

It will be shown in the following pages that from the two parameter family of space extremals passing through the point 0, a set containing E_{01} and depending upon a single parameter may be selected, each curve of which is tangent to the surface S at a point of a curve C through the point 1. Through the points of C a one parameter family of extremals on S can be passed, each one of which is tangent at a point of C to one of the space extremals through the point 0. Again at any point of one of the single infinitude of surface extremals so determined, there is a space extremal tangent to the surface extremal and lying on the required side of the surface. These form a two parameter family of space extremals tangent to S.

Two new conditions, analogues of the Jacobi condition in the plane case, will be established; (1) the arc E_{12} must not contain a point of contact with the enveloping curve of the one-parameter family of surface extremals above mentioned; and (2) the arc E_{23} must not contain a contact point with the enveloping surface of the two parameter family of space extremals constructed as above tangent to the surface S.

The necessary conditions hitherto derived and the two analogues to the Jacobi condition described in the preceeding paragraph, when strengthened in a manner customary in the calculus of variations, are sufficient to insure that the curve E actually minimizes the integral J.

1. Necessary Conditions Hitherto Derived

Let the curve E joining the points 0 and 3, whose minimizing properties are to be studied, have the equations

$$(E) \qquad x = x(s), \qquad y = y(s), \qquad z = z(s) \qquad (s_0 \leqq s \leqq s_3),$$

where s is the length of arc measured from the point 0. The curve is supposed

* See a remark by Hadamard in his *Leçons sur le calcul des variations*, p. 457.

not to intersect itself and is made up of three pieces, each of class C''',* as shown in Fig. 1.

The surface S, on which E_{12} lies, has equations of the form

$$(S) \qquad x = \xi(\alpha, \beta), \qquad y = \eta(\alpha, \beta), \qquad z = \zeta(\alpha, \beta).$$

It has no points in common with E_{01} and E_{23} except 1 and 2, and is further supposed to have no multiple or singular points and to be of class C'''' in the vicinity of E_{12}.

The function F in the integral J is assumed to have the following properties in a region R of points (x, y, z, x', y', z') for which (x, y, z) ranges over a neighborhood of E, while (x', y', z') are any values not all zero:

$$(A) \qquad F \text{ is of class } C''';$$

$$(B) \qquad F(x, y, z, \kappa x', \kappa y', \kappa z') = \kappa F(x, y, z, x', y', z') \qquad (\kappa > 0);$$

$$(C) \qquad F_1 \neq 0.$$

The property (B) is the usual homogeneity property of the function F which makes the value of the integral independent of the parametric representation of the curve along which it is taken, while (C) specifies that the problem is a regular one.†

If E is to minimize J, then it must satisfy a number of necessary conditions which are already known and which may be stated as follows:

(I) The arcs E_{01} and E_{23} must be space extremals satisfying the differential equations

$$(1) \qquad \begin{gathered} J^{(x)} = F_x - \frac{d}{ds} F_{x'} = 0, \qquad J^{(y)} = F_y - \frac{d}{ds} F_{y'} = 0, \\ J^{(z)} = F_z - \frac{d}{ds} F_{z'} = 0; \end{gathered}$$

(II) Along the arc E_{12} the equations

$$(2) \qquad \frac{J^{(x)}}{X} = \frac{J^{(y)}}{Y} = \frac{J^{(z)}}{Z} = \lambda(s)$$

must be satisfied, where X, Y, Z are the direction cosines of the normal of the surface S in the direction toward the side of S on which E_{01} and E_{23} lie, while $\lambda(s)$ is merely the notation for the function of s defined by the three ratios;

* An arc is said to be of class $C^{(n)}$ if the derivatives of the functions defining it up to and including those of order n are continuous, and if the expression $\sqrt{x'^2 + y'^2 + z'^2}$ is everywhere different from zero along it.

† For the definition of F_1 and a "regular problem" see a paper by Mason and Bliss, these Transactions, vol. 9 (1908), p. 444. This paper will be referred to hereafter as M. and B.

(III) Along E_{12} the inequality $\lambda(s) \gtreqless 0$ must hold;

(IV) The function

$$E(x, y, z; x', y', z'; p, q, r)$$
$$= F(x, y, z, p, q, r) - pF_{x'}(x, y, z, x', y', z')$$
$$- qF_{y'}(x, y, z, x', y', z') - rF_{z'}(x, y, z, x', y', z')$$

must be positive or zero for all values (x, y, z, x', y', z') on the arc E_{03} and for arbitrary values $(p, q, r) \neq (0, 0, 0)$;

(V) At the points 1 and 2, E_{01} and E_{23} must be tangent to the arc E_{12};

(VI) The arc E_{01} must not contain a point conjugate to 0 between 0 and 1.*

From the property IV it follows that the quadratic form

$$Q(x, y, z; x', y', z'; p, q, r)$$
$$= F_{x'x'}\, p^2 + F_{y'y'}\, q^2 + F_{z'z'}\, r^2$$
$$+ 2F_{y'z'}\, qr + 2F_{z'x'}\, rp + 2F_{x'y'}\, pq$$

must be positive or zero for the arguments described for E.† The sign of F_1 must therefore be positive, and it can be shown that $Q \geqq 0$ for all values (x, y, z, x', y', z') in the region R and for arbitrary directions (p, q, r).‡ Furthermore Q vanishes only when the directions x', y', z' and p, q, r are in the same line.§ The same properties hold for the E-function, with the further restriction that E vanishes only when x', y', z' and p, q, r are identical (not opposite) directions.‖

The condition II can also be stated in terms of the integrand

$$f(\alpha, \beta, \alpha', \beta') = F(\xi, \eta, \zeta, \xi_\alpha \alpha' + \xi_\beta \beta', \eta_\alpha \alpha' + \eta_\beta \beta', \zeta_\alpha \alpha' + \zeta_\beta \beta')$$

of the integral

$$J = \int f(\alpha, \beta, \alpha', \beta')\, ds \tag{3}$$

* For the conditions (I), (IV), (VI) on the arcs E_{01} and E_{23}, see M. and B., loc. cit., pages 443, 455, 453, respectively. For (II), (III), (V), see Hadamard, loc. cit., pages 133, 179, 456. There is an exceptional case to (VI) mentioned in M. and B., p. 453, which can, however, be treated by a consideration of the second variation instead of the geometric proof used by them. The proof that the condition (IV) must hold along the arc E_{12} can be made in a manner analogous to that used for the other arcs. It is somewhat more difficult because the variations are restricted to lie on one side of the surface S.

† M. and B., p. 455.

‡ The argument is like that of M. and B., p. 456, for the functions $F_{x'x'}, F_{y'y'}, F_{z'z'}$.

§ M. and B., p. 456, formulas (40).

‖ The proof of this statement may be made by means of the formulas in M. and B., but more elegantly by the use of the results of Behaghel, Mathematische Annalen, vol. 73 (1913), p. 596. The functions A, B, C used by him may become infinite for directions parallel to the axes and the validity of his proof is questionable in such cases, though his result is correct. The justification for these statements will appear in the near future in a paper by Bliss.

taken along curves on the surface S. Along E_{12} the parameters α and β are functions

$$\alpha = \alpha(s), \qquad \beta = \beta(s) \qquad (s_1 \leqq s \leqq s_2)$$

which are of class C''. After an easy calculation the Euler equations for the integral (3) are seen to have the form

$$(4) \qquad \begin{aligned} f_\alpha - \frac{d}{ds} f_{\alpha'} &= J^{(x)} \xi_\alpha + J^{(y)} \eta_\alpha + J^{(z)} \zeta_\alpha = 0, \\ f_\beta - \frac{d}{ds} f_{\beta'} &= J^{(x)} \xi_\beta + J^{(y)} \eta_\beta + J^{(z)} \zeta_\beta = 0, \end{aligned}$$

and these are satisfied along E_{12} on account of the condition II.

Furthermore the function*

$$f_1 = \frac{f_{\alpha'\alpha'}}{\beta'^2} = -\frac{f_{\alpha'\beta'}}{\alpha'\beta'} = \frac{f_{\beta'\beta'}}{\alpha'^2}$$

has the values

$$f_1 = \frac{Q(x, y, z; x', y', z'; \xi_\alpha, \eta_\alpha, \zeta_\alpha)}{\beta'^2} = \frac{Q(x, y, z; x', y', z'; \xi_\beta, \eta_\beta, \zeta_\beta)}{\alpha'^2},$$

where

$$x' = \xi_\alpha \alpha' + \xi_\beta \beta', \qquad y' = \eta_\alpha \alpha' + \eta_\beta \beta', \qquad z' = \zeta_\alpha \alpha' + \zeta_\beta \beta'.$$

One of these expressions is always well defined and different from zero, since the derivatives x', y', z' can coincide with at most one of the coördinate directions on the surface. The problem of minimizing the integral (3) on the surface is therefore a regular one, and E_{12} is one of its extremals.

If a space extremal satisfying the condition I is tangent to a surface extremal satisfying II, then at the point of contact there is a relation between their curvatures which will be of service later. Let the elements of the second order on the two extremals at the point of contact be, respectively,

$$(x, y, z, x', y', z', \bar{x}'', \bar{y}'', \bar{z}'')$$

and

$$(x, y, z, x', y', z', x'', y'', z'').$$

From equations (1) and (2) it follows that

$$(5) \qquad \lambda X = J^{(x)} - \bar{J}^{(x)}, \qquad \lambda Y = J^{(y)} - \bar{J}^{(y)}, \qquad \lambda Z = J^{(z)} - \bar{J}^{(z)},$$

and

$$\begin{aligned} \lambda\{X(x'' - \bar{x}'') + Y(y'' - \bar{y}'') + Z(z'' - \bar{z}'')\} \\ = (x'' - \bar{x}'')(J^{(x)} - \bar{J}^{(x)}) + (y'' - \bar{y}'')(J^{(y)} - \bar{J}^{(y)}) \\ + (z'' - \bar{z}'')(J^{(z)} - \bar{J}^{(z)}) \\ = Q(x, y, z; x', y', z'; x'' - \bar{x}'', y'' - \bar{y}'', z'' - \bar{z}''). \end{aligned}$$

* See Bolza, *Vorlesungen über Variationsrechnung*, p. 196.

This last expression can vanish, however, only when

$$x'' - \bar{x}'' = \kappa x', \qquad y'' - \bar{y}'' = \kappa y', \qquad z'' - \bar{z}'' = \kappa z',$$

where κ is a value which may be zero. In case it does vanish the equations (5) show that $\lambda = 0$, as a result of the homogeneity properties of the function F.* Hence the following lemma is true:

LEMMA 1: *If $\lambda \neq 0$ at the point of contact of a surface extremal with a space extremal, then at that point the inequality*

$$Xx'' + Yy'' + Zz'' \neq X\bar{x}'' + Y\bar{y}'' + Z\bar{z}''$$

holds; or, in other words, the projections of the first curvatures, $1/r$ and $1/\bar{r}$, of the two extremals on the normal to the surface are different.

From this point in the paper it will be assumed that the arc E_{03} satisfies the conditions I, II, IV, V and the modified conditions

(III′) $\lambda(s) > 0$ along E_{12};

(VI′) the arc E_{01} does not contain a point conjugate to 0 between 0 and 1 or at 1.†

On the basis of these assumptions further conditions, analogous to Jacobi's conditions for the usual form of the problem, will be derived.

2. THE THREE FAMILIES OF EXTREMALS ASSOCIATED WITH E_{03}

The totality of solutions of the differential equations (1) passing through 0 contains two arbitrary parameters. As a result of the hypotheses made upon the function F they may be represented in the form

$$(6) \quad x = \phi(s, u, v, w), \qquad y = \psi(s, u, v, w), \qquad z = \chi(s, u, v, w),$$
$$u^2 + v^2 + w^2 = 1,$$

the parameters u, v, w being the direction cosines of the positive tangent to the extremal at the point 0. The particular extremal E_{01} is contained in this set for $u = u_0$, $v = v_0$, $w = w_0$, and the functions $\phi, \psi, \chi, \phi_s, \psi_s, \chi_s$ are of class C' in neighborhoods of the two sets of parameter values

$$s = 0, \qquad u^2 + v^2 + w^2 = 1;$$

$$0 \leqq s \leqq s_1, \qquad u = u_0, \qquad v = v_0, \qquad w = w_0$$

of which the second set defines the curve E_{01}. At the point 0, where $s = 0$, the following identities in u, v, w hold

$$x_0 = \phi(0, u, v, w), \qquad y_0 = \psi(0, u, v, w), \qquad z_0 = \chi(0, u, v, w).$$

* M. and B., p. 441, formula (5).

† The condition VI′ is necessary except in one singular case. See M. and B., p. 453.

Let the notation $\Delta(s, u, v, w)$ designate the determinant

$$\Delta(s, u, v, w) = \begin{vmatrix} \phi_s & \phi_u & \phi_v & \phi_w \\ \psi_s & \psi_u & \psi_v & \psi_w \\ \chi_s & \chi_u & \chi_v & \chi_w \\ 0 & u & v & w \end{vmatrix}.$$

On account of the condition VI′ it follows* that

$$\Delta(s, u_0, v_0, w_0) \neq 0 \qquad (0 < s \leqq s_1). \tag{7}$$

The equations

$$\begin{aligned} \xi(\alpha, \beta) + \omega X &= \phi(s, u, v, w), \\ \eta(\alpha, \beta) + \omega Y &= \psi(s, u, v, w), \\ \zeta(\alpha, \beta) + \omega Z &= \chi(s, u, v, w), \end{aligned}$$

have an initial solution $(\alpha, \beta, \omega; s, u, v, w) = (\alpha_1, \beta_1, 0; s_1, u_0, v_0, w_0)$ corresponding to the point 1, at which the functional determinant for α, β, ω is different from zero. Hence these equations have solutions

$$\alpha = \alpha(s, u, v, w), \qquad \beta = \beta(s, u, v, w), \qquad \omega = \omega(s, u, v, w) \tag{8}$$

of class C'' near (s_1, u_0, v_0, w_0) and reducing to $(\alpha_1, \beta_1, 0)$ at these values. The derivatives of ω at the point 1 are readily calculated to be

$$\begin{aligned} \omega_s &= X\phi_s + Y\psi_s + Z\chi_s, \qquad \omega_u = X\phi_u + Y\psi_u + Z\chi_u, \\ \omega_v &= X\phi_v + Y\psi_v + Z\chi_v, \qquad \omega_w = X\phi_w + Y\psi_w + Z\chi_w, \\ \omega_{ss} &= X\phi_{ss} + Y\psi_{ss} + Z\chi_{ss} - L\alpha_s^2 - 2M\alpha_s\beta_s - N\beta_s^2\dagger \\ &= Xx'' + Yy'' + Zz'' - (X\bar{x}'' + Y\bar{y}'' + Z\bar{z}''), \end{aligned} \tag{9}$$

where the notations in the last line are the same as those of the preceding section. The set (x'', y'', z'') belongs to E_{01} at the point 1, and the set $(\bar{x}'', \bar{y}'', \bar{z}'')$ to E_{12}.

The solutions (s, u, v, w) of the equations

$$\omega(s, u, v, w) = 0, \qquad \omega_s(s, u, v, w) = 0, \qquad u^2 + v^2 + w^2 = 1 \tag{10}$$

determine points where the extremal (6) is tangent to the surface S. On account of the tangency of E_{01} and E_{12}, it is evident that among the solutions are the values (s_1, u_0, v_0, w_0). On the other hand the determinants of the matrix

$$\left\| \begin{matrix} \omega_{ss} & \omega_{su} & \omega_{sv} & \omega_{sw} \\ \omega_s & \omega_u & \omega_v & \omega_w \\ 0 & u & v & w \end{matrix} \right\|$$

* See M. and B., §3 and p. 453.

† For the definitions of L, M, N and their relations to the first curvature, see Scheffers, *Anwendung der Differential und Integral-Rechnung auf Geometrie*, vol. 2, page 494 (D).

cannot all vanish at (s_1, u_0, v_0, w_0). At these values $\omega_s = 0$ since E_{01} is tangent to the surface S; but from equations (9) and the property (7) it follows readily that the determinants of the last two rows can not all vanish; and from III′ and the lemma of the preceding section, ω_{ss} is different from zero. The equations (10) therefore have solutions for s and two of the variables u, v, w in terms of the third, passing through the values (s_1, u_0, v_0, w_0) and of class C' near them.

These solutions and the equations (8) *determine a one parameter family of values*

$$\alpha = \alpha(\tau), \qquad \beta = \beta(\tau), \qquad s = \sigma(\tau),$$
$$u = u(\tau), \qquad v = v(\tau), \qquad w = w(\tau)\,*$$

of class C' near $\tau = 0$, which in turn determine a curve

$$(C) \qquad \alpha = \alpha(\tau), \qquad \beta = \beta(\tau)$$

on the surface S and a one parameter family

$$(11) \qquad \begin{aligned} x &= \phi[s, u(\tau), v(\tau), w(\tau)] = \phi(s, \tau), \\ y &= \psi[s, u(\tau), v(\tau), w(\tau)] = \psi(s, \tau), \\ z &= \chi[s, u(\tau), v(\tau), w(\tau)] = \chi(s, \tau), \end{aligned}$$

of the extremals (6). *Through each point of C there passes a unique extremal of the set tangent to the surface S. The functions $\phi(s, \tau)$ and $\phi_s(s, \tau)$ are of class C' near the values $s_0 \leqq s \leqq s_1$, $\tau = 0$, which define the arc E_{01}, with similar properties for ψ and χ.*

On account of the hypotheses on the function F and the resulting properties of f, it is possible to prove the existence of a one parameter family of extremals on the surface S satisfying the differential equations II or (4), and such that each extremal passes through a point of C, and has the direction determined by the corresponding space extremal of the set (11) tangent to S at that point. Let the length of arc along the surface extremal be denoted by σ and be measured so that $\sigma - \sigma(\tau)$ is the length of arc from C.

There exists, then, a one parameter family of surface extremals with equations on the surface of the form

$$(12) \qquad \alpha = A(\sigma, \tau), \qquad \beta = B(\sigma, \tau),$$

or, in space, of the form

$$(13) \qquad \begin{aligned} x &= \xi[A(\sigma, \tau), \ B(\sigma, \tau)] = \xi(\sigma, \tau), \\ y &= \eta[A(\sigma, \tau), \ B(\sigma, \tau)] = \eta(\sigma, \tau), \\ z &= \zeta[A(\sigma, \tau), \ B(\sigma, \tau)] = \zeta(\sigma, \tau), \end{aligned}$$

* The parameter τ will be $u - u_0$, $v - v_0$ or $w - w_0$.

where the functions $\xi(\sigma, \tau)$ *and* $\xi_\sigma(\sigma, \tau)$ *are of class* C' *near the values*

$$s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0,$$

defining the arc E_{12}. *Along the curve* C *the relations*

$$\phi[\sigma(\tau), \tau] = \xi[\sigma(\tau), \tau], \qquad \phi_s[\sigma(\tau), \tau] = \xi_\sigma[\sigma(\tau), \tau],$$

with similar ones for ψ *and* χ, *are identities in* τ. *They express the fact that at a point of* C *the surface extremal* (12) *and the space extremal* (11) *are tangent to each other.*

Through each point of an extremal on the surface S determined by the equations (12), there passes a space extremal whose direction at the point of intersection coincides with that of the extremal on the surface. This extremal must satisfy the differential equations I. If its length of arc s is measured from the initial value σ at the point of contact with the surface, the existence theorems for differential equations and the properties of the function F justify the following theorem:

THEOREM. *There exists a two parameter family of space extremals tangent to the extremals* (13) *on the surface* S, *and having equations of the form*

$$x = \Phi(s, \sigma, \tau), \qquad y = \Psi(s, \sigma, \tau), \qquad z = \mathrm{X}(s, \sigma, \tau), \tag{14}$$

where Φ *and* Φ_s *are of class* C' *near the values*

$$s = \sigma, \qquad s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0,$$

$$s_2 \leqq s \leqq s_3, \qquad \sigma = s_2, \qquad \tau = 0,$$

defining, respectively, the arcs E_{12} *and* E_{23}. *Similar properties hold for* Ψ *and* X. *The identities*

$$\Phi(\sigma, \sigma, \tau) \equiv \xi(\sigma, \tau), \qquad \Phi_s(\sigma, \sigma, \tau) \equiv \xi_\sigma(\sigma, \tau), \tag{15}$$

with similar ones for the other functions, express the tangency of (14) *with the surface extremal* (13) *at their point of intersection.*

The identities (15) give rise to a number of further relations important in the following pages. By differentiating the first one and applying the second, it follows that

$$\Phi_\sigma(\sigma, \sigma, \tau) \equiv 0, \qquad \Phi_\tau(\sigma, \sigma, \tau) = \xi_\tau(\sigma, \tau),$$

and by differentiating the second,

$$\Phi_{s\sigma}(\sigma, \sigma, \tau) \equiv \xi_{\sigma\sigma}(\sigma, \tau) - \Phi_{ss}(\sigma, \sigma, \tau) \equiv \bar{x}'' - x''$$

with similar relations for Ψ and X. The determinant

$$\Delta(s, \sigma, \tau) = \begin{vmatrix} \Phi_s & \Phi_\sigma & \Phi_\tau \\ \Psi_s & \Psi_\sigma & \Psi_\tau \\ \mathrm{X}_s & \mathrm{X}_\sigma & \mathrm{X}_\tau \end{vmatrix}$$

has therefore the derivative

$$(16)\qquad \Delta_s(s,\sigma,\tau) = \begin{vmatrix} \Phi_s & \Phi_{s\sigma} & \Phi_\tau \\ \Psi_s & \Psi_{s\sigma} & \Psi_\tau \\ X_s & X_{s\sigma} & X_\tau \end{vmatrix} = \begin{vmatrix} x'' - \bar{x}'' & \xi_\sigma & \xi_\tau \\ y'' - \bar{y}'' & \eta_\sigma & \eta_\tau \\ z'' - \bar{z}'' & \zeta_\sigma & \zeta_\tau \end{vmatrix}$$

for the values (σ, σ, τ) along the surface S. But at a point where the functional determinant of the functions A and B is different from zero, the relations (13) show that this has the value

$$\Delta_s(\sigma,\sigma,\tau) = K\{X(x'' - \bar{x}'') + Y(y'' - \bar{y}'') + Z(z'' - \bar{z}'')\}$$

where K is a factor of proportionality which does not vanish. According to the lemma of § 1, the following result may be stated:

LEMMA 2. *The derivative $\Delta_s(s,\sigma,\tau)$ is different from zero at any point of the curve E_{12} on the surface S at which the determinant*

$$\begin{vmatrix} A_\sigma & A_\tau \\ B_\sigma & B_\tau \end{vmatrix}$$

is different from zero.

3. AUXILIARY THEOREMS

The computations of the following pages can be considerably simplified by the use of some auxiliary theorems, the first of which may be stated as follows:

THEOREM. *If a family of curves*

$$(17)\qquad x = \phi(s,\lambda), \qquad y = \psi(s,\lambda), \qquad z = \chi(s,\lambda)$$

satisfies the relation

$$J^{(x)}\phi_\lambda + J^{(y)}\psi_\lambda + J^{(z)}\chi_\lambda \equiv 0,$$

then the derivative of the integral

$$\int_{s_0}^{s_1} F(\phi,\psi,\chi,\phi_s,\psi_s,\chi_s)\,ds,$$

when s_0, s_1, and λ are functions $s_0(v)$, $s_1(v)$, $\lambda(v)$, is

$$[F_{x'}x_v + F_{y'}y_v + F_{z'}z_v]_0^1.$$

The arguments of the derivatives of F are the same as those in the integral, and

$$(18)\qquad x_v = \phi_s s_v + \phi_\lambda \lambda_v, \qquad y_v = \psi_s s_v + \psi_\lambda \lambda_v, \qquad z_v = \chi_s s_v + \chi_\lambda \lambda_v.$$

This theorem follows at once by known methods* provided that the func-

* See M. and B., §2.

tions s_0, s_1, λ are of class C' for $|v| \leqq \epsilon$, and ϕ, ϕ_s of class C' in the region

$$s_0(v) \leqq s \leqq s_1(v), \qquad \lambda = \lambda(v) \qquad (|v| \leqq \epsilon),$$

with similar properties for ψ and χ.

THEOREM. *If the curves of a second family*

$$x = \Phi(t, \mu), \qquad y = \Psi(t, \mu), \qquad z = \mathrm{X}(t, \mu) \tag{19}$$

adjoin those of the first family for parameter values

$$s = s_1(v), \qquad \lambda = \lambda(v); \qquad t = t_1(v), \qquad \mu = \mu(v),$$

and if corresponding arcs of the two families have the same direction at their common point, then the function

$$G(v) = \int_{s_0}^{s_1} F(\phi, \psi, \chi, \phi_s, \psi_s, \chi_s)\, ds + \int_{t_1}^{t_2} F(\Phi, \Psi, \mathrm{X}, \Phi_t, \Psi_t, \mathrm{X}_t)\, dt, \tag{20}$$

in which t_2 is a function of v as well as t_1 and μ, has the derivative

$$G'(v) = [F_{x'} x_v + F_{y'} y_v + F_{z'} z_v]_0^2. \tag{21}$$

The expressions in the bracket must be formed at the point 0 *for the first family, and at the point* 2 *for the second, as described in the first theorem.*

The derivative of $G(v)$ has the value

$$[F_{x'} x_v + F_{y'} y_v + F_{z'} z_v]_0^1 + [F_{x'} x_v + F_{y'} y_v + F_{z'} z_v]_1^2,$$

from the first theorem. At the point 1 the two brackets cancel on account of the homogeneity properties of F and the relations

$$\phi[s_1(v), \lambda(v)] = \Phi[t_1(v), \mu(v)],$$

$$\phi_s[s_1(v), \lambda(v)] = \kappa(v)\Phi_t[t_1(v), \mu(v)] \qquad [\kappa(v) > 0],$$

with similar ones for the other functions, which express the intersection and the tangency of the two families. It is understood that the continuity properties of the two families are like those described above for the first.

A similar theorem evidently holds when the number of families is more than two. If the curves of the first family all pass through a fixed point 0 for $s = s_0(v)$, $\lambda = \lambda(v)$, the bracket in (21) vanishes at 0 since the same is true of the derivatives of (17). If the curves of the second family similarly pass through a fixed point 2 for $t = t_2(v)$, $\mu = \mu(v)$, then $G(v)$ is a constant.

The family (19) may be independent of μ, in which case it will be simply an enveloping curve of the family (17). If in this case $t_2(v)$ is a constant, the second of the integrals (20) will be simply the value of J taken along the envelope from 1 to a fixed point 2, and the function $G(v)$ will again be a constant.

4. Jacobi's Conditions for Unilateral Variations

As a result of the assumptions made on the curve E_{03} at the end of §1, it is possible to prove the following further necessary conditions for a minimum:

(VII) *The curve E_{03} must not have a point of contact between* 1 *and* 2 *with an enveloping curve d of the family* (13) *of extremals on the surface S.*

The proof will be made here for the case where the enveloping curve d does not have a singular point at its point of contact with E_{02}. In that case a stronger condition holds, namely,

(VII′) *The curve E_{03} must not have a point of contact with an enveloping curve d either between* 1 *and* 2 *or at* 2.

If the curve d has a singular point at its contact point with E_{02}, the condition VII still holds and can be proved by other methods; but the case where d touches E at the point 2 requires, as usual, a special investigation.

The equation which defines d on the surface S is

$$\Delta(\sigma, \tau) = \begin{vmatrix} A_\sigma & A_\tau \\ B_\sigma & B_\tau \end{vmatrix} = 0,$$

where A and B are the functions (12) defining the extremals on the surface. If this is satisfied at a point $(\sigma, \tau) = (\sigma_6, 0)$ of the arc E_{12}, the derivative Δ_σ will be different from zero at that point, as is known from the theory of the plane problem applied to the integral (3). The last equation has therefore a solution

$$\sigma = \Sigma(u), \qquad \tau = \mathrm{T}(u),$$

reducing to $(\sigma_0, 0)$ for $u = 0$ and of class C' near that value. The equations of the envelope in space will be

$$(d) \quad x = \xi[\Sigma(u), \mathrm{T}(u)], \quad y = \eta[\Sigma(u), \mathrm{T}(u)], \quad z = \zeta[\Sigma(u), \mathrm{T}(u)].$$

The parameter u is chosen as one of the values $\pm\tau$, but so as to make the positive direction along d the same as that of the surface extremal tangent to it.

For an arbitrary value of u the corresponding value of τ defines an extremal E_{04} of the set (11) which is tangent to the corresponding extremal E_{45} of the set (13) at a point 4 of the curve C as shown in Fig. 2. Further, E_{45} is tangent to d at a point 5, and d_{56} touches E_{03} at the point 6. The families (11), (13), and d are then of the form described in §3 when $u = v$, and the function

$$G(u) = J(E_{04}) + J(E_{45}) + J(d_{56}) + J(E_{63})$$

is a constant, where E_{63} denotes the arc of E_{03} between 6 and 3. An argument precisely like that used for problems in the plane, shows that d_{56} can not be an extremal on the surface, and since

$$G(0) = J(E_{03})$$

it follows that E_{03} can not minimize J if VII is not satisfied.

If the condition VII′ is satisfied together with those postulated at the end of §1, a final necessary condition is the following:

(VIII) *The curve E_{03} must not have a point of contact between* 2 *and* 3 *with an enveloping surface D of the family* (14) *of extremals tangent to S.*

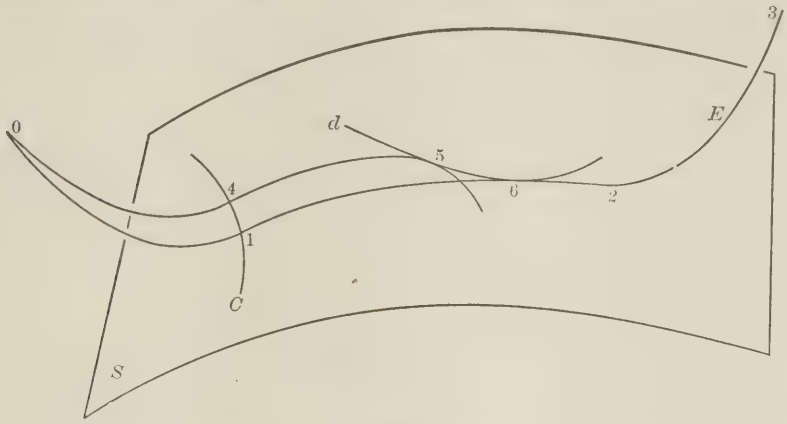

FIG. 2.

Here again the proof will be made for the case where D does not have a singular point at its contact with E_{03}, and the stronger condition holding under those circumstances is

(VIII′) *The curve E_{03} must not have a point of contact with D either between* 2 *and* 3 *or at* 3.

The points on the enveloping surface are defined by the equation

$$\Delta(s, \sigma, \tau) = \begin{vmatrix} \Phi_s & \Phi_\sigma & \Phi_\tau \\ \Psi_s & \Psi_\sigma & \Psi_\tau \\ X_s & X_\sigma & X_\tau \end{vmatrix} = 0. \tag{22}$$

On account of Lemma 2 of §2, the determinant is different from zero on the arc E_{23} near the point 2, though it vanishes at 2 itself. If the equation is satisfied at a point 7 of the arc E_{23} for which $(s, \sigma, \tau) = (s_7, s_2, 0)$, and further defines a non-singular enveloping surface D, then the argument of Mason and Bliss* shows that there is a curve on D,

$$x = x(u), \qquad y = y(u), \qquad z = z(u), \tag{d}$$

passing through the point 7 for $u = 0$, and which is tangent to a one parameter family of the extremals (15) at points defined by equations of the form

$$s = S(u), \qquad \sigma = \Sigma(u), \qquad \tau = T(u),$$

* See M. and B., §3.

where S, Σ, T reduce to s_7, s_2, 0 for $u = 0$. The parameter u can be so chosen that the positive direction on the curve d coincides with the positive direction of the extremal (15) at the point defined by these functions.

An arbitrarily chosen value u defines an extremal E_{04} of the set (11) tangent to an extremal E_{45} of the set (12) at a point 4 of the curve C; and E_{45} is in turn tangent at a point 5 to the extremal E_{56} defined by the equations

$$(23) \qquad x = \Phi(s, \Sigma, \mathrm{T}), \qquad y = \Psi(s, \Sigma, \mathrm{T}), \qquad z = \mathrm{X}(s, \Sigma, \mathrm{T}).$$

Finally this last curve is tangent to d_{67} at a point 6. When u is set equal to v, the families (11), (13), (23), and d_{67} are of the kind described in §3, and the function

$$G(u) = J(E_{04}) + J(E_{45}) + J(E_{56}) + J(d_{67}) + J(E_{73})$$

is a constant. Since

$$G(0) = J(E_{03})$$

and d can not be an extremal, it follows that E_{03} can not minimize J if VIII is not satisfied.

5. The Existence of a Field

If the arc E_{03} satisfies the stronger set of eight conditions described in §§1, 4, it will surely minimize the integral J with respect to variations sufficiently near to it and lying on one side of the surface S. But before proceeding to the proof of this fact it is necessary to show how a field may be constructed about the minimizing arc.

For this purpose consider first the family (14) which contains the arcs E_{12} and E_{23}, respectively, for the parameter values

$$(\Pi) \qquad \begin{aligned} &s = \sigma, \qquad s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0; \\ &s_2 \leqq s \leqq s_3, \qquad \sigma = s_2, \qquad \tau = 0. \end{aligned}$$

The symbol Π_ϵ will be used to denote the set of points (s, σ, τ) which satisfy either of the following sets of conditions:

$$(\Pi_\epsilon) \qquad \begin{aligned} &\sigma \leqq s \leqq \sigma + \epsilon, \qquad \sigma(\tau) \leqq \sigma \leqq s_2 + \epsilon, \qquad |\tau| \leqq \epsilon; \\ &\sigma + \epsilon \leqq s \leqq s_3 + \epsilon, \qquad |\sigma - s_2| \leqq \epsilon, \qquad |\tau| \leqq \epsilon. \end{aligned}$$

If the constant ϵ is chosen sufficiently small, any two distinct points of Π_ϵ will be transformed by the equations (14) *into distinct points* (x, y, z); *and in the interior of Π_ϵ the determinant* (22) *will be everywhere different from zero.*

To prove this it will first be shown that in a suitably chosen neighborhood of a point $\pi_0(s_0, \sigma_0, \tau_0)$ satisfying the second of the conditions (Π), no two distinct points $\pi(s, \sigma, \tau)$, $\pi'(s', \sigma', \tau')$ of the region Π_ϵ can define

the same point (x, y, z) by means of the equations (14). By Taylor's formula,*

$$(24)\qquad \Phi(s', \sigma', \tau') - \Phi(s, \sigma, \tau) = (s' - s) A_1 + (\sigma' - \sigma) A_2 + (\tau' - \tau) A_3$$

where, for example,

$$(25)\qquad A_2 = \int_0^1 \Phi_\sigma(s'', \sigma'', \tau'')\, du,$$

$$s'' = s + u(s' - s), \qquad \sigma'' = \sigma + u(\sigma' - \sigma), \qquad \tau'' = \tau + u(\tau'' - \tau),$$

and the other coefficients have similar values. The expression (24) and the similar ones for Ψ and X can vanish simultaneously only when the determinant of the coefficients vanishes. But this is impossible near π_0 since the determinant (22) is different from zero at π_0 and the derivatives of Φ, Ψ, X are continuous.

Near a point π_0 satisfying the first of the conditions Π the expression (25) for A_2 can be put in the form

$$A_2 = \int_0^1 (s'' - \sigma'') \int_0^1 \Phi_{s\sigma}[\sigma'' + v(s'' - \sigma''), \sigma'', \tau'']\, dv\, du,$$

since Φ_σ vanishes when its first two arguments are the same. The point (s'', σ'', τ'') is always in Π_ϵ when π and π' are in Π_ϵ and $0 \leqq u \leqq 1$, and hence $s'' - \sigma''$ is never negative. By a double application of the mean value theorem for a definite integral

$$A_2 = (s_1 - \sigma_1)\Phi_{s\sigma},$$

where the arguments of $\Phi_{s\sigma}$ contain properly chosen intermediate values for u and v, and (s_1, σ_1, τ_1) is the middle point of the segment $\pi\pi'$.

Near a point π_0 satisfying the second of the conditions (Π) the difference (24) can therefore be put into the form

$$(26)\qquad (s' - s) A_1 + (\sigma' - \sigma)(s_1 - \sigma_1) A_2' + (\tau' - \tau) A_3,$$

where A_1, A_2', A_3 approach continuously the values of Φ_s, $\Phi_{s\sigma}$, Φ_τ at the point π_0 when π and π' approach π_0. This expression and the similar ones for Ψ and X can vanish simultaneously, for π and π' sufficiently near to π_0, only when

$$s' - s = \tau' - \tau = (\sigma' - \sigma)(s_1 - \sigma_1) = 0,$$

since at the point π_0 the determinant (22) is different from zero. But if $s' = s$ and $\tau' = \tau$, the values of σ' and σ must be distinct since π and π' are distinct. The difference $s_1 - \sigma_1$ can not vanish since $s_1 = s = s'$ is greater than the larger of σ and σ' on account of the conditions (Π_ϵ). Hence near π_0 the

* See Jordan, *Cours d'Analyse*, 2d ed., vol. 1, p. 247.

difference (24) and the similar ones for Ψ and X can not all vanish if π and π' are distinct.

Suppose now that it should not be possible to choose a constant ϵ such as is described in the first part of the italicized statement, and consider a sequence $\{\epsilon_k\}$ of decreasing positive constants having zero as its limit. For any choice of k there would exist at least one pair of distinct points,

$$(27) \qquad (s, \sigma, \tau)_k, \qquad (s', \sigma', \tau')_k$$

in the region Π_{ϵ_k} and defining the same point $(x, y, z)_k$ by means of the equations (14). The two sequences (27) would have points of condensation

$$(28) \qquad (s_0, \sigma_0, \tau_0), \qquad (s_0', \sigma_0', \tau_0')$$

necessarily lying in the region Π, and these two points must be the same. Otherwise they would define distinct points of the arc E_{13}, and on account of the continuity of the functions (14) the pairs of values (27) which lie sufficiently near to the values (28) could not define the same point $(x, y, z)_k$. Neither could the pair of sets of values (27) define the same point $(x, y, z)_k$ in case the two condensation points (28) were identical, as has been seen in the paragraphs just preceding.

A similar argument shows that ϵ can be chosen so as to satisfy the last part of the italicized statement. Otherwise there would be a sequence $(s, \sigma, \tau)_k$ of interior points of the respective regions Π_{ϵ_k} at each of which the determinant (22) vanishes. Such a sequence must have a condensation point π_0 in Π. But π_0 could not satisfy the second of the conditions (Π), since at such points $\Delta(s, \sigma, \tau)$ is different from zero. Neither could it satisfy the first condition; for near such a point π_0 the value of $\Delta_s(s, \sigma, \tau)$ is different from zero, on account of Lemma 2 and because for $s > \sigma$

$$\Delta(s, \sigma, \tau) = (s - \sigma)\Delta_s[s + \theta(s - \sigma), \sigma, \tau] \neq 0.$$

The points of the sequence $(s, \sigma, \tau)_k$ which condense on π_0 have $s > \sigma$, since they are interior points, and hence the existence of the sequence is contradicted.

If the region Π_ϵ is chosen as described above, its image in the xyz-space, after transformation by equations (14), *is a region $\mathcal{F}_2$, a continuum and its boundary, whose boundary points are the images of the boundary points of Π_ϵ. The correspondence between $\mathcal{F}_2$ and Π_ϵ is one-to-one. The single valued functions*

$$(29) \qquad s = s(x, y, z), \qquad \sigma = \sigma(x, y, z), \qquad \tau = \tau(x, y, z)$$

so defined are continuous over $\mathcal{F}_2$, and in the interior of $\mathcal{F}_2$ they have continuous derivatives.

The first part of this statement follows at once from some theorems proved

by Bliss,* since the images of the interior and boundary of Π_ϵ are distinct. The preceding paragraphs of this paper show that the correspondence between $\mathcal{F}_2$ and Π_ϵ is one-to-one. An interior point of $\mathcal{F}_2$ corresponds to an interior point of Π_ϵ at which $\Delta(s, \sigma, \tau) \neq 0$, and the fundamental theorem of the implicit function theory shows at once that the functions (29) are of class C' at such points. If a sequence of points $(x, y, z)_k$ of $\mathcal{F}_2$ approaches a boundary point (x', y', z') of $\mathcal{F}_2$, the corresponding values $(s, \sigma, \tau)_k$ defined by the equations (29) must approach the values (s', σ', τ') which define (x', y', z') by means of equations (14), as may readily be seen from the continuity of the functions Φ, Ψ, X. Hence the functions (29) are continuous everywhere in $\mathcal{F}_2$.

There exists a field $\mathcal{F}_1$ about the arc E_{01} through each point of which there passes a unique extremal of the set (6) *defined by equations of the form*

$$(30)\qquad \begin{aligned} s &= s(x, y, z), \qquad & u &= u(x, y, z), \\ v &= v(x, y, z), \qquad & w &= w(x, y, z). \end{aligned}$$

These functions are continuous everywhere in $\mathcal{F}_1$ and have continuous derivatives except at the point 0.†

The symbol $\mathcal{F}_{12}$ will be used to denote the set of points (x, y, z) consisting of the interior of $\mathcal{F}_2$, the points of S in $\mathcal{F}_2$, and the points interior to $\mathcal{F}_1$. Let V be a continuous curve

$$(V)\qquad x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad z = \bar{z}(t) \qquad (t_0 \leqq t \leqq t_3)$$

consisting of a finite number of non-singular arcs of class C', joining the points 0 and 3, and lying in the region $\mathcal{F}_{12}$ on the same side of S with the arc E_{01}.

If $\mathcal{F}_1$ is taken sufficiently small the only points where V can enter $\mathcal{F}_2$ are on the boundary B of $\mathcal{F}_2$ which corresponds to the parameter values

$$(31)\qquad \sigma \leqq s < \sigma + \epsilon, \qquad \sigma = s(t), \qquad |\tau| < \epsilon.$$

A neighborhood of the point 1 can be chosen so small that the only boundary points of $\mathcal{F}_2$ in it are points of the surface S and points on the boundary B defined by the conditions (31). Further, $\mathcal{F}_1$ can be taken so small that the only points common to it and $\mathcal{F}_2$ are in the same neighborhood. Under these circumstances the curve V can enter $\mathcal{F}_2$ only at points of B or at points of S

* *The Princeton Colloquium Lectures on Mathematics*, p. 38.

† For the existence of the field in the neighborhood of the point 0, see Bliss and Mason, *Fields of extremals in space*, these Transactions, vol. 11 (1910), p. 328; also references at the beginning of the same paper; also Szücs, Mathematische Annalen, vol. 71 (1911), p. 380. The existence of the field about the rest of the arc follows by the usual methods. A proof analogous to that just preceding in the text is more elegant than any o these.

distinct from B. The latter are, however, excluded since V lies entirely on one side of S near the point 1.

6. Sufficient Conditions for a Minimum

It is proposed to prove in this section that an arc E_{03} of the kind described at the end of §5, will give the integral J a value at least equal to $J(E_{03})$. To each value of t defining a point 6 of V_{03} in $\mathcal{F}_2$, there will correspond a unique curve

$$E_{06} = E_{04} + E_{45} + E_{56},$$

shown in Fig. 1, where E_{04}, E_{45}, E_{56} belong respectively to the families (11), (13), and (14); while if t defines a point 6 which does not lie in $\mathcal{F}_2$ it will determine a unique extremal E_{06} of the family (6).

The integral $J(E_{06})$, is a continuous function of t which has the derivative

$$\frac{d}{dt} J(E_{06}) = F_{x'}\,\bar{x}' + F_{y'}\,\bar{y}' + F_{z'}\,\bar{z}'\,|^6.* \tag{32}$$

In this expression the arguments of the derivatives of F are the values (x, y, z, x', y', z') which define the point and direction of E_{06} at the point 6, while $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ are the derivatives at the same point of the functions (V) defining V.

To prove this consider the one parameter family of curves E_{06} determined by the points of V_{03} interior to $\mathcal{F}_2$. It is determined by substituting the functions defining V in the expressions (29) for s, σ, τ in terms of x, y, z. The functions $s(t)$, $\sigma(t)$, $\tau(t)$ so determined are of class C' near a value of t defining an interior point of $\mathcal{F}_2$ unless t is one of the exceptional values at which V has a corner point. By substituting $\sigma(t)$, $\tau(t)$ in the equations (11), (13), and (14), the three families E_{04}, E_{45}, E_{56} are determined with properties similar to those of the auxiliary theorems of §3. It follows readily that the derivative of $J(E_{06})$ has the value given in the theorem.

The argument is the same for a point 6 in $\mathcal{F}_1$ but not in $\mathcal{F}_2$, except that the equations whose solutions determine the family E_{06} are then

$$x(t) = \phi(s, u, v, w), \qquad y(t) = \psi(s, u, v, w), \qquad z(t) = \chi(s, u, v, w),$$
$$u^2 + v^2 + w^2 = 1.$$

When the point 6 lies on the surface S at the point 0, or on the boundary B, it is more difficult to show the existence of the derivative. But the desired result can be obtained by the use of expansions of the functions (14) similar to (26), by means of the properties of the functions (6) at the point 0 described in the paper referred to in the preceding section, and by special considerations at a point of B.

* At corner points of the curve V this should be thought of as a forward derivative.

If the stronger set of eight properties described in the preceding sections is satisfied by the arc E_{03}, then

$$J(E_{03}) < J(V_{03}) \tag{33}$$

for any curve V_{03} distinct from E_{03} in the field $\mathcal{F}_{12}$ and on the same side of S with the arc E_{03}. It is understood that V_{03} consists of a finite number of non-singular arcs of class C'.

For consider the continuous function

$$W(t) = J(E_{06}) + J(V_{63}),$$

where 6 is the point of V determined by the parameter value t. With the help of the last theorem but one, the derivative of $W(t)$ exists except at a finite number of values of t, and has itself the value

$$W'(t) = -E(x, y, z;\ x', y', z';\ \bar{x}', \bar{y}', \bar{z}'). \tag{34}$$

On account of IV and the properties of the function E this is always negative except when the positive directions along V_{63} and E_{06} coincide at the point 6. Furthermore

$$W(t_0) = J(V_{03}), \qquad W(t_1) = J(E_{03}).$$

The inequality (33) must therefore be satisfied unless the positive directions along E_{06} and V_{63} coincide at every point 6 of V where the derivative (34) is well defined.

If the positive directions along V coincide with those of E_{06} at every point of V, then V_{03} and E_{03} must be coincident. The proof is somewhat complicated but may be made with the help of the following lemma:

LEMMA 3. *Any arc of V defined by an interval $t' \leqq t \leqq t''$ and containing no points of S or B, is an extremal of one of the families* (6), (14) *along which s increases from s' to s'' as t traverses the interval $t'\ t''$. An arc of V lying entirely on the surface S must be an arc of one of the surface extremals* (13) *with similar properties.*

An arc of V with which the lemma is concerned must lie in the interior of $\mathcal{F}_1$ or $\mathcal{F}_2$, or else on the surface S. The proofs in the three cases are quite similar, so that it will be sufficient to suppose the arc V to lie entirely on S. The values $s = \sigma, \sigma, \tau$ are then continuous functions $\sigma(t), \tau(t)$ defined over the interval $t' \leqq t \leqq t''$ by equations (V) and (29), and they satisfy the equations

$$\begin{gathered} x(t) = \xi[\sigma(t), \tau(t)], \qquad y(t) = \eta[\sigma(t), \tau(t)], \\ z(t) = \zeta[\sigma(t), \tau(t)]. \end{gathered} \tag{35}$$

At least one of the determinants of the matrix

$$\begin{Vmatrix} \xi_\sigma & \eta_\sigma & \zeta_\sigma \\ \xi_\tau & \eta_\tau & \zeta_\tau \end{Vmatrix}$$

is different from zero at every value of t on account of VII′ and the relations (13). Hence the theory of implicit functions applied to a suitably chosen pair of the equations (35) shows that $\sigma(t)$ and $\tau(t)$ are of class C'. If the direction of V coincides with that of the surface extremal at each point of the arc in question, then

$$x'(t) = \kappa\,\xi_\sigma, \qquad y'(t) = \kappa\,\eta_\sigma, \qquad z'(t) = \kappa\,\zeta_\sigma \qquad (\kappa > 0),$$

and by differentiating equations (35) the relations

$$\sigma'(t) = \kappa, \qquad \tau'(t) = 0$$

are readily deduced. Hence the lemma is true for an arc on S, and for the other two cases by similar arguments.

LEMMA 4. *If t_4 is a lower bound of values of t such that for $t_4 \leqq t \leqq t_3$ the curve V_{03} coincides with E_{03}, then no value $t' < t_4$ can define a point of V interior to $\mathcal{F}_2$, or a point of V on the surface S and distinct from B.*

There must be a lower bound t_4 of the kind described since near the point 3 the curve V_{03} coincides with an extremal of the field passing through the point 3, according to Lemma 3, and E_{03} is the only such extremal.

If t' defined a point interior to $\mathcal{F}_2$, then there would be a first value $t'' > t'$ such that the arc $t'\,t''$ of V would be entirely interior to $\mathcal{F}_2$ and distinct from E_{03} except at the value t''. The value t'' would necessarily define a point either of B, or S, or E_{23}. But the arc $t'\,t''$ of V would in that case be an extremal arc along which s increases with t from a value s' to a value s''. This is, however, impossible, since any such arc which is distinct from B, S, E_{23} for $s = s'$ retains this property for all values $s > s'$.

With the help of this result and by similar arguments it follows that t' could not define a point of S distinct from B.

The identity of V_{03} and E_{03} now follows without difficulty. The value $t = t_4$ must in fact define a point 4 on the arc E_{01}, since in case this is not true there would certainly be values $t' < t_4$ satisfying the conditions of Lemma 4. The arc $t_0\,t_1$ of V then lies entirely in the field $\mathcal{F}_1$, and no point of it can be distinct from E_{01}. Otherwise there would be an extremal arc of the field $\mathcal{F}_1$ distinct from E_{01} whose end point, as s increased toward the parameter value defining it, would lie on E_{01}. There is, however, no such arc.

ON A METHOD OF COMPARISON FOR TRIPLE-SYSTEMS*

BY

LOUISE D. CUMMINGS

INTRODUCTION

1. If n elements $1, 2, 3, \cdots n$ can be distributed in triads in such a way that every pair of elements appears in one and only one triad, the totality of triads forms a triple-system of n elements. Reiss† has shown that it is possible to form a triple-system of n elements provided n is of the form $6m + 1$ or $6m + 3$.

Different methods for constructing triple-systems have been given by Reiss,† Netto,‡ Heffter,§ and E. H. Moore;|| but methods for testing the non-congruency of these systems when formed are lacking. The group of substitutions that transform a triple-system into itself has hitherto been adopted as the abstract mark of the system, and Zulauf¶ has shown, by a consideration of the groups, that the four systems on 13 elements of Kirkman, Reiss, De Vries, and Netto are reducible to two incongruent systems whose groups are different. In the present paper it is shown however that for $n = 15$ two incongruent triple-systems, Δ_{15}, may have the same group.

By a simple process non-congruent triple-systems are constructed. These are used to illustrate a new method of comparison, by means of a new sort of abstract marks. This method requires no knowledge of the group but incidentally facilitates its determination. No exhaustive determination of every Δ_{15} which may be obtained by this process is here undertaken, but the 24 systems discussed include 12 apparently not hitherto constructed.

Application of the method to a Reiss triple-system on 15 elements

2. A triple-system formed by the Reiss method** is shown in the accompanying table, which gives 28 triples of the system; the remaining 7, formed

* Presented to the Society, April, 1914.

† *Ueber eine Steinersche kombinatorische Aufgabe*, Crelle's Journal, vol. 56 (1859), pp. 326–344.

‡ *Zur Theorie der Tripelsysteme*, Mathematische Annalen, vol. 42 (1892), pp 143–152.

§ *Ueber Tripelsysteme*, Mathematische Annalen, vol. 49 (1897), pp. 101–112.

|| *Concerning Triple Systems*, Mathematische Annalen, vol. 43 (1893), pp. 271–285.

¶ *Ueber Tripelsysteme von 13 Elementen.* Dissertation. Marburg, 1897.

** M. Reiss, l. c.

from the elements which head the columns, are abd, acg, aef, bce, bfg, cdf, deg.

TABLE I

a	b	c	d	e	f	g
87	86	85	84	83	82	81
62	71	76	75	74	73	72
53	52	31	61	65	64	63
41	43	42	32	21	51	54

This system may be arranged in the following rectangular array. Each element heads one column; below it are placed the 7 pairs of elements that occur with it in triads of the system.

a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
87	86	85	84	83	82	81	$a4$	$a6$	$a5$	$a1$	$a3$	$a2$	$a8$	$a7$
62	71	76	75	74	73	72	$b7$	$b5$	$b4$	$b3$	$b2$	$b8$	$b1$	$b6$
53	52	31	61	65	64	63	$c3$	$c4$	$c1$	$c2$	$c8$	$c7$	$c6$	$c5$
41	43	42	32	21	51	54	$d6$	$d3$	$d2$	$d8$	$d7$	$d1$	$d5$	$d4$
bd	ad	ag	ab	af	ae	ac	$e2$	$e1$	$e8$	$e7$	$e6$	$e5$	$e4$	$e3$
cg	ce	be	cf	bc	bg	bf	$f5$	$f8$	$f7$	$f6$	$f1$	$f4$	$f3$	$f2$
ef	fg	df	eg	dg	cd	de	$g8$	$g7$	$g6$	$g5$	$g4$	$g3$	$g2$	$g1$

To examine the relations existing among the elements of the system, consider any triple, as $a87$ and the 21 pairs of elements which are united with the 3 elements of this triple.

a	8	7
87	$a7$	$a8$
62	$b6$	$b1$
53	$c5$	$c6$
41	$d4$	$d5$
bd	$e3$	$e4$
cg	$f2$	$f3$
ef	$g1$	$g2$

Begin with a pair in the column headed a, for example 62, written thus: call 6 the first element and 2 the second element of this pair; proceed from one column to the next in cyclical order, selecting in successive columns the pair which contains the second element of the preceding pair; continue this process until the pair to be selected from the first column is the initial pair. For the triple $a87$ the pairs would then appear as follows: 62, $2f$, $f3$, 35, $5c$, $c6$, 62, $2f$, the pairs repeating after each of the three columns has been examined twice. The occurring pairs, the periodicity, and the fact that the *period* consists of two rounds of the three columns can be shown conveniently by the notation $62f/35c/6$. The bar between two elements indicates the completion of one round of the three columns. Such a succession of pairs, repeating after 2 rounds of the columns, forms a *sequence* of period 2.

Since the elements in the pair 62 can be written either as 62 or 26, examine next this same pair with the two elements written in reverse order. Beginning with the pair 26 the pairs are selected from the columns as follows: 26, $6b$, $b1$; 14, $4d$, $d5$; 53, $3e$, $e4$; 41, $1g$, $g2$; 26, showing that after 4 rounds of the columns, the first column is again reached with the initial pair, or in sequence notation $26b/14d/53e/41g/2$.

Apply this process to the remaining pairs 53, 41, bd, cg, ef, of column a. The pair 53 appears in the sequence of period 4 and does not require further examination, since 53 as initial pair would furnish the sequence $53e/41g/26b/14d/5$, which is identical with the above sequence of period 4 except as to starting point; hence any pair which has appeared in a sequence can be omitted from further examination. The pair 41 appears in the sequence of period 4; the pair bd furnishes the sequence $bd4/ef2/gc5/db6/cg1/b$, of period 5; the pair fe leads to the sequence $fe3/f$ of period 1; the pair 87 to the sequence $87a/8$. The pair 78 leads from the body of the first column up to the element 8 at the head of the second column and the process cannot be continued. Therefore in examining the pairs of a column we omit the pair which, with the element at the head of the column, completes the triple under examination.

For the triple $a87$ the sequences are $62f/35c/6$, $26b/14d/53e/41g/2$, $bd4/ef2/gc5/db6/cg1/b$, $fe3/f$; and all possible pairs for the column a, namely 62, 26, 53, 35, 41, 14, bd, db, cg, gc, ef, fe, appear in these sequences. The number of rounds of the columns required to close a sequence (the period) is recorded for each sequence belonging to the triple under examination, and we shall call these numbers the *index* of that triple, and the longest sequence the *major sequence.*

The index of the triple $a87$ consists of the numbers 2, 4, 5, 1. The index shows that the sequences for the triple $a87$ consist of a sequence closed in two cycles of three elements, hence containing two triads, a second sequence containing four triads, a third sequence containing five triads and a fourth sequence containing one triad. In any column twelve pairs must be considered and each pair occurs in one and only one round of the columns and therefore in one and only one triad in the sequences. Hence for a triple-system on 15 elements, the sum of the periods for every index is 12. Indices of the types 111117, 111144, 66 are written in the abbreviated form $1^5\,7$, $1^4\,4^2$, 6^2.

The sequences are closed cycles, in which any element may be selected as initial element and the sequence may be read in direct or reverse order. The sequences for $a87$, $87a$, and $8a7$ illustrate the following

THEOREM. *The index of every triple in a given triple-system is invariant under all permutations of its triad.*

ANALYSIS OF THE MAJOR SEQUENCES

3. An element which appears once, twice, three times in a sequence is designated as a single, double, triple point, respectively, for that sequence. It is evident that a sequence of period 1 contains no double point and hence that no duplicate element occurs in a triad of a sequence.

LEMMA. *A sequence of period greater than one contains no repetition in any quintad of elements.*

x	y	z
ab	bc	cd
de		

Let the sequence for xyz begin in the column x with the initial pair ab, and for convenience number the places in the sequence. The elements in places 1, 2, and 3 may be a, b, and c, but the element in the place 4 is not a, since that would close the sequence. No repetition occurs in a triad of a sequence, hence the element in place 4 is neither b nor c, and therefore must be a new element d. The element to fill place 5 forms with d a pair in column x and so is neither a nor b; since this required element forms a triad with c, d, it must be a new element e. Hence no repetition occurs in a quintad of elements. By applying the above Lemma and reading the sequence forward and backward we obtain the following

THEOREM. *A sequence of period equal to or less than three contains no multiple point.*

4. This process applied to the 35 triples of the Reiss Δ_{15} gives indices and sequences only of certain types, namely:

	TRIPLE	INDEX	SEQUENCES
(1)	abd	$1^5\,7$	$871/625/786/143/268/417/534/8$, $352/3$, cgf/c, gce/g, efg/e, fec/f;
(2)	$a87$	1245	$62f/35c/6$, $26b/14d/53e/41g/2$, $bd4/ef2/gc5/db6/cg1/b$, $fe3/f$;
(3)	$a41$	6^2	$87e/26f/53b/78d/62c/35g/8$, $bd8/gc2/ef6/db3/cg5/fe7/b$.

The indices have separated the 35 triples of the Reiss system into 3 classes, (1), (2), and (3). This separation into classes will be found to simplify the problem of determining the group of the system. For every substitution which converts the system into itself must permute among themselves the triples of a class, since formation of cycles depends on identity or difference of elements in the different columns and these are not changed by any substitution.

The 35 triples arranged in classes according to their indices are shown in the following table.

Classification of the Reiss Δ_{15} by indices

$1^5\,7$			1245							6^2		
abd	bce	deg	$a26$	$b25$	$c24$	$d23$	$e38$	$f28$	$g27$	$a14$	$d16$	$g18$
acg	bfg		$a35$	$b34$	$c58$	$d48$	$e47$	$f37$	$g36$	$b17$	$e12$	
aef	cdf		$a78$	$b68$	$c67$	$d57$	$e56$	$f46$	$g45$	$c13$	$f15$	
(1)			(2)							(3)		

In the major sequence 871/625/786/143/268/417/534/8 of the triple *abd*, the second 8 enters in the seventh place after the first 8, the third 8 in the seventh place after the second 8, and the first 8 in the seventh place after the third 8; we denote this by the symbol $8(7,7,7)$; with this notation the elements of the sequence are characterized as follows: $8(7,7,7)$, $7(5,11,5)$, $1(7,7,7)$, $6(5,5,11)$, $2(8,13)$, $5(13,8)$, $4(5,5,11)$, $3(8,13)$. For a sequence of period 7 containing 21 elements the symbols $(5,11,5)$, $(5,5,11)$, $(11,5,5)$ are identical and $(8,13)\equiv(13,8)$. Hence this sequence contains two distinct types of triple points and one type of double points.

The analysis of the major sequences of the triples of class (1) is given in the following table:

Class (1)	(7, 7, 7)	(5, 11, 5)	(7, 7, 7)	(5, 5, 11)	(8, 13)	(13, 8)	(5, 5, 11)	(8, 13)
abd	8	7	1	6	2	5	4	3
bce	4	3	1	2	5	8	7	6
cdf	7	6	1	5	8	4	3	2
deg	3	2	1	8	4	7	6	5
efa	6	5	1	4	7	3	2	8
fgb	2	8	1	7	3	6	5	4
gac	5	4	1	3	6	2	8	7

Determination of the group for the Reiss Δ_{15}

5. Any operation of the group that leaves a triple-system invariant can only transform any triad and its sequences into itself, or into another of the same class. Therefore to find the group, we examine first for substitutions that transform into itself one of the longest sequences, and secondly for those that transform it into the remaining sequences of its class. Substitutions when determined must be tested on the whole triple-system.

An examination of the table of indices for this system shows that the 15 elements do not enter symmetrically as members of the triads in the classes, for example, the 7 triads in class (1) contain only the elements a, b, c, d, e, f, g. Since only those elements may be permuted which occur the same number of times in a class, the enumeration of the appearances of each element in the classes, as in the following table, shows the possible sets of transitive elements.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
(1)	3	3	3	3	3	3	3								
(2)	3	3	3	3	3	3	3		6	6	6	6	6	6	6
(3)	1	1	1	1	1	1	1	7	1	1	1	1	1	1	1

The systems of transitivity for the group are therefore a, b, c, d, e, f, g; 1; 2, 3, 4, 5, 6, 7, 8.

The substitutions may be determined from any class in the system, but most easily from the class containing the longest sequences since these exhibit

more repetitions of the elements. In the present case class (1) is therefore selected.

(1) Examine for substitutions to transform the triple *abd* into itself. The major sequence of *abd* is 871625786143268417534, and a consideration of the possible rotations of this sequence into itself will determine the substitutions which transform the triple *abd* into itself. The element 1 forms a transitive system; hence a rotation of period 3 given by (1) (647) (235) (8) transforms this sequence into itself. The triple-system will now enable us to complete the substitution, for if two elements in one triple transform into two elements in a second triple, the third element in the first triple transforms into the third element in the second triple, for example, 18*g* transforms into 18*g*. An application of both cycles (235) and (647) to the two triples 26*a* and 24*c* gives respectively 26*a*, 34*b*, 57*d*; 24*c*, 37*f*, 56*e* and shows for the required substitution additional cycles (*abd*) and (*cfe*). Hence the substitution (*abd*) (*cfe*) (*g*) (1) (235) (476) (8) transforms *abd* and its sequences into itself.

Any rotation of period 2 which transforms the sequence of *abd* into itself must keep fixed one of the elements 1 and interchange elements at equal distances from this 1, for example, (1) (46) (38) (27) $\cdots$, but in the sequence 3 is a double point and 8 is a triple point and it is not possible to convert the sequence into itself by the interchange of two points of different multiplicities; hence no substitution of period 2 exists which will convert this sequence into itself. Therefore only a sub-group of order 3 transforms the triple *abd* into itself.

(2) Examine for substitutions to transform the triple *abd* into the other triples in its class. The table of class (1) exhibits the cycles (8473625) (1). If now we apply these two cycles to any triple in the system, as *a*14, then the triples of the system appear in the following succession: 14*a*, 17*b*, 13*c*, 16*d*, 12*e*, 15*f*, 18*g*, showing the cycle (*abcdefg*) and this completes the substitution $s_2 \equiv$ (*abcdefg*) (1) (2584736) which transforms *abd* in succession into every other triple in its class. Therefore the group for the Reiss system is of order 21 and is generated by

$s_1 \equiv$ (*abd*) (*cfe*) (*g*) (1) (235) (476) (8), $s_2 \equiv$ (*abcdefg*) (1) (2584736).

The method of construction employed by Reiss involves a cyclic substitution of order 7, hence the appearance of s_2 was to be expected; but it did not show the substitution of order 3.

Construction of new systems

6. The seven triples containing only the elements which head the columns in the seven-by-four array, Table I, §2, form a triple-system on 7 elements, which is designated as a *head* for this array. The seven triples in the head

and the twenty-eight triples in the array make up the thirty-five triples in the system. New systems may be formed by retaining the twenty-eight triples of the seven-by-four array and changing the seven triples of the head or by changing the seven-by-four array. An index determined from a seven-by-four array is designated as a *partial* index to distinguish it from a complete index determined from a fifteen-by-seven array.

Formation of new systems by changing the head of the seven-by-four array

7. From the seven elements a, b, c, d, e, f, g thirty-five triples can be formed. We determine for each of these triples its partial index from the seven-by-four array I. For example:

Triple	Index	Sequences
abc	$1^2, 2, 4$	786/7, 871/352/417/625/8, 268/534/2, 143/1;
abd	1, 7	871/625/786/143/268/417/534/8, 352/3.

Table of partial indices

$1^2, 2, 4$							1, 7				
abc	*ace*	*adf*	*bcd*	*bdg*	*cde*	*cfg*	*abd*	*acg*	*bde*	*aef*	*deg*
abe	*acf*	*adg*	*bcf*	*bef*	*cdg*	*efg*	*abf*	*bce*	*cdf*	*aeg*	*dfg*
abg	*ade*	*afg*	*bdf*	*beg*	*ceg*	*def*	*acd*	*bcg*	*cef*	*bfg*	

Select any triple, as *abd*. Construct all the triple-systems on the seven elements a, b, c, d, e, f, g, which contain *abd*, and retain one of each type. Six systems Δ_7 can be formed, but the sets of partial indices of four of those systems are identical. Hence this triple furnishes only three types of head for the seven-by-four array, namely A, B, and C, as in the table.

A		B		C	
abd	1, 7	*abd*	1, 7	*abd*	1, 7
acg	1, 7	*acg*	1, 7	*ace*	$1^2, 2, 4$
aef	1, 7	*aef*	1, 7	*afg*	$1^2, 2, 4$
bce	1, 7	*bcf*	$1^2, 2, 4$	*bcf*	$1^2, 2, 4$
bfg	1, 7	*beg*	$1^2, 2, 4$	*beg*	$1^2, 2, 4$
cdf	1, 7	*dce*	$1^2, 2, 4$	*cdg*	$1^2, 2, 4$
deg	1, 7	*dfg*	1, 7	*def*	$1^2, 2, 4$

Since triples with different partial indices may give different types of heads, we examine in the same way the triple *abe*; this furnishes 4 heads of type C and 2 heads of type B.

The 35 triples formed of the elements a, b, c, d, e, f, g contain only 2 partial indices, 1^2 24 and 17, and all triple-systems of seven elements have been formed for one triple selected from each class of partial indices, and three types of heads A, B, and C have appeared. The twenty-eight triples in the seven-by-four array completed with any one of these heads form a triple-system on 15 elements.

The array, Table I, completed by these three heads, is denoted by IA, IB, IC; of these IA is the Reiss system already investigated.

	a	b	c	d	e	f	g
I	87	86	85	84	83	82	81
	62	71	76	75	74	73	72
	53	52	31	61	65	64	63
	41	43	42	32	21	51	54
IA	abd	acg	aef	bce	bfg	cdf	deg
IB	abd	acg	aef	bcf	beg	cde	dfg
IC	abd	ace	afg	bcf	beg	cdg	def

To show that the 3 systems IA, IB, IC, are non-congruent, it is necessary to write each system in the form of the fifteen-by-seven array and to obtain the indices for the 35 triples of each system; since it is possible that triples with different partial indices may have the same complete index and *vice versa*.

No two congruent systems have been admitted, but it does not follow that non-congruent systems have not been excluded, and in an exhaustive determination of all systems Δ_{15} every head must be applied to the seven-by-four array, and the indices for each system determined from its fifteen-by-seven array.

8. Classification of IB by indices and transitive elements

$1^6\,24$	$1^5\,7$	$1^3\,234$	$1^2\,2^3\,4$	$1^2\,46$	1245			$2^3\,6$	$2^2\,4^2$		6^2
bcf	abd	$d23$	$b25$	$b17$	$a87$	$b68$	$f28$	$d16$	$b34$	$d48$	$a14$
beg	acg	$f37$	$c67$	$c13$	$a26$	$e56$	$g36$	$f15$	$c24$	$f46$	
cde	aef	$g27$	$e38$	$e12$	$a35$	$c58$	$d57$	$g18$	$e47$	$g45$	
	dfg										
(1)	(2)	(3)	(4)	(5)		(6)		(7)	(8)		(9)

The enumeration of the entrances of the elements in the 9 classes shows that for the group of the system the sets of transitive elements are a; b, c, e; d, f, g; 1; 2, 3, 7; 5, 6, 8; 4. These sets of transitive elements subdivide the classes into sets of triples which are not transformable into one another by operations of the group of the system; the subdivisions are shown by lines separating the triples in a class, and the system IB contains 13 subdivisions. The group is generated by $s \equiv (a)\,(bec)\,(dfg)\,(1)\,(237)\,(586)\,(4)$, and is of order 3.

9. Classification of IC by indices and transitive elements

$1^6\,24$		$1^5\,7$	$1^3\,234$	$1^2\,2^3\,4$		$1^2\,46$		1245		$2^2\,4^2$	$23^2\,4$	6^2
ace	beg	abd	$c58$	$a35$	$g27$	$a14$	$c13$	$a78$	$c24$	$a26$	$c67$	$g18$
def	cdg		$e38$	$b25$	$g36$	$b17$	$e12$	$b68$	$e56$	$b34$	$e47$	
bcf	afg		$f28$	$d23$	$g45$	$d16$	$f15$	$d48$	$f37$	$d57$	$f46$	
(1)		(2)	(3)	(4)		(5)		(6)		(7)	(8)	(9)

The sets of transitive elements for the system are $a, b, d; c, e, f; g; 1; 2, 3, 5; 4, 6, 7; 8$. These with the indices separate the system into 13 non-permutable subdivisions. The group is generated by $s \equiv (adb)(cef)(g)(1)(253)(467)(8)$, and is of order 3.

10. A comparison of IB and IC shows that the groups of these two non-congruent systems are the same. For these groups are generated respectively by $s \equiv (a)(bec)(dfg)(1)(237)(4)(586)$, $t \equiv (adb)(cef)(g)(1)(253)(467)(8)$, and are abstractly the same, since each of these substitutions permutes 12 elements, in 4 cycles, each of order 3, and leaves the remaining 3 elements fixed. But the system IB contains 4 triples with index $1^5\,7$, while the system IC contains only 1 triple with index $1^5\,7$. Now in a given system the index of a triple is invariant under all the substitutions of the symmetric group; two systems are, therefore, certainly incongruent if their sets of indices are unlike, whether their groups are different or identical.* *Hence no substitution exists which converts* IB *into* IC, *and the two systems are non-congruent. That is, as pointed out in* §1, *the same group belongs to two non-congruent systems.*

Up to the present time the group has been considered as the abstract mark of a system. It is true that two systems with different groups are distinct, but as shown above, the converse is not true, a fact which apparently has not been suspected before.

Formation of new systems by changing the fundamental seven-by-four array

11. If a seven-by-four array shows 3 columns of the types x, y, and z containing the 3 sets of pairs 12, 34: 13, 24: 14, 23 which can be formed from the four elements 1, 2, 3, 4, the four elements constitute a *quadrangular array* which may be denoted by the symbol $\left[\begin{smallmatrix}12\\34\end{smallmatrix}\right]$. The remaining four elements 5, 6, 7, 8 must necessarily form also a quadrangular array; hence quadrangular arrays enter into a seven-by-four array always by couples, and it is easily shown that a seven-by-four array must contain 0, 2, 6, or 14 quadrangular arrays; — a fact hitherto not noticed, but of considerable importance in the following discussion.

x	y	z
12	13	14
34	24	23
56	57	58
78	68	67

An examination of every combination of two pairs occurring in the 7 columns of Table I shows that this seven-by-four array contains no quadrangular array.

Let us form, then, a new fundamental Table II, which shall contain two and only two of these quadrangular arrays. In constructing columns suitable for a seven-by-four array, the first pair in the column may be selected in 7 ways (since with the first element we may unite any one of the remaining

* The truth of the converse is not here asserted, though it seems probable.

7 elements), the second pair in 5 ways, the third pair in 3 ways, and the fourth pair only in 1 way; hence 105 suitable columns may be formed.

From the 105 columns we select 3 columns a, b, c, containing the 2 quadrangular arrays $\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}5&6\\7&8\end{smallmatrix}\right]$. The remaining 4 columns d, e, f, g, must be selected so that no more quadrangular arrays are introduced, and the entering columns must not contain any pair that appears in the columns already in the table.

TABLE II

a	b	c	d	e	f	g
87	86	85	84	83	82	81
65	75	76	73	72	71	74
43	42	41	61	64	63	62
21	31	23	25	15	45	53

The Table II shows one of the several possible ways of constructing a seven-by-four array of the desired type. The determination, from the seven-by-four array II, of the partial indices for the 35 triples which may be formed from the 7 elements a, b, c, d, e, f, g, shows 5 different partial indices. The re-heading process is now applied to Table II. A triple of each index furnishes 6 systems Δ_7. By the selection of 1 system of each type from these 30, the 6 different heads A, B, C, D, E, F shown in the following table are obtained for the seven-by-four array II.

	a	b	c	d	e	f	g	Number of triples of the head with partial indices				
II	87 65 43 21	86 75 42 31	85 76 41 23	84 73 61 25	83 72 64 15	82 71 63 45	81 74 62 35	1^8	$1^4\,2^2$	$1^2\,24$	2^4	4^2
A	*bde*	*abc*	*bfg*	*adf*	*cdg*	*aeg*	*cef*	1	0	4	2	0
B	*bde*	*abf*	*bcg*	*acd*	*dfg*	*aeg*	*cef*	0	0	2	2	3
C	*bde*	*abf*	*bcg*	*adg*	*cdf*	*ace*	*efg*	0	2	1	1	3
D	*bde*	*abg*	*bcf*	*acd*	*dfg*	*aef*	*ceg*	0	2	1	2	2
E	*bde*	*bac*	*bfg*	*dag*	*cdf*	*aef*	*ceg*	1	4	2	0	0
F	*bde*	*abg*	*bcf*	*adf*	*cdg*	*ace*	*efg*	0	0	2	3	2

If a system Δ_{15} contains more than one Δ_7, then for every such contained Δ_7 there is an arrangement of the Δ_{15}, showing this Δ_7 as a head to a suitable seven-by-four array. These different arrangements may furnish dissimilar partial indices. For example, the system IIA contains three systems Δ_7 formed of the elements a, b, c, d, e, f, g; $a, b, c, 1, 2, 3, 4$; and $a, b, c, 5, 6, 7, 8$. The second of these heads, placed over the seven-by-four array formed of the 28 remaining triples of the system IIA, gives for the triples abc, $a43$, $a12$, $b42$, $b31$, $c41$, $c23$, respectively, the following partial indices 1^8, $1^4\,2^2$, $1^4\,2^2$, 2^4, 2^4, 4^2, 4^2. Hence IIA shows dissimilar sets of partial indices for different heads. Therefore if a system Δ_{15} contains more than one Δ_7, a determination of the *partial indices* of the seven triples of one head is not a sufficient characterization of the system.

To determine if the six systems with fundamental Table II are distinct, we compare the indices of the thirty-five triples of each system, since systems with different sets of indices are incongruent. The indices for the 6 systems with fundamental seven-by-four array II are shown in the following table:

II	1^{12}	$1^8 2^2$	$1^6 24$	$1^5 2^2 3$	$1^4 2^4$	$1^4 26$	$1^4 4^2$	$1^3 2^2 5$	$1^3 234$	$1^2 2^3 4$	$1^2 2^2 3^2$	$1^2 46$	1236
A	1		12		10						12		
B			2		2	4	3	4	4	4	8		4
C		2	1	6	5		3	4	4		4	2	4
D		2	1	6	4		2	6	2	2	8	2	
E	1	10	4	8	6		2				4		
F			2		5	4	2		4	4	10		4

This table shows 6 distinct systems differing from the 3 systems with fundamental seven-by-four array I.*

The sets of transitive elements for IIA are $a, b, c; d, f, 2, 4, 5, 8; e, g, 1, 3, 6, 7$; these with the classes separate the 35 triples of the system into 6 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (ab)(d2f4)(e1g3)(58)$, $s_3 \equiv (abc)(d48)(e17)(f25)(g36)$, and is of order 24.

The sets of transitive elements for IIB are $a; b, c; d, f; e; g; 1, 3, 6, 7; 2, 4, 5, 8$; these with the classes separate the system into 13 non-permutable subdivisions. The group is generated by $s_1 \equiv (bc)(1637)(2548)(df)$, and is of order 4.

The sets of transitive elements for IIC are $a; b; c; d; e; f; g; 1, 3; 2, 4; 5, 8; 6, 7$; these with the classes separate the system into 23 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, and is of order 2.

The sets of transitive elements for IID are $a; b; c; d; e; f; g; 1, 3; 2, 4; 5, 8; 6, 7$; these with the classes separate the system into 23 non-permutable subdivisions. The group is generated by $s \equiv (13)(24)(58)(67)$, and is of order 2.

The two systems IIC *and* IID *are distinct, since the number of triples appearing under each index is not the same for the two systems; but the group is identical for the two systems.* This is another example of the fact that two distinct systems may have the same group.

The sets of transitive elements for IIE are $a; b; c; d, f, 6, 7; e, g, 5, 8; 1, 3; 2, 4$; these with the classes separate the system into 12 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (d7f6)(e5g8)(13)(24)$, and is of order 8.

* For the sake of brevity the work involved in the determination of the indices for every system is suppressed, and a brief outline of the process employed in obtaining the groups is given only for the system IA. A fuller elaboration is contained in a manuscript volume deposited in the Vassar College Library.

The sets of transitive elements for IIF are a; b, c; d; f; e, g; 1, 3, 6, 7; 2, 4, 5, 8; these with the classes separate the system into 15 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (bc)(eg)(17)(28)(36)(45)$, and is of order 4.

Seven-by-four array with 14 quadrangular arrays

12. We construct next a seven-by-four array containing 14 quadrangular arrays, the greatest possible number, and we apply to this new array the re-heading process. The partial indices show 4 types of heads and we obtain the following systems:

	a	b	c	d	e	f	g	Number of triples with indices			
III	12	13	14	15	16	17	18				
	34	24	23	26	25	28	27				
	56	58	57	38	37	36	35				
	78	67	68	47	48	45	46	1^{12}	$1^8 2^2$	$1^6 3^2$	$1^4 2^4$
A	abc	ade	afg	bdg	bef	cdf	ceg	35			
B	abc	ade	afg	bdf	beg	cdg	cef	7	24		4
C	abc	adg	aef	bdf	beg	cde	cfg	1	12	16	6
D	abd	acg	aef	bcf	beg	cde	dfg			28	7

The system IIIA contains one set of transitive elements a, b, c, d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8, and the thirty-five triples belong to one class.

The analysis of the Kirkman system by the method of sequences shows the index of each triple to be 1^{12}, and the system IIIA is congruent to the Kirkman system. The group for the system IIIA is by far the most extensive of the 24 groups determined in this paper, and belongs to the earliest known Δ_{15}.

The group for IIIA is generated by $s_1 \equiv (12)(34)(56)(78)$, $s_2 \equiv (13)(24)(58)(67)$, $s_3 \equiv (15)(26)(38)(47)$, $s_4 \equiv (acb)(efg)(243)(678)$, $s_5 \equiv (bc)(fg)(34)(78)$, $s_6 \equiv (bc)(dfeg)(34)(5768)$, $s_7 \equiv (adgfceb)(1347286)$, $s_8 \equiv (ad7gfc5368b1e42)$, and is of order $8!/2$.*

The sets of transitive elements for IIIB are a; b, c, d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group for the system is generated by $s_1 \equiv (beg)(cdf)(12)(358467)$, $s_2 \equiv (df)(eg)(57)(68)$, $s_3 \equiv (15)(26)(38)(47)$, $s_4 \equiv (de)(fg)(56)(78)$, and is of order 96.

The sets of transitive elements for IIIC are a, b, c; d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group for the system IIIC is generated by $s_1 \equiv (abc)(dfg)(234)(578)$, $s_2 \equiv (16)(25)(37)(48)$, $s_3 \equiv (12)(34)(56)(78)$, $s_4 \equiv (dg)(ef)(58)(67)$, and is of order 96. The groups for the systems IIIB and IIIC are of the same order 96, but the sets of transitive elements for the two groups differ, hence the groups are distinct.

* C. Jordan (*Traité des substitutions et des équations algébriques* (1870), p. 380) states that the group for the Kirkman system is of order $8!/2$, which agrees with the result obtained by the sequence method of analysis.

The sets of transitive elements for IIID are a, b, c, d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group is generated by $s_1 \equiv (acg)(bef)(248)(367)$, $s_2 \equiv (15)(26)(38)(47)$, $s_3 \equiv (abedgcf)(3658472)$, $s_4 \equiv (13)(24)(58)(67)$, and is of order 168.

Seven-by-four array with six quadrangular arrays

13. We construct finally a seven-by-four array containing 6 quadrangular arrays, the only remaining number (cf. §11) and we obtain the 4 following systems:

	a	b	c	d	e	f	g
IV	87	86	85	84	83	82	81
	65	75	76	73	74	71	72
	43	41	42	62	61	64	63
	21	23	31	15	25	53	54
A	bdf	abc	beg	adg	cde	afe	cfg
B	bdf	abe	bcg	adg	cde	acf	efg
C	bdf	abe	cef	bcg	acd	deg	afg
D	bdf	abc	beg	ade	cdg	afg	ecf

IV	1^{12}	$1^8 2^2$	$1^6 24$	$1^6 3^2$	$1^5 2^2 3$	$1^4 2^4$	$1^4 26$	$1^4 4^2$	$1^2 2^2 3^2$
A	1	8			16	10			
B		1		4	8	6	8	2	6
C	1	10	4		8	6		2	4
D	7	24				4			

The systems IVC and IVD show the same indices as the systems IIE and IIIB, respectively, and are congruent to them.

The sets of transitive elements for IVA are $a, b; c; d, e, f, g$, 1, 2, 3, 4; 5, 6, 7, 8; and the system contains 7 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (ab)(d1)(e3)(f2)(g4)(67)$, $s_2 \equiv (ab)(d2f1)(e4g3)(5687)$, $s_3 \equiv (13)(24)(57)(68)$, and is of order 32.

The systems of transitive elements for IVB are $a; b; c; d; e; f; g$; 1, 2, 3, 4; 5, 6, 7, 8; and the system separates into 17 non-permutable subdivisions. The group is generated by $s_1 \equiv (12)(34)(56)(78)$, $s_2 \equiv (14)(23)(58)(67)$, and is of order 4.

Formation of new systems by varying the number of partial quadrangular arrays

14. The three sets of pairs 12, 34; 13, 24; 14, 23 entering the seven-by-four array in 3 columns form a quadrangular array which has been denoted by the symbol $[{12 \atop 34}]$. Two of these sets of pairs occurring in the seven-by-four array in 2 columns form a *partial* quadrangular array, and may be denoted by the symbol $\{{12 \atop 34}\}$. A partial quadrangular array considered with respect to the two elements which head the columns in which the two sets of pairs enter, is designated in a recent paper as an "interlacing"* of the two ele-

* F. N. Cole, *The triad systems of thirteen elements.* These Transactions, vol. 14 (1913), pp. 1–5.

ments which head the columns. Non-congruent triple-systems may be formed by keeping constant the number of quadrangular arrays in the seven-by-four array, and varying the number of partial quadrangular arrays.

The seven-by-four array II contains two quadrangular arrays $\left[\begin{smallmatrix}12\\34\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}56\\78\end{smallmatrix}\right]$ and eight partial quadrangular arrays $\left\{\begin{smallmatrix}12\\56\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}12\\78\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}13\\58\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}13\\67\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}24\\58\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}24\\67\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}34\\56\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}34\\78\end{smallmatrix}\right\}$. The seven-by-four array V, given below, contains the same number of quadrangular arrays, $\left[\begin{smallmatrix}12\\58\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}34\\67\end{smallmatrix}\right]$, but a different number of partial quadrangular arrays, the twelve $\left\{\begin{smallmatrix}12\\46\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}12\\37\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}13\\48\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}13\\56\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}14\\57\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}16\\78\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}23\\45\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}23\\68\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}24\\78\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}25\\67\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}35\\78\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}45\\68\end{smallmatrix}\right\}$. The re-heading process applied to this seven-by-four array shows six types of heads.

	a	*b*	*c*	*d*	*e*	*f*	*g*	Number of triples with partial indices			
V	12 58 34 67	15 28 37 46	18 25 36 47	14 26 38 57	17 23 45 68	13 27 48 56	16 24 35 78	1^8	$1^4\,2^2$	2^4	4^2
A	*abc*	*ade*	*afg*	*bdf*	*beg*	*cdg*	*cef*	1		6	
B	*abd*	*ace*	*afg*	*bcf*	*beg*	*cdg*	*def*			4	3
C	*abd*	*ace*	*afg*	*bcg*	*bef*	*cdf*	*deg*		2	2	3
D	*abd*	*acf*	*aeg*	*bcg*	*bef*	*cde*	*dfg*		3	1	3
E	*abc*	*ade*	*afg*	*bdg*	*bef*	*cdf*	*ceg*	1	4	2	
F	*abc*	*adf*	*aeg*	*bdg*	*bef*	*cde*	*cfg*	1	6		

A determination of the indices of the 6 systems with fundamental seven-by-four array V gives the following result:

V	1^{12}	$1^8\,2^2$	$1^6\,3^2$	$1^5\,2^2\,3$	$1^4\,2^4$	$1^4\,35$	$1^4\,4^2$	$1^3\,234$	$1^3\,3^3$	$1^2\,2^2\,3^2$
A	1				18				16	
B					4		3	12	4	12
C		2		4	6	8	3	4		8
D		3	4		7	12	3			6
E	1	8		16	10					
F	1	12	16		6					

The triple-systems V*E* and V*F* show the same sets of indices as IV*A* and III*C*, respectively, and are congruent to them. The analysis follows for the 4 systems V*A*, V*B*, V*C*, V*D*, which are distinct from the preceding 15 triple-systems.

The sets of transitive elements for V*A* are $a, b, c; d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8$. The group is generated by $s_1 \equiv (abc)\,(efg)\,(258)\,(367)$, $s_2 \equiv (bc)\,(d1)\,(e2)\,(f8)\,(g5)\,(67)$, $s_3 \equiv (d14)\,(e23)\,(f56)\,(g87)$, $s_4 \equiv (acb)\,(dfg)\,(182)\,(376)$, and is of order 288.

The sets of transitive elements for V*B* are $a, b, c; d, e, f; g; 1, 2, 3, 4, 5, 6, 7, 8$; these with the classes separate the system into 6 non-permutable subdivisions. The group is generated by $s_1 \equiv (ab)\,(ef)\,(1327)\,(4865)$, $s_2 \equiv (ab)\,(ef)\,(14)\,(26)\,(35)\,(78)$, $s_3 \equiv (acb)\,(def)\,(182)\,(467)$, and is of order 24.

The sets of transitive elements for VC are a; b, c; d, e; f; g; 1, 2, 3, 4, 5, 6, 7, 8; these with the classes separate the system into 11 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (bc)(de)(16)(27)(35)(48)$, $s_2 \equiv (bc)(de)(1457)(2386)$, and is of order 8.

The sets of transitive elements for VD are a, b, c; d, f, g; e; 1, 2, 5, 8; 3, 4, 6, 7; these with the classes separate the system into 7 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (12)(37)(46)(58)$, $s_2 \equiv (18)(25)(34)(67)$, $s_3 \equiv (acb)(dfg)(125)(347)$, and is of order 12.

Analysis of triple-systems formed by other methods

15. A few of the well-known systems Δ_{15} are now examined, in order to determine, if possible, systems incongruent to the 19 already given.

(i) *Analysis of the Carpmael systems.* – The indices for the systems given by Carpmael* show two non-congruent systems. One of these yields a set of indices identical with that of VA; hence this system is omitted from further examination. The other Carpmael system yields a new set of indices; therefore we determine in this system a Δ_7, arrange the system in the form of a seven-by-four array, and apply the re-heading process, thus obtaining four systems incongruent to the preceding 19, VIA, VIB, VIC, and VID of the following table:

	a	b	c	d	e	f	g							
VIA	12	17	16	15	18	14	13	*abe*	*acf*	*adg*	*bdf*	*cde*	*efg*	*bcg*
VIB	34	23	24	28	27	25	26	*abc*	*adg*	*aef*	*bde*	*bfg*	*cdf*	*ceg*
VIC	58	48	38	36	35	37	45	*abd*	*acg*	*aef*	*bce*	*bfg*	*cdf*	*deg*
VID	67	56	57	47	46	68	78	*abg*	*ace*	*adf*	*bcd*	*bef*	*cfg*	*deg*

A comparison of the two seven-by-four arrays I and VI shows that neither contains any quadrangular array, but that while I contains no partial quadrangular array, VI contains the following six $\left\{\begin{smallmatrix}12\\58\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}14\\68\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}15\\67\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}23\\48\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}23\\57\end{smallmatrix}\right\}$, and $\left\{\begin{smallmatrix}34\\67\end{smallmatrix}\right\}$. Hence the two sets of systems I and VI may be formed by constructing seven-by-four arrays, which contain a fixed number (zero) of quadrangular arrays and a variable number of partial quadrangular arrays. The analysis and the groups for the system VI are given below.

	$1^8 2^2$	$1^6 24$	$1^5 2^2 3$	$1^5 7$	$1^4 2^4$	$1^4 35$	$1^4 4^2$	$1^3 2^2 5$	$1^3 234$	$1^2 2^3 4$	$1^2 2^2 3^2$	$1^2 5^2$	1236	1245	$13^2 5$	$2^3 6$	2^6
VIA				4	3			6					12		6		4
VIB	1	2		2	1	2	1	3	2	2	8	2	2	4	1	2	
VIC		3		1			3			3	6	9	6	3			1
VID	3	3	6	1		3		3		3	6	6					1

* *Some solutions of Kirkman's 15-school-girl problem*, Proceedings of the London Mathematical Society, vol. 12 (1881), pp. 148–156.

The sets of transitive elements for VIA are $a, b, c, d, e, f; g; 1, 2, 4, 7; 3, 5, 6, 8$. The group is generated by $s_1 \equiv (abe)(cfd)(172)(386)$, $s_2 \equiv (ad)(bc)(14)(27)(35)(68)$, $s_3 \equiv (bc)(ef)(12)(36)(47)(58)$, and is of order 12.

The sets of transitive elements for VIB are $a; b, c; d; e, f; g; 1, 2; 3, 6; 4, 7; 5, 8$. These with the classes separate the system into 21 non-permutable subdivisions. The group is of order 2 and is generated by $s \equiv (bc)(ef)(12)(36)(47)(58)$.

The sets of transitive elements for VIC are $a, b, e; c, d, f; g; 1, 2, 7; 4; 5; 3, 6, 8$. These with the classes separate the system into 13 non-permutable subdivisions. The group for the system is generated by $s \equiv (aeb)(cdf)(127)(368)$, and is of order 3.

The sets of transitive elements for VID are $a, c, e; b, d, f; g; 1, 4, 7; 2; 6; 3, 5, 8$. These with the classes separate the system into 13 non-permutable subdivisions. The group for the system is generated by $s \equiv (ace)(bfd)(147)(358)$, and is of order 3.

A comparison of the systems IB, IC, VIC, VID, shows different sets of indices but identical groups. *Here then are four incongruent systems with the same group* (cf. §8 and §9). *Therefore it is evident that the identity of the groups, though a necessary condition for the congruency of two triple-systems, is not sufficient.*

(ii) *Analysis of the Heffter systems.* — The Heffter* method of constructing triple-systems of $6m + 3$ elements furnishes two systems for $6m + 3 = 15$. The first of these contains 15 systems Δ_7, the index of each triple is 1^{12}, and the system shows the same set of indices as the system here denoted by IIIA. The second system contains no Δ_7, and is therefore distinct from the preceding 23 systems, each of which contains at least one Δ_7. This second system is now denoted by VII. Its analysis is shown in the following table:

Classification of VII by indices

4, 8		3, 9									
*af*4	*d*27	*a*58	*ag*7	*bc*7	*b*24	*ce*4	*df*5	*eg*6	*f*17	*g*28	367
*bg*5	*e*38	*ade*	*ac*2	*bef*	*cd*8	*c*35	*d*46	*e*57	*f*68	145	478
*c*16		*ab*6	*a*13	*b*18	*cfg*	*dg*1	*e*12	*f*23	*g*34	256	*bd*3

The system contains 1 set of transitive elements: $a, b, c, d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8$, and separates into 2 non-permutable subdivisions. The group is generated by $s_1 \equiv (ad63)(c8f2)(e471)$, $s_2 \equiv (abcdefg12345678)$, and is of order 60.

(iii) Netto† describes two methods for constructing triple-systems of 15

* L. Heffter, *Ueber Tripelsysteme*, Mathematische Annalen, vol. 49 (1897), pp. 101–112.

† L. c.

elements. The analysis of the two systems by the method of sequences shows that the index of every triple in each system is 1^{12} and the two systems are of the type here designated by IIIA.

(iv) E. H. Moore* gives different methods for constructing triple-systems. A system formed by one of these methods was found to be of the type IIIA. Under the re-heading process a system formed by a second method yielded 4 systems congruent, respectively, to the systems here designated as IIE, IIIB, IIIC, IVA.

16. We have shown for the two fundamental seven-by-four arrays containing, respectively, zero and two quadrangular arrays, that non-congruent systems may be formed by varying the number of partial quadrangular arrays. It seems possible that additional systems Δ_{15}, each containing at least one Δ_7, might be formed by the same method from these two arrays, and also from the two fundamental arrays containing, respectively, six and fourteen quadrangular arrays. The purpose of this investigation is primarily to exhibit a new method of comparison for triple-systems, and while several systems Δ_{15} not hitherto constructed have been obtained, no exhaustive determination of all possible systems Δ_{15} has been undertaken.

The method of comparison here investigated is applicable to any system Δ_{2t+1} by merely arranging the system in the form of a $2t + 1$-by-t array and determining the indices for this array. In constructing systems Δ_{15} which contain no Δ_7, the number of partial quadrangular arrays admitted into a fifteen-by-seven array must be considered, but for systems Δ_{15} which contain one or more systems Δ_7, the number of complete and partial quadrangular arrays admitted into a seven-by-four array is to be considered. Exactly similar conditions exist in the case of systems Δ_{2t+1}. In constructing systems Δ_{2t+1} which contain no Δ_t, the consideration of a $2t + 1$-by-t array is necessary, but for systems Δ_{2t+1} which contain one or more systems Δ_t, the consideration of a t-by-$\frac{1}{2}(t + 1)$ array is sufficient.

April, 1913

* L. c.

AN EXISTENCE THEOREM FOR A CERTAIN DIFFERENTIAL EQUATION OF THE nTH ORDER*

BY

WILLIAM RAYMOND LONGLEY

1. The usual method for investigating the solutions of a differential equation of the nth order in the neighborhood of a singular point has been to reduce the problem to a system of n equations of the first order. This procedure was adopted by Poincaré† and others for the treatment of equations of a general form, and the results obtained have been summarized and simplified by Dulac.‡ No explicit statement has been made concerning an equation of the form

$$\frac{d^n y}{dx^n} = \frac{N}{D} = \frac{ax + by + \cdots}{\alpha x + \beta y + \cdots}, \tag{1}$$

where N and D are convergent power series in x and y vanishing for $x = 0$, $y = 0$, but an application of the general theory yields the following information concerning the integrals in the neighborhood of the values $x = 0$, $y = 0$.

Equation (1) may be reduced to the system

$$\frac{dy}{dx} = c + y_1, \qquad \frac{dy_1}{dx} = c_2 + y_2, \qquad \cdots, \qquad \frac{dy_{n-1}}{dx} = \frac{N}{D}, \tag{2}$$

where $c, c_2, \cdots, c_{n-1}$ are arbitrary finite constants and the initial values of the variables are all zero. If $\lambda = \alpha + c\beta$ is not zero then the variables can be expressed as power series in terms of a single quantity z; that is,

$$x = x(z), \qquad y = y(z), \qquad \cdots, \qquad y_{n-1} = y_{n-1}(z), \tag{3}$$

such that if z satisfies an equation

$$\frac{dz}{dt} = \lambda z + z^2 P, \tag{4}$$

then the functions (3) will satisfy the system (2). The quantity P is a power

* Presented to the Society, December 30, 1913.

† Journal de mathématiques, ser. 4, vol. 1 (1885), p. 167, and vol. 2 (1886), p. 151.

‡ Bulletin de la société mathématique de France, vol. 40 (1912), p. 324.

series in z whose coefficients depend upon the system (2). The solution z of equation (4) can be expressed as a power series in $\theta = ke^{\lambda t}$, k being an arbitrary constant. After substitution of this expression in the system (3), θ can be found in terms of x from the first equation and, when this value is put in the second, y is obtained as a function of x which satisfies equation (1). This solution involves the $n-1$ finite constants $c, c_2, \cdots, c_{n-1}$, which are arbitrary except for the condition $\alpha + c\beta \neq 0$. The exact form of the solution is not given and its construction by this general method would involve a number of transformations and considerable calculation.

The object of the present paper is to obtain this result directly in explicit form and to derive results in some cases for which $\alpha + c\beta = 0$.

2. By setting $y = x(c+v)$, where v is to be determined as a function of x vanishing with x, and c is a constant, equation (1) becomes

$$x\frac{d^n v}{dx^n} + n\frac{d^{n-1} v}{dx^{n-1}} = \frac{a + b(c+v) + xN'}{\alpha + \beta(c+v) + xD'}.$$

If $\alpha + c\beta \neq 0$ the second member can be expanded as a power series in x and v, and the equation takes the form

$$x\frac{d^n v}{dx^n} + n\frac{d^{n-1} v}{dx^{n-1}} = a_{00} + a_{10}x + a_{01}v + \cdots + a_{ij}x^i v^j + \cdots. \tag{5}$$

Suppose a solution is assumed in the form

$$v = c_2 x + c_3 x^2 + \cdots + c_{n-1}x^{n-2} + c_n x^{n-1} + \cdots + c_{m+1}x^m + \cdots. \tag{6}$$

On substituting this value of v in equation (5) and equating the coefficients of corresponding powers of x, it follows that $c_2, c_3, \cdots, c_{n-1}$ may be chosen arbitrarily, while the remaining coefficients of the series (6) are uniquely determined by

$$\begin{aligned} &\underline{|n}\, c_n = a_{00}, \\ &\cdots\cdots\cdots\cdots, \\ &\{m(m-1)\cdots(m-n+1) \\ &\qquad + nm(m-1)\cdots(m-n+2)\}c_{m+1} \\ &\qquad\qquad = F_{m+1}(a_{ij}, c_2, \cdots, c_{n-1}), \end{aligned} \tag{7}$$

where F_{m+1} is a rational integral function of $c_2, \cdots, c_{n-1}$ and the coefficients $a_{ij}\,(i+j < m-n)$.

The convergence of the series (6) can be demonstrated by comparing it with the solution of

$$\frac{d^{n-1}u}{dx^{n-1}} = a'_{00} + a'_{10}x + a'_{01}u + \cdots + a'_{ij}x^i u^j + \cdots, \tag{8}$$

the coefficients a'_{ij} being positive constants equal to the absolute values of the a_{ij}. By Cauchy's fundamental theorem it is known that this equation has a solution for u as a power series in x which is convergent for values of x whose moduli are sufficiently small. The solution is

$$(9) \qquad u = \alpha_2 x + \cdots + \alpha_{n-1} x^{n-2} + \alpha_n x^{n-1} + \cdots + \alpha_{m+1} x^m + \cdots,$$

where $\alpha_2, \cdots, \alpha_{n-1}$ are arbitrary. The remaining coefficients are determined by

$$(10) \qquad \begin{aligned} &\underline{|n-1}\, \alpha_n = a'_{00}, \\ &\cdots\cdots\cdots\cdots\cdots, \\ &m(m-1)\cdots(m-n+2)\,\alpha_{m+1} = F'_{m+1}(a'_{ij}, \alpha_2, \cdots, \alpha_{n-1}). \end{aligned}$$

The function F'_{m+1} in equations (10) is a rational integral function of the same form as F_{m+1} in equations (7), the coefficients being all positive and greater than the absolute values of the corresponding coefficients in F_{m+1}. It follows then that the absolute value of c_{m+1} is less than α_{m+1} for every m provided $\alpha_2 = c_2, \cdots, \alpha_{n-1} = c_{n-1}$. Hence the series (6) converges for at least the same range of values as the series (9).

The solution of equation (1) is now given in the explicit form

$$y = cx + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + \cdots + c_m x^m + \cdots,$$

where $c, c_2, \cdots, c_{n-1}$ are finite constants, arbitrary except for the condition $\alpha + c\beta \neq 0$. The remaining coefficients are uniquely determined in terms of the arbitrary constants and the coefficients in the differential equation. This statement contains the information which may be derived from the general theory as summarized by Dulac.

3. The preceding method may be applied without alteration to the case when in equation (1) the lowest terms in N are of degree p, and the lowest terms in D are of degree q if $p \geqq q$. If the terms of lowest degree in D are

$$D_0 x^q + D_1 x^{q-1} y + \cdots + D_q y^q,$$

the assumption concerning the constant c is

$$D_0 + D_1 c + \cdots + D_q c^q \neq 0.$$

Geometrically this assumption is expressed by saying that the integral curve is not tangent at the origin to a branch of the curve $D = 0$.

The existence theorem may now be stated in the following form:

THEOREM. *Given the differential equation*

$$(E) \qquad \frac{d^n y}{dx^n} = \frac{N}{D} = \frac{N_0 x^p + N_1 x^{p-1} y + \cdots + N_p y^p + \cdots}{D_0 x^q + D_1 x^{q-1} y + \cdots + D_q y^q + \cdots}$$

in which N and D are convergent series in positive integral powers of x and y

and $p \geqq q$. *If*

$$(C) \qquad D_0 + D_1 c + \cdots + D_q c^q \neq 0,$$

then there exists a solution expressible in the form

$$(S) \qquad y = cx + \sum_{i=2}^{\infty} c_i x^i,$$

which is convergent for values of x *whose moduli are sufficiently small, and the coefficients of the first* $n - 1$ *powers of* x *are arbitrary except for the condition* (C).

The fundamental theorem of Cauchy applies to an equation of the form

$$(11) \qquad \frac{d^n y}{dx^n} = P,$$

in which P is a series in positive integral powers of x and y, and asserts the existence of a solution of the form (S) in which the coefficients of the first $n - 1$ powers of x are entirely arbitrary. The difference in the results may be stated geometrically by saying that for equation (11) the regular integral curves through the origin may go in *any* direction, while for the equation (E) they may go in any direction not tangent to a branch of the curve $D = 0$ (excluding always the direction of the y-axis).

4. We now return to equation (1) in which the numerator and denominator contain terms of the first degree and inquire for integral curves tangent at the origin to the curve $D = 0$. We ask if the equation can be satisfied by expressing y as a series in positive integral powers of x^σ, where σ is a positive constant. Since $dy/dx = -\alpha/\beta = c$ for $x = 0$, we make the substitution $y = -\alpha x/\beta + v$, and v is to be determined as a series in x^σ containing x^k as a factor, where k is a positive number greater than unity, and satisfying the equation

$$(12) \qquad \frac{d^n v}{dx^n} = \frac{Ax + bv + P_2}{\beta v + Q_2}.$$

In this equation P_2 and Q_2 contain no first degree terms, and $\beta A = a\beta - b\alpha$. Let the term in Q_2 of lowest degree independent of v be x^r. Then r is an integer greater than unity. Suppose that A is not zero, which means that the curves $N = 0$ and $D = 0$ are not tangent at the origin.

Let a solution of equation (12) be assumed in the form

$$(13) \qquad v = lx^k + mx^{k+\sigma} + \cdots \qquad (l \neq 0).$$

After multiplying equation (12) by the denominator of the second member and substituting the value of v, the index of the lowest power of x must be the same on each side of the equation. Two cases are to be considered.

I. Suppose k is not an integer less than n. If k is not greater than r, the lowest term in the left member is of degree $2k - n$, and in the right member is of the first degree. Hence $k = (n + 1)/2$, which is possible only if n is even. If k is greater than r, the index equation is $k - n + r = 1$, which is contrary to the assumption that k is not an integer less than n.

II. Suppose k is an integer less than n. If σ is not an integer, the index equation is $2k + \sigma - n = 1$ or $k + \sigma + r - n = 1$, which contradicts the assumption that σ is not an integer. If σ is an integer it may be taken as unity without loss of generality. The lowest power of x in $d^n v/dx^n$ will be x^q, where q is zero or a positive integer, and the index equation becomes $k + q = 1$, which contradicts the assumption that k is greater than unity, or $r + q = 1$, which is impossible.

The preceding argument shows that equation (1) may admit a solution for y expressible as a positive power series in x^σ $(\sigma > 0)$ such that $dy/dx = -\alpha/\beta$ when $x = 0$, only if n is even. If such a series exists the first two terms will be

$$(N) \qquad y = -\frac{\alpha}{\beta}x + lx^{(n+1)/2} + \cdots.$$

5. We shall consider in detail the case when $n = 2$, and show that equation (1) admits a solution for y as a power series in $x^{\frac{1}{2}}$. On introducing ξ as the independent variable by the relation $x = \xi^2$, equation (1) becomes

$$(14) \qquad \frac{1}{4\xi^3}\left\{\xi\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\right\} = \frac{a\xi^2 + by + \cdots}{\alpha\xi^2 + \beta y + \cdots}.$$

Following the condition (N) it is convenient to set

$$y = -\frac{\alpha}{\beta}\xi^2 + \xi^3(l + v),$$

where v is to be determined as a power series in ξ vanishing with ξ. Then equation (14) takes the form

$$(15) \qquad \xi^2\frac{d^2v}{d\xi^2} + 5\xi\frac{dv}{d\xi} + 3(l + v) = \frac{4A + \xi P_0}{\beta l + \beta v + \xi Q_0},$$

in which P_0 and Q_0 are series in ξ and v. The constant l is determined by setting $\xi = 0$, $v = 0$ which gives $3\beta l^2 = 4A$. The second member of equation (15) can be expanded as a power series in ξ and v, and the equation becomes

$$(16) \qquad \xi^2\frac{d^2v}{d\xi^2} + 5\xi\frac{dv}{d\xi} + 6v = A\xi + \cdots + A_{ij}\,\xi^i v^j + \cdots \qquad (i + j > 1).$$

Let a solution be assumed in the form

$$(17) \qquad v = a_1\xi + a_2\xi^2 + \cdots + a_n\xi^n + \cdots.$$

The equations for the determination of the coefficients are

$$(18) \qquad (5+6)a_1 = A, \qquad \cdots, \qquad \{n(n-1)+5n+6\}a_n = F_n.$$

The quantities F_n are rational integral functions of $A_{ij}\,(i+j \leqq n)$, and the coefficients in equation (17) are uniquely determined.

The convergence of the series (17) can be demonstrated by comparing it with the solution of

$$(19) \qquad 5\xi\frac{du}{d\xi} + 6u = A'\xi + \cdots + A'_{ij}\,\xi^i u^j + \cdots \qquad (i+j>1),$$

the coefficients A'_{ij} being positive constants equal to the absolute values of the A_{ij}. If a solution is assumed in the form

$$(20) \qquad u = \alpha_1\xi + \cdots + \alpha_n\xi^n + \cdots,$$

the equations for the determination of the coefficients are

$$(21) \qquad (5+6)\alpha_1 = A', \qquad \cdots, \qquad (5n+6)\alpha_n = F'_n.$$

In equations (21), F'_n is a rational integral function of the same form as F_n in equations (18), the coefficients being all positive and greater than the absolute values of the corresponding coefficients in F_n. Evidently the absolute value of a_n is less than α_n for every n. Now equation (19) is the well-known equation of Briot and Bouquet* which admits a solution of the form (20) if the coefficient of u (in this case 6/5) is not a negative integer. The series (20) is therefore known to be convergent for values of x whose moduli are sufficiently small. Consequently the series (17) is convergent for at least the same range of values.

The conclusion is that if $a\beta - b\alpha \neq 0$, then equation (1), in which $n = 2$, admits a solution of the form

$$y = -\frac{\alpha}{\beta}x + x^{\frac{3}{2}}P,$$

where P is a power series in $x^{\frac{1}{2}}$.

When this result is combined with that of the theorem above, we can make the following geometric statement concerning a differential equation of the form (1) and of the second order. If the origin is an ordinary point on each of the curves $N = 0$ and $D = 0$, and if these curves have distinct tangents at the origin, then, in every direction through the origin (except, perhaps, tangent to the y-axis) there passes one integral curve. These integral curves are all regular except the one tangent to the curve $D = 0$. This one has a branch point at the origin.

It has been said above that for n even the equation (1) may possibly admit a

*Journal de l'école polytechnique, vol. 21 (1856), p. 161.

solution of the form (N). For n greater than 2 the search for such solutions involves the terms in D of order higher than the first and the investigation will not be undertaken in this paper.

Sheffield Scientific School,
New Haven, Connecticut,
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THE CONIC AS A SPACE ELEMENT*

BY

ROGER A. JOHNSON

Introduction

In this article it is proposed to determine a system of coördinates for the conic in space of three dimensions, analogous to the line coördinates of Plücker, and the similar systems for the circle in two and in three dimensions; and to study systems of conics by means of these coördinates.

The conic in space has been considerably studied in the last sixty years, but no systematic theory has been developed. In 1908, the Belgian Royal Academy announced the offer of a prize for a discussion of the subject, but so far as the writer knows, no award was made.

It will be of interest to summarize briefly the work that has been done in this field.

In 1861 Grunert† treated the general theory of the conic as a space curve, in connection with the study of planetary orbits. Defining a conic by choosing arbitrarily a line as directrix, a point as focus, and the eccentricity, he deduces a number of analytic relations. He also treats a number of other problems, chiefly of interest to the astronomer.

About the same time Chasles‡ laid the foundations of a theory of *characteristics* of conics in space, analogous to the well-known theory for conics in the plane. Hierholzer§ and Lüroth‖ discuss similar problems.

The earliest attempt to determine a set of coördinates for a conic is doubtless that of Spottiswoode.¶ His coördinates are not the simplest nor the most natural ones, however. We shall touch on them later (§ 3, 6, footnote).

A treatment somewhat analogous to parts of ours is given by P. van Geer.** His methods, however, are clumsy, and such of his results as are of value we can obtain by more direct methods.

* Presented to the Society, September 8, 1914.

† Grunert's Archiv, 1861, pp. 1–104.

‡ Comptes Rendus, 1865, pp. 389–397.

§ Mathematische Annalen, vol. 2 (1870), pp. 563–580.

‖ Mathematische Annalen, vol. 3 (1871), pp. 124–133.

¶ Proceedings of the London Mathematical Society, 1878, pp. 185–196.

** Archives Néerlandaises, 1888, pp. 58–90.

Some parts of this paper are based on an article by Reye* on linear systems of quadrics. It will turn out most desirable for us to regard a conic as a special quadric envelope, and hence the methods of studying systems of quadrics are applicable to the study of conics. In the article in question, Reye does not give much special attention to systems of conics; but from his results we can derive at once considerable information concerning linear systems of conics.

For the most part, however, systems of conics have been treated by pure geometrical methods. Several writers have studied particular systems, each defined in a special way. For example, Humbert† discusses the following system: through five fixed points there are ∞^2 cubic curves; of the ∞^1 that touch a fixed plane, the points of contact in general lie on a conic. Thus, a conic is determined in every plane of space; and the paper is a very interesting study of this system. The methods, however, are applicable only to this particular system, and the results no more general than this system. The same remarks apply to the papers of Pieri,‡ de Vries,§ and the first article of Montesano.‖

Since 1890, however, there has been developed a general theory of two and three parameter systems of certain types, the authors being Prof. Domenico Montesano of Pisa and M. Lucien Godeaux of Liège. The former¶ has given a complete treatment of linear congruences,—that is, of two-parameter families with one member through an arbitrary point. In a paper read before the Rome Congress,** he sketched the theory of "bilinear" complexes or three parameter families.

Godeaux, in a number of short notes,†† has considered the same problems. In a later note,‡‡ he called attention to some alleged errors in Montesano's work, and offered corrections. This elicited from Prof. Montesano a rather spirited reply §§ showing deficiencies in the reasoning of M. Godeaux. Apparently they are in agreement as to the facts of the case.

We shall not consider these matters further, for in the present paper we shall adopt a different point of view. Namely, these writers regard a conic as a curve of order 2, while we shall mean by a conic a degenerate envelope of class 2. Hence our totality of conics is a different entity from theirs, and in

*Crelle, Journal für Mathematik, vol. 82 (1876), pp. 54–83.

†Journal de l'école polytechnique, vol. 62 (1894), pp. 123–149.

‡Torino Atti, vol. 28, pp. 289–303.

§Amsterdam Akademie, Verslagen, 1904, pp. 281–284.

‖Rendiconti del Reale Istituto Lombardo, 1893, pp. 589–604.

¶Napoli Rendiconti, 1895–6, pp. 155–181.

**Atti del IV Congresso Internazionale, Roma, 1908, II, pp. 231–233.

††Bulletin de l'Académie de Belgique, 1908–1910.

‡‡Paris Comptes Rendus, vol. 152 (1911), pp. 1149–1151.

§§Paris Comptes Rendus, vol. 153 (1911), pp. 45–48.

dealing with systems of conics, which in general include degenerate members, the results will be different.

In addition to his work on congruences and complexes Godeaux* has attempted to devise coördinates for the conic in space. His most successful method is that of defining a conic as the intersection of a plane with a quadric linearly dependent on six fixed quadrics. The coördinates of the conic are then the four coördinates of the plane and the six multipliers of the system of quadrics. He makes some use of these coördinates in studying linear systems, but to us they seem to have little value. He suggests several other schemes, but they appear to be of even less use than the one we have discussed.

1. Determination of Coördinates

In studying conics in space, it is desirable to determine a system of coördinates, such that to a conic there always corresponds a unique set of coördinates, and conversely.

Two simple methods of defining a conic suggest themselves.

(*a*) If (u_1, u_2, u_3, u_4) be homogeneous coördinates of a plane, the equation

$$\sum_{i,j=1}^{4} a_{ij}\, u_i\, u_j = 0 \qquad (a_{ij} = a_{ji}) \tag{1}$$

represents an envelope of class two, which, when the matrix

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{Vmatrix} \tag{2}$$

is of rank *three*, is a *conic*. Conversely, any non-degenerate conic of space may be represented by an equation of the type (1) with vanishing discriminant.

This suggests the possibility of taking as coördinates of a conic a set of ten homogeneous numbers

$$(a_{11}, a_{12}, \cdots, a_{44}) \qquad (a_{ij} = a_{ji}), \tag{3}$$

which satisfy the quartic identity

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0. \tag{4}$$

*Bulletin de l'Académie Belgique, 1908, pp. 896–902.

For we have established a one-to-one correspondence between non-degenerate conics and those sets (a) whose matrix is of rank three.

(b) The second method of representing a conic is the parametric form

$$x_i = a_i s^2 + 2b_i st + c_i t^2 \qquad (i = 1, \cdots, 4), \tag{5}$$

where the a's, b's, and c's are constants, and (s, t) are homogeneous parameters. Any set of twelve numbers (a, b, c) define a conic uniquely, in general; but the converse is not true, since these coefficients may be changed by a linear transformation of the parameters which does not change the conic.

The last remark suggests that we may get a desirable set of coördinates by choosing functions of these coefficients that are invariant under all proper linear transformations of the parameters. The simplest invariants of the forms

$$a_i s^2 + 2b_i st + c_i t^2 \qquad (i = 1, \cdots, 4), \tag{6}$$

are the expressions

$$w_{ij} = \tfrac{1}{2}(a_i c_j + c_i a_j - 2b_i b_j) \tag{7}$$

whereof there are sixteen, reducing to ten by virtue of the obvious relation $w_{ij} = w_{ji}$. We have then, for any conic given by (5), a homogeneous set of numbers (7). Clearly they cannot be independent coördinates of a conic; for since the conics of space form an eight-parameter family, we have one superfluous coördinate.

Let us see how the numbers (w) are related to the coördinates indicated under (a). We will find the condition that the conic (5) touches an arbitrary plane

$$(vx) \equiv v_1 x_1 + v_2 x_2 + v_3 x_3 + v_4 x_4 = 0.$$

The two points in which the conic cuts the plane are given by the equation

$$(va)s^2 + 2(vb)st + (vc)t^2 = 0. \tag{8}$$

These points coincide, and the plane touches the conic, if and only if

$$\begin{vmatrix} (va) & (vb) \\ (vb) & (vc) \end{vmatrix} = 0. \tag{9}$$

Expanding this equation, we have by means of (7)

$$\sum_{i,j=1}^{4} w_{ij} v_i v_j = 0, \tag{10}$$

and we see that the ten numbers w_{ij} are precisely the same coördinates to which the first treatment led us. Thus it appears that they are very desirable coördinates for a conic in space.

We next wish to extend our notion of conic in such a way that the correspondence between conics and sets of coördinates shall be without exception. Accordingly, we formulate the following

DEFINITION. *Let*

$$\left\| \begin{matrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{matrix} \right\| \qquad (w_{ji} = w_{ij}) \tag{11}$$

be a set of numbers not all zero, satisfying the relation

$$\begin{vmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{vmatrix} = 0. \tag{12}$$

Then the envelope of class 2, *defined by the equation* $\Sigma\, w_{ij}\, u_i\, u_j = 0$ *shall be called a* CONIC.

Obviously conics are to be classified according to the rank of the matrix (w).

Case 1. (w) is of rank 3. Then the envelope is a non-degenerate plane conic, or, as we shall say, a *proper conic.*

Case 2. (w) is of rank 2. In this case the left-hand side of equation (12) is reducible into two linear factors, and the planes satisfying (12) are the totality of planes passing through either of two fixed points. Hence the envelope consists of these two points, and we shall call such a conic a *point-pair.*

Case 3. (w) is of rank 1. Then the left-hand side of (12) is the square of a linear form, and the envelope is a single point counted twice. Such a conic we shall call a *point-double.*

From this definition it follows that there is a complete one-to-one correspondence between conics of space and sets of coördinates w with vanishing determinant. But sets of $n + 1$ homogeneous coördinates can always be put in one-to-one correspondence with points of a projective linear space of n dimensions. Hence we have the following representation of conics:

Since a quadric envelope is determined by ten homogeneous coördinates w_{ij}, *the totality of such quadrics correspond in a one-to-one manner with points of a linear space of nine dimensions* S_9. *In this domain the conics of space are represented by points of an eight-dimensional hyperspace of order four, whose equation is given by* (12). *We shall denote this spread* S_8^4 *by* Σ.

2. ELEMENTARY PROPERTIES

1. Consider a conic (w) of rank 3. Write

$$\Delta(w) \equiv \begin{vmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{vmatrix} = 0. \tag{1}$$

We shall denote the co-factor of any element w_{ij} of this determinant by W_{ij}. Then by hypothesis the W's do not all vanish, and

$$\Delta \equiv \sum_{i=1}^{4} w_{ij} W_{ij} \equiv \sum_{j=1}^{4} w_{ij} W_{ij}. \tag{2}$$

THEOREM. *The adjoint matrix*

$$\left\| \begin{matrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{matrix} \right\| \tag{3}$$

is of rank unity.

For at least one member is not zero, by hypothesis; but every two-rowed determinant of (3) has Δ as a factor, and hence vanishes.

It follows that the W's are dependent on four parameters; and in fact, if we set

$$W_{11} = D_1^2, \qquad W_{12} = D_1 D_2, \qquad W_{13} = D_1 D_3, \qquad W_{14} = D_1 D_4 \tag{4}$$

we obtain

$$W_{ij} = D_i D_j \qquad (i, j = 1, \cdots, 4). \tag{5}$$

We can show at once that the D's so defined are the coördinates of the plane of the conic. For in general the quadric

$$\sum_{i,j=1}^{4} w_{ij} u_i u_j = 0 \tag{6}$$

has one and only one singular plane, if the system

$$\begin{aligned} w_{11} u_1 + w_{12} u_2 + w_{13} u_3 + w_{14} u_4 &= 0, \\ w_{21} u_1 + w_{22} u_2 + w_{23} u_3 + w_{24} u_4 &= 0, \\ w_{31} u_1 + w_{32} u_2 + w_{33} u_3 + w_{34} u_4 &= 0, \\ w_{41} u_1 + w_{42} u_2 + w_{43} u_3 + w_{44} u_4 &= 0, \end{aligned} \tag{7}$$

has one and essentially only one solution not consisting of zeros; i. e., if the matrix (w) is of rank 3. When this occurs, there are three of the equations (7) that are linearly independent; and by solving them we get the coördinates of the plane. For example, if $W_{11} \neq 0$, so that the last three equations are independent, we have from these three

$$u_1 : u_2 : u_3 : u_4 = W_{11} : W_{12} : W_{13} : W_{14} = D_1 : D_2 : D_3 : D_4, \tag{8}$$

as was to be shown.

2. Consider next a conic (w) of rank 2. In this case, the equations (7)

are equivalent to *two* equations; and therefore the system has ∞^1 solutions, linearly dependent on two. That is to say, a conic of rank two has a pencil of singular planes.

We have noted that in this case the envelope consists of two points; and it is obvious that the singular planes are the planes through both of these points. Hence the axis of the pencil is the line joining the two points. We shall call this line the *axis of the conic.* Its coördinates may be found as follows.

Each of the equations (7), in which not all the coefficients vanish, is the equation of a point, whose coördinates are the coefficients of the u's. Further, since every singular plane satisfies (7), these points lie on the axis of the conic. For definiteness assume that

$$\begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} \neq 0.$$

Then $(w_{11}, w_{12}, w_{13}, w_{14})$ and $(w_{21}, w_{22}, w_{23}, w_{24})$ are coördinates of two distinct points on the axis, and the plücker coördinates of the latter are accordingly given by

$$\rho_{12}\, p_{ij} = \begin{vmatrix} w_{1i} & w_{1j} \\ w_{2i} & w_{2j} \end{vmatrix} \qquad (i = 1, \cdots, 4), \tag{9}$$

where ρ_{12} is a constant factor not zero.

Similarly, for any two distinct equations of (7),

$$\rho_{mn}\, p_{ij} = \begin{vmatrix} w_{mi} & w_{mj} \\ w_{ni} & w_{nj} \end{vmatrix} \qquad (i, j = 1, \cdots, 4). \tag{10}$$

To determine ρ_{mn}, set $i = 1, j = 2$. Then

$$\rho_{mn}\, p_{12} = \begin{vmatrix} w_{m1} & w_{m2} \\ w_{n1} & w_{n2} \end{vmatrix} \equiv \begin{vmatrix} w_{1m} & w_{1n} \\ w_{2m} & w_{2n} \end{vmatrix} = \rho_{12}\, p_{mn}, \tag{11}$$

whence if we take $\rho_{12} = p_{12}$, we have $\rho_{mn} = p_{mn}$.

Obviously also when $p_{mn} = 0$, then $\rho_{mn} = 0$ also; and we have the

THEOREM. *Let (w) be a conic of rank two, and let (p) be the plücker coördinates of its axis. Then*

$$p_{ij}\, p_{mn} = \begin{vmatrix} w_{im} & w_{in} \\ w_{jm} & w_{jn} \end{vmatrix}. \tag{12}$$

3. Consider a conic (w) of rank 1. We have

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j \equiv (x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4)^2 \tag{13}$$

and hence

$$w_{ij} = x_i\, x_j. \tag{14}$$

Hence when a conic is a point-double, we can at once write the coördinates of its point; and conversely, we have at once the w-coördinates of the point-double at any point.

3. Relations to Point, Line, and Plane

It is now proposed to determine the representation in S_9 of simple systems of conics in space.

1. First let us recall that a set of coördinates (w) define on the one hand a quadric envelope, and on the other hand a point of a nine dimensional space S_9.

If, in particular,

$$\Delta(w) = 0, \tag{1}$$

the envelope is a conic, and the corresponding point lies on a quartic hypersurface which we shall call Σ.

2. The condition that a conic be tangent to a plane (u) is obviously

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0. \tag{2}$$

When (u) is given, this is linear in (w). Hence we have the result:

The totality of conics that touch a fixed plane (u) are represented in S_9 by the intersection of Σ with a hyperplane whose equation is (2).

Evidently there are ∞^3 such hyperplanes. We will now show that they are precisely the tangent hyperplanes to Σ. For the equation of Σ is $\Delta(w) = 0$, and the polar hyperplane, with regard to Σ, of any point $(\bar{w})$ is

$$\sum_{i,j=1}^{4} \frac{\partial \Delta(\bar{w})}{\partial \bar{w}_{ij}}\, w_{ij} = 0;$$

and this may be written

$$\sum_{i,j=1}^{4} \overline{W}_{ij}\, w_{ij} = 0. \tag{3}$$

In particular, this hyperplane contains $(\overline{w})$ itself if and only if $\Delta(\bar{w}) = 0$, and hence $(\bar{w})$ is a point on Σ. In this case $(\bar{w})$ corresponds to a conic, and the plane of the latter is given by $\overline{W}_{ij} = v_i\, v_j$. Hence at any point $(\bar{w})$ of Σ, the tangent hyperplane is given by

$$\sum_{i,j=1}^{4} v_i\, v_j\, w_{ij} = 0, \tag{4}$$

where (v) is the plane of the conic $(\bar{w})$. It follows that *the hypersurface Σ has the same tangent hyperplane at all points which correspond to conics in any given plane; and hence it has only ∞^3 tangent hyperplanes, each touching it at ∞^5 points.* The S_7^4 (spread of 7 dimensions and order 4) in which a tangent hyperplane cuts Σ corresponds to the totality of conics that touch the plane.

It is clear that these remarks apply only to points of Σ that correspond to proper conics; for at other points the tangent hyperplane is indeterminate. That is, the points that correspond to point-pairs and point-doubles generate a singular locus on Σ; but at ordinary points Σ is of a nature analogous to that of a developable surface. To complete the analogy, we observe that the ∞^5 points of tangency of a tangent hyperplane form a *linear* spread. For the conditions that the plane (v) be a singular plane of the envelope

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0$$

are clearly the four equations

$$w_{i1}\, v_1 + w_{i2}\, v_2 + w_{i3}\, v_3 + w_{i4}\, v_4 = 0 \qquad (i = 1, \cdots, 4) \tag{5}$$

If (v) be given, and the w's be regarded as independent variables except for the conditions $w_{ij} = w_{ji}$, then these equations are independent. From them, however, $\Delta(w) = 0$ is a consequence, so that any quadric (w) which satisfies (5) is a conic in the plane (v). Hence the conics in a given plane are the totality of solutions of four linear equations, and so they compose a linear five-parameter family S_5. Summarizing, we have the

THEOREM. *The points of S_9 corresponding to the conics in a given plane form a linear S_5. Through each ordinary point of Σ there passes one and only one such S_5, which is the totality of points of contact of the tangent hyperplane to Σ at that point.*

3. In an article on linear systems of quadrics,* Reye deduces a number of results which give us at once several important theorems. He determines the dimension (Stufe) and order (Grad) of the family of conics which satisfy a given condition; from which we have at once the nature of the spread in S_9 that represents the family. We shall not reproduce his work, which is rather long; but presently we shall summarize his results.

4. We next determine the equation, in point coördinates, of the cone from a point (y) to a conic (w).

First let us find the cone from a fixed point (y) that envelopes a proper quadric (a)

$$\sum_{i,j=1}^{4} a_{ij}\, u_i\, u_j = 0. \tag{6}$$

Let (x) be any point, then the condition that a point $\lambda(x) + \mu(y)$ on the line (yx) also lie on the quadric (6) is found by means of the point-equation of the latter,

* Loc. cit., Introduction.

$$(7)\qquad \begin{vmatrix} 0 & \lambda x_1 + \mu y_1 & \lambda x_2 + \mu y_2 & \lambda x_3 + \mu y_3 & \lambda x_4 + \mu y_4 \\ \lambda x_1 + \mu y_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ \lambda x_2 + \mu y_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ \lambda x_3 + \mu y_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ \lambda x_4 + \mu y_4 & a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0.$$

This is quadratic in $\lambda : \mu$. Its roots will be equal and the line will touch the quadric, if and only if

$$(8)\qquad \begin{vmatrix} 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & a_{11} & \cdot & \cdot & a_{14} \\ x_2 & \cdot & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot & \cdot \\ x_4 & a_{41} & \cdot & \cdot & a_{44} \end{vmatrix}^2 - \begin{vmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & a_{11} & \cdot & \cdot & a_{14} \\ x_2 & \cdot & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot & \cdot \\ x_4 & a_{41} & \cdot & \cdot & a_{44} \end{vmatrix} \begin{vmatrix} 0 & y_1 & y_2 & y_3 & y_4 \\ y_1 & a_{11} & \cdot & \cdot & a_{14} \\ y_2 & \cdot & \cdot & \cdot & \cdot \\ y_3 & \cdot & \cdot & \cdot & \cdot \\ y_4 & a_{41} & \cdot & \cdot & a_{44} \end{vmatrix} = 0.$$

Since $\Delta(a) \neq 0$, this equation is equivalent to the following (cf. Bôcher's *Higher Algebra*, p. 31).

$$(9)\qquad \begin{vmatrix} 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & y_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ x_2 & y_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ x_3 & y_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ x_4 & y_4 & a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0,$$

and this is accordingly the equation of the cone.

If we start with a proper conic instead of a quadric, this method fails. Let the conic be

$$(10)\qquad \sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0, \qquad \Delta(w) = 0.$$

We form the equation

$$(11)\qquad \sum_{i,j=1}^{4} (w_{ij} + \rho y_i\, y_j)\, u_i\, u_j = 0, \qquad \rho \neq 0,$$

which is a quadric through (w) and having the point (y) as vertex of the tangent cone along (w). Setting $a_{ij} = w_{ij} + \rho y_i\, y_j$, we have

$$(12)\qquad \Delta(a) \equiv \Delta(w) + \rho \sum_{i,j=1}^{4} y_i\, y_j\, W_{ij} \equiv \rho \sum_{i,j=1}^{4} d_i\, d_j\, y_i\, y_j \equiv \rho\,(dy)^2.$$

If (y) is not in the plane (d) of (w), and we take $\rho \neq 0$, then $\Delta(a) \neq 0$, and (a) is a proper quadric. Applying the previous result, we get the desired equation in the form

$$(13)\qquad \begin{vmatrix} 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & y_1 & w_{11} + \rho y_1^2 & w_{12} + \rho y_1 y_2 & w_{13} + \rho y_1 y_3 & w_{14} + \rho y_1 y_4 \\ x_2 & y_2 & w_{21} + \rho y_2 y_1 & \cdot & \cdot & \cdot \\ x_3 & y_3 & w_{31} + \rho y_3 y_1 & \cdot & \cdot & \cdot \\ x_4 & y_4 & w_{41} + \rho y_4 y_1 & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

This may be simplified by subtracting multiples of the second row from later rows in an obvious manner. This leads to the

THEOREM. *The equation of the cone drawn to any proper conic from a point (y) not in its plane is*

$$(14)\qquad \begin{vmatrix} 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & y_1 & w_{11} & \cdot & \cdot & w_{14} \\ x_2 & y_2 & \cdot & \cdot & \cdot & \cdot \\ x_3 & y_3 & \cdot & \cdot & \cdot & \cdot \\ x_4 & y_4 & w_{41} & \cdot & \cdot & w_{44} \end{vmatrix} = 0.$$

Remark. It is easily shown that if (y) be a point in the plane of (w), then (14) becomes $(yx)^2 = 0$, where $(yx) = 0$ is the equation of the plane.

5. We shall next determine the order of the S_6 that represents conics through a fixed point.

In the first place, if (w) be a conic whose plane does not pass through $(0, 0, 0, 1)$, then the cone from this point to (w) is given by

$$(15)\qquad \begin{vmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & w_{11} & w_{12} & w_{13} \\ x_2 & w_{21} & w_{22} & w_{23} \\ x_3 & w_{31} & w_{32} & w_{33} \end{vmatrix} = 0.$$

Conversely, if $P:(x_1, x_2, x_3, x_4)$ be a fixed point, the conics (w) that satisfy (15) are those that meet the line from $P_0:(0, 0, 0, 1)$ to P. For

(*a*) if $W_{44} \neq 0$, the plane of the conic does not pass through P_0: and since P lies on the cone, the element $P_0 P$ of the cone meets the conic;

(*b*) if $W_{44} = 0$, (15) is the equation of the plane of (w); and since both P and P_0 lie in this plane, their line meets (w) twice.

Now a conic passes through P if and only if

(1) its plane passes through P, and (2) it intersects one line through P, not lying in its plane.

We shall consider the line $P_0 P$. The conics that meet it satisfy (15); and this defines in S_9 a hypersurface S_8^2. The conics whose planes pass through a fixed point P form, as Reye (loc. cit.) shows, a S_7^6. The points common to a S_8^2 and a S_7^6 form a S_6^{12}. We have just seen that these include two sets of conics: first, the conics through P, and second, those whose planes pass through $P_0 P$. The latter, as Reye (loc. cit.) shows, correspond to a S_6^4. Hence the conics through P are represented by a S_6^8, and we have the

THEOREM. *The conics which pass through a given point P are represented in S_9 by a S_6^8.*

6. By methods similar to those of part 4, we can prove the

THEOREM. *Let*

$$\sum_{i,j=1}^{4} a_{ij}\, u_i\, u_j = 0$$

be any quadric envelope. Let the plücker coördinates of a line be given by

$$p_{01} = \pi_{23}, \qquad p_{02} = \pi_{31}, \qquad p_{03} = \pi_{12}, \qquad \textit{etc.}$$

Then the equation

$$\sum_{1}^{4} \begin{vmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{vmatrix} \pi_{ik}\, \pi_{jl} = 0$$

has the following interpretations:

(*a*) *If (a) is a proper quadric, (π) touches it.*

(*b*) *If (a) is a proper conic, (π) meets it.*

(*c*) *If (a) is a point-pair, (π) meets its axis.*

(*d*) *If (a) is a point-double, the equation is satisfied identically.*

For convenience we make the following

DEFINITION. *A line (π) shall be said to intersect a conic that is a point-pair, when it intersects the axis of the latter.*

We have then the

THEOREM. *The totality of conics that intersect a given line (π) are represented in S_9 by the total intersection of Σ with the S_8^2 whose equation is*

$$\sum_{1}^{4} \begin{vmatrix} w_{ij} & w_{il} \\ w_{kj} & w_{kl} \end{vmatrix} \pi_{ik}\, \pi_{jl} = 0.^* \tag{16}$$

* In equation (16), if (w) be regarded as fixed, the equation is that of the complex of lines that meet the conic (w). The coefficients may thus be regarded as coördinates of the conic. This is essentially the system of coördinates used by Spottiswoode. (See Introduction.)

There are ∞^4 such S_8^2's, corresponding to the lines of space; and they have in common the singular locus (see below) that represents point-doubles.

7. We will close the section with a tabular summary of the correspondence we have developed, including the results obtained by Reye. The latter are indicated by the symbol #.

In S_3	In S_9
Quadric envelope	Point
Conic	Point of $S_8^4 \equiv \Sigma$
Conics in fixed plane	S_5 lying in Σ
Quadrics touching a fixed plane	S_8. Touching Σ at all points of a S_5
Conics whose planes pass through fixed point	S_7^6 #
Conics whose planes pass through fixed line	S_6^4 #
Conics which pass through fixed point	S_6^8 (5)
Conics that meet a fixed line	S_7^8 (6)
Point-pairs	S_6^{10} #
Point-pairs whose lines meet a fixed line	S_5^{10} #
Point-pairs with one member on given line	S_4^4 #
Point-pairs with both members on given line	S_2^1 #
Point-pairs with both members in given plane	S_4^3 #
Point-doubles	S_3^8 #
Point-doubles in given plane	S_2^4 #
Point-doubles on given line	S_1^2 #

4. Relations Between Conic and Quadric

Let us determine the conic in which a given plane cuts a given quadric. We observe that this is exactly dual to a problem already treated (§ 3, 4); hence we need not repeat the work. The result is as follows:

THEOREM. *The condition that a plane* (u) *touch the conic* (w) *in which a given quadric*

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0 \tag{1}$$

is cut by a given plane (A_1, A_2, A_3, A_4) *is*

$$\begin{vmatrix} 0 & 0 & u_1 & u_2 & u_3 & u_4 \\ 0 & 0 & A_1 & A_2 & A_3 & A_4 \\ u_1 & A_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ u_2 & A_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ u_3 & A_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ u_4 & A_4 & a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0. \tag{2}$$

Hence the coördinates of the conic are immediately written down. If, in particular, the plane (A) touches the quadric, equation (2) reduces to the equation of the point of contact, counted twice.

We next determine conditions that a conic lie on a given quadric. These conditions do not seem capable of very simple expression. We will treat the problem in two ways. Of these, the former gives a redundant set of conditions; but as usual in such cases, it is impossible to choose a set which will in all cases be sufficient and independent. The second method, on the other hand, gives exactly the right number of equations; but they are complicated in form.

FIRST METHOD. If

$$\sum_{i,j=1}^{4} A_{ij}\, u_i\, u_j = 0 \tag{3}$$

be a proper quadric, and

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0$$

a conic lying on it, then one member of the pencil linearly dependent on them will be the pole of the plane of (w), counted twice. (This is well known in the theory of quadrics.) Hence (w) lies on (A) if and only if there is a number k such that the matrix

$$\left\| \begin{matrix} A_{11} + kw_{11} & A_{12} + kw_{12} & A_{13} + kw_{13} & A_{14} + kw_{14} \\ A_{21} + kw_{21} & \cdot & \cdot & \cdot \\ A_{31} + kw_{31} & \cdot & \cdot & \cdot \\ A_{41} + kw_{41} & \cdot & \cdot & \cdot \end{matrix} \right\| \tag{4}$$

is of rank 1. This gives a number of quadratic equations in k, of the type

$$k^2 \begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} + k \left\{ \begin{vmatrix} w_{11} & A_{12} \\ w_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & w_{12} \\ A_{12} & w_{22} \end{vmatrix} \right\} + \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = 0. \tag{5}$$

The required conditions are that these equations all have a common root, and we may obtain them by equating to zero the resultant of each pair of them. These equations are necessary and sufficient, but are not independent.

SECOND METHOD. This consists in making a non-singular linear transformation that carries the plane of the conic into the plane $(0, 0, 0, 1)$. Let a proper quadric be given by the point equation

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0, \qquad \Delta(a) \neq 0, \tag{6}$$

and hence by the tangential equation

$$\sum_{i,j=1}^{4} A_{ij}\, u_i\, u_j = 0, \tag{7}$$

where, as usual, A_{ij} is the cofactor of a_{ij} in $\Delta(a)$. Let (w) be a conic,

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0, \qquad \Delta(w) = 0, \tag{8}$$

and denote the coördinates of its plane by (d). Then

$$W_{ij} = d_i\, d_j, \qquad \sum_{i=1}^{4} w_{ij}\, d_i = 0 \qquad (j = 1, \cdots, 4). \tag{9}$$

Case 1. $d_4 = 0$. Then without making the transformation indicated under case 2, we can carry on the later work directly, as will be obvious.

Case 2. $d_4 \neq 0$. The linear transformation

$$\left.\begin{aligned} u_1' &= d_4\, u_1 & & & & - d_1\, u_4 \\ u_2' &= & d_4\, u_2 & & & - d_2\, u_4 \\ u_3' &= & & d_4\, u_3 & & - d_3\, u_4 \\ u_4' &= & & & & - d_4\, u_4 \end{aligned}\right\} \tag{10}$$

is non-singular and involutory. We will apply it to the quadratic equations

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0 \tag{11}$$

$$\sum_{i,j=1}^{4} A_{ij}\, u_i\, u_j = 0. \tag{12}$$

We find that the former goes into

$$\sum_{i,j=1}^{3} w_{ij}\, u_i'\, u_j' = 0, \tag{13}$$

while the latter goes into an equation whose matrix is

$$\left\| \begin{matrix} d_4^2\, A_{11} & d_4^2\, A_{12} & d_4^2\, A_{13} & -d_4 \sum_{i=1}^{4} d_i\, A_{i1} \\ d_4^2\, A_{21} & d_4^2\, A_{22} & d_4^2\, A_{23} & -d_4 \sum_{i=1}^{4} d_i\, A_{i2} \\ d_4^2\, A_{31} & d_4^2\, A_{32} & d_4^2\, A_{33} & -d_4 \sum_{i=1}^{4} d_i\, A_{i3} \\ -d_4 \sum_{i=1}^{4} d_i\, A_{i1} & -d_4 \sum_{i=1}^{4} d_i\, A_{i2} & -d_4 \sum_{i=1}^{4} d_i\, A_{i3} & +\sum_{i,j=1}^{4} d_i\, d_j\, A_{ij} \end{matrix} \right\|, \tag{14}$$

so that this equation may be written

$$d_4^2 \sum_{i,j=1}^{3} A_{ij}\, u_i'\, u_j' - 2d_4 \sum_{\substack{i=1\\ j=1}}^{i=4,\, j=3} d_i\, A_{ij}\, u_j'\, u_4' + \left(\sum_{i,j=1}^{4} d_i\, d_j\, A_{ij} \right) u_4'^2 = 0. \tag{15}$$

We shall for brevity represent this equation by

$$\sum_{i,j=1}^{4} k_{ij}\, u_i'\, u_j' = 0,$$

and denoting the cofactors of the elements k_{ij} of (14) by K_{ij}, we get the result:

The effect of the transformation (10) on the quadric

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0 \tag{7}$$

is to carry it into the quadric

$$\sum_{i,j=1}^{4} K_{ij}\, x_i'\, x_j' = 0. \tag{16}$$

We next find the conic in which this quadric cuts the plane $(0, 0, 0, 1)$. In formula (2) set $(A) \equiv (0, 0, 0, 1)$, and $a_{ij} \equiv K_{ij}$. We get

$$\begin{vmatrix} 0 & u_1' & u_2' & u_3' \\ u_1' & K_{11} & K_{12} & K_{13} \\ u_2' & K_{21} & K_{22} & K_{23} \\ u_3' & K_{31} & K_{32} & K_{33} \end{vmatrix} = 0 \tag{17}$$

as the equation of the conic. But

$$\begin{vmatrix} K_{ij} & K_{im} \\ K_{nj} & K_{nm} \end{vmatrix} = M\Delta(k), \tag{18}$$

where M is the algebraic complement of $k_{ij}\, k_{mn}$ in $\Delta(k)$. Hence (17) reduces to

$$\sum_{i,j=1}^{3} \begin{vmatrix} k_{ij} & k_{i4} \\ k_{j4} & k_{44} \end{vmatrix} u_i'\, u_j' = 0, \tag{19}$$

and this is the conic in which the quadric cuts the plane of the given conic, after the transformation (10).

This coincides with the given conic if and only if

$$\rho w_{ij} = \begin{vmatrix} k_{ij} & k_{i4} \\ k_{j4} & k_{44} \end{vmatrix} \qquad (i, j = 1, 2, 3), \tag{20}$$

or, when we replace the k's by their values from (14), and set $d_i\, d_j = W_{ij}$,

$$\rho w_{ij} = \begin{vmatrix} W_{44}\, A_{ij} & -\sum_{l=1}^{4} W_{l4}\, A_{lj} \\ -\sum_{l=1}^{4} W_{4l}\, A_{li} & \sum_{l,m=1}^{4} W_{lm}\, A_{lm} \end{vmatrix} \qquad (i, j = 1, 2, 3). \tag{21}$$

These equations are six in number, but they involve an inessential factor ρ. By eliminating it, we get exactly the correct number of equations, namely five.

5. Relations Between Two Conics

Let (a) and (b) be any two conics. The quadrics linearly dependent on them,

$$\sum_{j=1}^{4} (\lambda a_{ij} + \mu b_{ij}) u_i u_j = 0 \tag{1}$$

form a one-parameter family called a *series*. Among them there are in general four conics, given by values of $\lambda : \mu$ that satisfy the equation

$$\Delta(\lambda a + \mu b) = 0. \tag{2}$$

It is known that this equation is invariant under all proper collineations; hence the vanishing of any one of its coefficients is the condition for some special projective relation between the two given conics.

We shall represent (2) by the equation

$$\lambda^4 \Delta(a) + \lambda^3 \mu \Theta(a, b) + \lambda^2 \mu^2 \Phi(a, b) + \lambda \mu^3 \Theta(b, a) + \mu^4 \Delta(b) = 0 \tag{3}$$

and shall refer to the functions $\Theta(a, b)$, $\Theta(b, a)$, $\Phi(a, b)$ as the *invariants* of the two conics. We observe that the coefficients of λ^4 and μ^4 are the discriminants of the conics (a) and (b) respectively, and hence are vanishing invariants. We propose to investigate the meaning of the vanishing of the invariants Θ and Φ.

These expressions are

$$\Theta(a, b) = \sum_{i,j=1}^{4} b_{ij} A_{ij}, \qquad \Theta(b, a) = \sum_{i,j=1}^{4} a_{ij} B_{ij},$$

$$\Phi(a, b) = \Phi(b, a) = \sum \begin{vmatrix} b_{11} & b_{12} & a_{13} & a_{14} \\ b_{21} & b_{22} & a_{23} & a_{24} \\ b_{31} & b_{32} & a_{33} & a_{34} \\ b_{41} & b_{42} & a_{43} & a_{44} \end{vmatrix}, \tag{4}$$

the summation being effected by replacing in turn every pair of columns of a's by the corresponding b's, and adding all these determinants.

1. Suppose $\Theta(a, b) = 0$.

Case a. Let (a) be a proper conic. Denoting the coördinates of its plane by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, we have $A_{ij} = \alpha_i \alpha_j$. Hence

$$\Theta(a, b) = \sum_{i,j=1}^{4} b_{ij} A_{ij} = \sum_{i,j=1}^{4} b_{ij} \alpha_i \alpha_j,$$

and so when $\Theta(a, b)$ vanishes, we have

$$\sum_{i,j=1}^{4} b_{ij}\,\alpha_i\,\alpha_j = 0, \tag{5}$$

and (α) is one of the planes that touch the envelope (b).

Case b. Let (a) be a degenerate conic. Then $A_{ij} = 0$ $(i, j = 1, \cdots, 4)$, and for every conic (b) we have $\Theta(a, b) = 0$.

Remark. It is obvious that the equation

$$\sum_{i,j=1}^{4} b_{ij}\,\alpha_i\,\alpha_j = 0$$

is satisfied whenever the conics (a), (b) are coplanar.

We have then the result:

THEOREM. *The equation* $\Theta(a, b) = 0$ *is satisfied if and only if one of the following occurs*:

(1) *The conics* (a) *and* (b) *are coplanar;* (2) (a) *is a proper conic whose plane touches* (b); (3) (a) *is a degenerate conic.*

2. If now we interchange (a) and (b), we have the interpretation of $\Theta(b, a)$.

3. We next consider the equation

$$\Phi(a, b) = 0.$$

Case a. The conics are coplanar. We have already mentioned that the functions Θ, Φ are invariant under any non-singular linear transformation. If we make such a transformation, carrying the common plane of the conics into the plane $(0, 0, 0, 1)$, their matrices assume the forms

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & 0 \\ a'_{21} & a'_{22} & a'_{23} & 0 \\ a'_{31} & a'_{32} & a'_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} & 0 \\ b'_{21} & b'_{22} & b'_{23} & 0 \\ b'_{31} & b'_{32} & b'_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and we see by inspection that $\Phi(a', b') = 0$. Hence the invariant Φ of two coplanar conics vanishes.

Case b. If either conic is a point pair, then as we saw in 2, the two-rowed determinants of its matrix are expressible as products of the plücker coördinates of its axis. If (b) is a point-pair, then $\Phi(a, b) = 0$ reduces to

$$\sum_{1}^{4} \begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} \pi_{ij}\,\pi_{kl} = 0,$$

where the π's are plücker coördinates of the axis of (b). But this (§ 3) is the condition that the line (π) meet the conic (a) when the latter is a

proper conic; and if it is a point-pair, the equation means that the axes of the two point-pairs intersect.

If either conic is a point-double, Φ vanishes identically.

Case c. The remaining case is that neither of the conics is degenerate, and they lie in distinct planes.

To simplify the work, we may without loss of generality take as the planes of the conics the planes $(0, 0, 0, 1)$ and $(1, 0, 0, 0)$ respectively. Then their matrices are

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix},$$

and $\Phi = 0$, reduces to the expression

$$(10) \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{vmatrix} - 2 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} b_{23} & b_{24} \\ b_{43} & b_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \begin{vmatrix} b_{22} & b_{24} \\ b_{42} & b_{44} \end{vmatrix} = 0.$$

We will determine the pair of points in which the line common to the planes of the conics cuts the conic (a).

The cone from the point $(0, 0, 0, 1)$ to the conic (a) is (§ 3)

$$(11) \qquad \begin{vmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & a_{11} & a_{12} & a_{13} \\ x_2 & a_{21} & a_{22} & a_{23} \\ x_3 & a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

This is cut by the line $x_1 = 0$, $x_4 = 0$ in the two points found by setting $x_1 = 0$ in this equation. We have

$$(12) \qquad x_2^2 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - 2x_2 x_3 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + x_3^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.$$

Similarly, the points in which (b) cuts this line are given by the equation

$$(12') \qquad x_2^2 \begin{vmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{vmatrix} - 2x_2 x_3 \begin{vmatrix} b_{23} & b_{24} \\ b_{43} & b_{44} \end{vmatrix} + x_3^2 \begin{vmatrix} b_{22} & b_{24} \\ b_{42} & b_{44} \end{vmatrix} = 0.$$

Then the well-known condition that the roots of two quadratic equations separate each other harmonically gives us

$$\Phi(a, b) = 0$$

by means of equation (10).

Summarizing, we have the

THEOREM. *Two conics* (a) *and* (b) *satisfy the relation* $\Phi(a, b) = 0$ *if and only if one of the following occurs*:

(1) *They are coplanar.* (2) *They are proper conics in different planes, and separate each other harmonically on the line of intersection of their planes.* (3) *One of them is a point-pair, and its axis meets the other.* (4) *They are both point-pairs, with intersecting axes (special case of* (1)). (5) *One conic is a point-double.*

4. The condition that two conics intersect at a point is simply expressed in terms of these invariants. Salmon* shows that if two cones have a common tangent plane, the two remaining cones of their pencil fall together.

Dually, if two conics have a common point, the other two conics linearly dependent on them will fall together, and conversely.

Let (a) and (b) be two conics, and let

$$c_{ij} = \lambda a_{ij} + \mu b_{ij}.$$

The four conics of the series (c) are given by

$$\Delta(c) = 0,$$

or, since we know that (a) and (b) are two of them, the others are given by

$$\lambda^2 \Theta(a, b) + \lambda\mu \Phi(a, b) + \mu^2 \Theta(b, a) = 0.$$

As we have remarked, (a) and (b) intersect if and only if this equation has equal roots, so that we have the result:

THEOREM. *Two proper conics,* (a) *and* (b), *intersect if and only if*

$$\begin{vmatrix} 2\Theta(a, b) & \Phi(a, b) \\ \Phi(a, b) & 2\Theta(b, a) \end{vmatrix} = 0.$$

It is seen that this equation is quartic in the coördinates of either conic; hence the conics that meet a fixed conic are represented in S_9 *by a* S_7^{16}, *the total intersection of* Σ *with a* S_8^4.

6. Complete and Incomplete Systems—the Incomplete Linear Systems

We have seen that every set of numbers $(w_{11}, w_{12}, \cdots, w_{44})$, $(w_{ij} = w_{ji})$, are the tangential coördinates of a quadric; and in particular, any set that satisfy the relation

(1) $$\Delta(w) = 0$$

* *Geometry of Three Dimensions*, 3d ed., § 202.

are coördinates of a conic. Any set of equations on the w's, of which $9 - r$ are independent, determine an r-parameter system of quadrics, and of these, the ones that satisfy (1) are conics. Hence in studying systems of conics defined by equations, we recognize two distinct cases.

Case 1. The equation (1) is not a necessary consequence of the given equations. Then the latter are satisfied by ∞^r quadrics, which include ∞^{r-1} conics given by the further equation (1). Such a system of conics we shall call an *incomplete system.*

Case 2. The equation (1) is satisfied by every solution of the given equations. Then every one of the solutions of the latter is a conic, and the system consists of ∞^r conics. A system of this sort we shall call a *complete system.*

In the nine-dimensional representation of conics and quadrics, a set of $9 - r$ independent equations define a spread of dimension r. In case this does not lie entirely on Σ, we have an incomplete system, represented by its intersection with Σ. In case we have a complete system, the S_r lies entirely on Σ.

Linear systems constitute an important case. The quadrics that satisfy $9 - r$ linearly independent linear equations may be given as linearly dependent on $r + 1$ quadrics and are represented in S_9 by a linear S_r. In general, this will cut Σ in a S^4_{r-1}, and we have an incomplete linear system; but it may happen that the S_r lies on Σ, so that every solution

$$w_{ij} = \lambda_1 a^{(1)}_{ij} + \cdots + \lambda_{r+1} a^{(r+1)}_{ij} \tag{2}$$

satisfies (1), and then we have a complete linear system.

We have already seen the existence of a complete linear system, namely the ∞^5 conics in a plane. In the following section we shall determine all possible complete linear systems. We take up first incomplete systems.

Evidently it is desirable to study these by means of linear systems of quadrics. The latter have been somewhat studied, and in the simplest cases they are well known. The general one, two and three parameter systems of quadric envelopes are treated in the *Encyklopädie,* III C2, § VIII, § XII, § XIII. The remaining cases are briefly treated in the same place, § XIV. The most important work on the subject is that of Reye, in the paper to which we have already referred (Introduction). In this paper he considers linear systems of point-quadrics and of envelope quadrics of all dimensions from one up to eight. From his results, obviously, we can get at once important facts concerning the systems of conics included in his systems of quadrics, though he says little specifically about these conics.

Before considering linear systems, we need to discuss the relation of *apolarity,* which is closely related.

DEFINITION. *Let*

$$\sum_{i,j=1}^{3} a_{ij}\, x_i\, x_j = 0, \tag{3}$$

$$\sum_{i,j=1}^{3} b_{ij}\, u_i\, u_j = 0 \tag{4}$$

be two conics in a plane. If the relation

$$\sum_{i,j=1}^{3} a_{ij}\, b_{ij} = 0 \tag{5}$$

is satisfied, the conics are said to be APOLAR.*

The geometrical meaning of the relation is that there exists a triangle self-conjugate with regard to (a) and circumscribed to (b); and dually, a triangle self-conjugate with regard to (b), and inscribed in (a). In either case, when there is one such triangle there are infinitely many.

A line pair or a point pair is apolar to another conic if its members are conjugate with regard to the latter. A line-double is apolar to a conic if the line touches the conic, and a point-double is apolar to a conic if the point lies on the conic.

We next define apolarity of quadrics.

DEFINITION. *Two quadrics*

$$\sum_{i,j=1}^{4} b_{ij}\, x_i\, x_j = 0, \tag{6}$$

$$\sum_{i,j=1}^{4} a_{ij}\, u_i\, u_j = 0 \tag{7}$$

that satisfy the relation

$$\sum_{i,j=1}^{4} a_{ij}\, b_{ij} = 0 \tag{8}$$

are said to be apolar to each other.

THEOREM. *Two apolar quadrics, neither of which is degenerate, have the property that there exist tetrahedra self-conjugate with regard to the one and inscribed in the other; and dually there exist tetrahedra self-conjugate with regard to the second, and circumscribed about the first.*†

This proof fails when either quadric is degenerate. We shall next determine

* Various other names are used for this relation. The word "apolar" is due to Reye, who also uses the expressions that (a) *supports* (stützt) (b), and that (b) *rests on* (ruht auf) (a). English writers call (a) *harmonically inscribed* in (b). Rosanes calls the two conics *conjugate*.

† For proof see Hesse's *Vorlesungen, Kap.* 16, p. 190.

the meaning of apolarity between a proper conic

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0, \qquad \Delta(w) = 0, \tag{9}$$

and an unrestricted point quadric

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0. \tag{10}$$

The relation is

$$\sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0, \tag{11}$$

and this is evidently an invariant for non-singular collineations of space. Hence there is no loss of generality if we take $x_4 = 0$ as the plane of the conic (w). Its equation then becomes

$$\sum_{i,j=1}^{3} w_{ij}\, u_i\, u_j = 0$$

and the relation of apolarity is

$$\sum_{i,j=1}^{3} a_{ij}\, w_{ij} = 0, \tag{12}$$

whence the

THEOREM. *The conic* (9) *is apolar to the quadric* (10) *if and only if it is apolar to the conic in which its plane cuts the quadric.*

Similarly it can be shown that if a point-pair is apolar to a quadric, its members are conjugate with regard to the quadric. For a point-double, the condition is that the point lie on the quadric.

Now it is clear that every quadric (w) whose coördinates satisfy the linear equations

$$\sum_{i,j=1}^{4} a_{ij}^{(k)}\, w_{ij} = 0 \qquad (k = 1, \cdots, r) \tag{13}$$

s apolar to each of the quadrics

$$\sum_{i,j=1}^{4} (\lambda_1\, a_{ij}^{(1)} + \cdots + \lambda_r\, a_{ij}^{(r)})\, x_i\, x_j = 0. \tag{14}$$

This system we shall call the *basal system* of the linear system of quadrics (w) defined by (13).

We shall denote the system of ∞^r conics (embedded in ∞^{r+1} quadrics) defined by a set of linear equations (13) of which $8 - r$ are linearly independent, by L_r, and we now proceed to study the systems L_r from $r = 7$ to $r = 0$.

1. The L_7 or seven-parameter family is defined by one equation

$$\sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0\,. \tag{15}$$

Then the basal system consists of a single quadric

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0\,, \tag{16}$$

and every conic of the family is apolar to this quadric.

We have then the result: *In the general L_7, the conics that lie in any plane are those which are apolar to a fixed conic in the plane, this fixed conic being the intersection of the plane with the base quadric* (16). *The system contains degenerate conics; in fact, the point-pairs of the general L_7 are ∞^5 in number. An arbitrary point of space may be associated with any one of the ∞^2 points of its polar plane in the base quadric to form a point-pair of the L_7. The point-doubles of the system consist of the points of the base quadric itself.*

The only planes, in general, in which the configuration is projectively different from that of the general plane are those tangent to the base quadric. In such a plane the base quadric cuts a pair of lines; and these are conjugate with regard to every conic of the L_7 in this plane.

Special types of L_7.

(A) The base quadric is a cone.

The system has then the following distinctive features:

(a) the ∞^2 special planes pass through the vertex; (b) there are ∞^1 of them, the tangents to the cone, in which all the ∞^4 conics of the L_7 touch a fixed line; (c) the vertex of the cone is singular, in that it can be associated with any point of space to form a point-pair of the L_7.

(B) The base quadric is a plane-pair.

In this case we have singular planes; namely the L_7 includes every conic in either of the planes of the basal plane-pair.

Conversely, if an L_7 includes all the conics of any plane, its basal quadric must be a plane-pair of which this plane is one member.

(C) The base quadric is a plane-double. Then (15) is of the form

$$\sum_{i,j=1}^{4} d_i\, d_j\, w_{ij} = 0\,.$$

We have studied this system (§ 3, 2) and have seen that it defines the quadrics that touch the plane (d); in particular, then, the conics of the system are those that touch (d), including those that lie in this plane. In S_9 this equation represents a hypersurface tangent to Σ at all points of a S_5.

2. The L_6, or 6-parameter family, is defined by two distinct equations

$$\sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} b_{ij}\, w_{ij} = 0. \tag{17}$$

The basal system consists of the pencil of quadrics

$$\sum_{i,j=1}^{4} (\lambda a_{ij} + \mu b_{ij})\, x_i\, x_j = 0.$$

These have, in general, a common quartic curve C_4. Conversely, any conic apolar to two quadrics of the system (21) is apolar to all, and hence belongs to L_6.

The conics of the L_6 in an arbitrary plane are ∞^3 in number, and are apolar to all the conics of the pencil in which the basal system cuts the plane. The latter conics pass through the four points P_1, P_2, P_3, P_4, in which C_4 intersects the plane; and among them are in general three line-pairs. Each of these is a conjugate pair with regard to any conic of the L_6. Conversely, if a conic has two of these as conjugate line-pairs, then it is apolar to two distinct quadrics of the basal system, and hence belongs to the L_6.

This gives us the following

THEOREM. *Let a plane* (π) *cut* C *in the four points* P_1, P_2, P_3, P_4. *Then the conics of* L_6 *that lie in* (π) *are those that have these points as a "polar quadrangle."* (Cf. Reye, *Geometrie der Lage*, 4th ed., vol. 1, p. 261 ff.)

Degenerate conics of the L_6. The point-doubles are necessarily common to all the basal quadrics, and hence are the points of the C_4. The members of a point-pair must be conjugate with regard to each of the basal quadrics. It is known that the polar planes of a point P with regard to a pencil of quadrics meet in a line l_p. Hence P and any point of this line form a point-pair of the L_6.

There are, however, four points which are exceptional to the above statement, namely the vertices of the cones of the basal pencil. We have then the result:

THEOREM. *The point-doubles of* L_6 *are the* ∞^1 *points of the curve* C_4 *common to the basal quadrics. There are four of them in an arbitrary plane* (cf. § 3 at end). *An arbitrary point determines a line, any point of which may be associated with it to form a point-pair of the* L_6. *But there are four singular points, the vertices of the cones of the basal system, each of which can be so associated with any point of the plane of the other three.*

We have seen (§ 3) that the point-pairs whose members lie on a given line are represented in S_9 by a S_2^1; hence any S_7 contains one of them. Hence the

THEOREM. *The* L_6 *includes one point-pair whose members lie on an arbitrary line.*

Special types of L_6. In the foregoing, we have tacitly assumed that the pencil of basal quadrics is of the most general type. It is clear that the classification of L_6's is the same as that of pencils of quadrics. The latter is well known, and we shall not give it here.

We observe, however, that the first important specialization occurs when the pencil includes a plane-pair. In this case we may define the basal system by means of the plane-pair and any other member. Then the L_6 is the totality of members of an L_7 with regard to which two fixed planes are conjugate.

Similarly we get other special types. In particular, the totality of conics that touch two fixed planes form a special L_6.

THEOREM. *An* L_6 *can be such as to include all the conics of one plane, but not of more than one plane.*

For consider the L_6 whose basal system is defined by two plane-pairs having a common plane; every conic in this plane belongs to the L_6. To prove the second part of the theorem, we note that an L_6 can be defined as the totality of conics common to two L_7's. But there is one and only one L_7 that contains all the conics of two planes, namely the one whose basal quadric is this plane-pair. Hence there is no L_6 that includes all these conics.

3. The L_5 is defined by three independent equations

$$\sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} b_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} c_{ij}\, w_{ij} = 0. \tag{18}$$

The basal system is a *net* or linear two-parameter system represented by

$$\sum_{i,j=1}^{4} (\lambda a_{ij} + \mu b_{ij} + \nu c_{ij})\, x_i\, x_j = 0,$$

and the L_5 consists of all the conics apolar to every member of this system.

The basal system cuts an arbitrary plane in a so-called "bundle" or linear two-parameter family of conics. The ∞^2 conics of the L_5 in this plane are all apolar to the conics of the bundle, and they comprise a linear family in tangential coördinates. Reye (loc. cit.) calls such a family a "Schaarschaar." The nature of these two systems is well known. The conics of a bundle have in general no common point, those of a "Schaarschaar" no common tangent. A special case consists of the conics that have three common points in the one case, or in the other have three common tangents.*

Degenerate conics of the L_5. Clearly the point-doubles are eight in number, being the points common to all the base-quadrics. Thus we have (cf. § 3) the

THEOREM. *The point-doubles of space correspond to points of a* S_3^8 *lying on* Σ.

* For further treatment see *Encyklopädie*, III C 1, § 81.

There are also ∞^1 singular points, each of which may be associated with any point of a line to form a point-pair of the system. These singular points are the vertices of the cones of the basal system, and Reye (loc. cit.) shows that they lie on a sextic curve called the "Kerncurve" of the bundle. This curve has been considerably studied by Reye, and also, as an adjunct to a net of quadrics, by Sturm.*

Special types of L_5. There are a large number of special cases, and as before we shall not classify them. In general the basal net contains no plane-pair nor plane-double. If it contains one or more of either, the L_5 is correspondingly specialized. In particular, the conics that touch three fixed planes form a special L_5.

THEOREM. *An* L_5 *can contain all the conics of a plane. In this case it contains no other conics, and the conics of the* L_5 *in an arbitrary plane are the point-pairs on the line in which it intersects the plane of the* L_5.

4. The L_4 is a system of conics determined by four independent linear equations

$$\begin{aligned} &\sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} c_{ij}\, w_{ij} = 0, \\ &\sum_{i,j=1}^{4} b_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} d_{ij}\, w_{ij} = 0. \end{aligned} \tag{19}$$

The basal system of quadrics is given by

$$\sum_{i,j=1}^{4} (\lambda_1 a_{ij} + \lambda_2 b_{ij} + \lambda_3 c_{ij} + \lambda_4 d_{ij})\, x_i x_j = 0.$$

This system is called in German a *Gebüsch*. Its ∞^3 members have in general no common point. They include ∞^2 cones, and in general ten plane-pairs.†

The Gebüsch being completely defined by any four of its members that are linearly independent, we may define it by means of four of its ten plane-pairs. If four plane-pairs be chosen arbitrarily, there is a Gebüsch linearly dependent on them. Hence we may define the L_4 as follows:

THEOREM. *The general* L_4 *has the property that there are ten pairs of planes, each a conjugate pair with regard to every conic of the* L_4. *Given four arbitrary pairs of planes, the totality of conics with regard to which each pair are conjugate constitute an* L_4 *of general type; and there are six additional pairs of planes, each pair conjugate with regard to all the conics of the system.*

Degenerate conics. In general the L_4 has no point-doubles. It has ∞^2 point-pairs. To determine the order of the surface on which they lie, we recall (§ 3) that the point-pairs with one member on a given line are repre-

* See *Encyklopädie*, III C 2, p. 248, note 543.

† For proof see Reye, loc. cit., § 6.

sented in S_9 by a S_4^4. Hence in an arbitrary S_5 there are four points of this locus, and so in the general L_4 there are *four* point-pairs of which one member lies on an arbitrary line. Hence the points of the point-pairs lie on a quartic surface Φ_4.

THEOREM. *The surface Φ_4 whose existence has just been established is the locus of vertices of the ∞^2 cones of the basal system.*

We observe that the L_4 has singular planes, in each of which there are ∞^2 conics instead of ∞^1. Namely, in any plane of the basal plane-pairs the conics of the L_4 form a *Schaarschaar* of the type considered in 3.

Of course there are numerous special types of L_4. In particular, the conics that touch four fixed planes form a special L_4. Thus we have the following corollary to an earlier theorem:

The conics that touch four fixed planes form a system L_4, and there exist six other pairs of planes each conjugate with regard to every conic of the L_4.

5. The L_3, or three parameter system, is defined by five independent equations:

$$\sum_{i,j=1}^{4} a_{ij}^{(k)} w_{ij} = 0 \qquad (k = 1, \cdots, 5). \tag{20}$$

These equations define ∞^4 envelope-quadrics, each apolar to the ∞^4 point-quadrics of the basal system

$$\sum_{i,j=1}^{4} \left(\sum_{k=1}^{5} \lambda_k a_{ij} \right) x_i x_j = 0. \tag{21}$$

There is complete duality between these two linear systems.* The former includes ∞^3 conics, which comprise the L_3, and the latter includes a dual system of ∞^3 cones.

We know that the conditions that a quadric be a conic lying in a given plane (p) are the four linear equations

$$\begin{aligned} \sum_{i=1}^{4} w_{1i} p_i &= 0, \qquad \sum_{i=1}^{4} w_{3i} p_i = 0, \\ \sum_{i=1}^{4} w_{2i} p_i &= 0, \qquad \sum_{i=1}^{4} w_{4i} p_i = 0. \end{aligned} \tag{22}$$

These, together with the five equations (20), form a system that has in general one and essentially only one solution; hence *the L_3 has one conic in an arbitrary plane.*

THEOREM. *The conics of the L_3 that lie in planes through a fixed line generate a surface of order 8, having the fixed line as a sextuple line.*

Degenerate conics of the L_3. We may state the

* Cf. Reye, loc. cit., § 7.

THEOREM. *The point-pairs of the L_3 are ∞^1 in number, and lie on a curve C_{10} of order ten. Their axes generate a ruled surface K_{10} of order ten. Dually, the basal system includes ∞^1 plane-pairs, which envelope a developable Γ_{10} of class ten; their lines of intersection lie on a ruled surface Φ_{10} of order ten.*

Reye (loc. cit., § 7) gives a number of very interesting relations among these four figures.

6. The L_2, or two-parameter linear system, is defined by a set of six independent equations. The totality of sets of solutions of such a system are linearly dependent on four, and define ∞^3 envelope-quadrics, which include the conics of the L_2.

Hence we have as alternative definition of the L_2 the equations

$$w_{ij} = \lambda_1 a_{ij} + \lambda_2 b_{ij} + \lambda_3 c_{ij} + \lambda_4 d_{ij} \quad (i, j = 1, \cdots, 4), \tag{23}$$

$$\Delta(w) = 0. \tag{24}$$

The system of quadrics (23) is dual to the *Gebüsch* discussed in 4; and the basal system is dual to the system of envelope-quadrics there discussed. We have then at once the following facts:

The conics of L_2 lie in ∞^2 planes, one in each plane in general, and these planes envelope a surface Φ_4 of class four.

The L_2 includes in general ten point-pairs; their axes all lie on Φ_4. The basal system contains ∞^2 plane-pairs, whose planes envelope the surface Φ_4. In other words, the planes of these plane-pairs are the planes that contain conics of the L_2. Since they can be associated into pairs that belong to the basal system, we have the

THEOREM. *The planes of the conics of the L_2 can be associated in pairs in such a way that the members of each pair are mutually conjugate with regard to every conic of the L_2.*

The general S_3 in S_9 meets in eight points the S_6^8 which represents the conics through an arbitrary point; hence the

THEOREM. *The L_2 includes 8 conics through an arbitrary point of space.*

Special types of L_2. As in the earlier cases, when the basal system contains any plane-doubles, the conics of L_2 all touch every such plane; and in particular, the conics that touch six fixed planes form an L_2 of very special type. The dual system (viz., the Gebüsch of quadrics through six fixed points) has been somewhat studied.*

In general, the L_2 will have no singular plane containing more than one conic. There may, however, be such planes. There are four types, of which one reduces to a complete S_3, and the others are incomplete L_2's with singular planes. Namely:

* Cf. *Encyklopädie*, III C 2, § 141.

(1) One singular plane containing ∞^1 conics.

(2) Two singular planes, each containing ∞^1 conics.

(3) One singular plane containing ∞^2 conics.

(4) If all four conics are coplanar, every linear combination of them is a conic, and we have a complete three-parameter linear system of conics in the plane in question.

7. The L_1, or one-parameter linear system, is most conveniently defined as a system linearly dependent on three sets of numbers:

$$w_{ij} = \lambda_1 a_{ij} + \lambda_2 b_{ij} + \lambda_3 c_{ij}, \tag{25}$$

$$\Delta(w) = 0. \tag{26}$$

THEOREM. *The planes of the conics of a general L_1 form a developable of class* 6.

For the conics whose planes pass through a given point are represented in S_9 by a S_7^6 (§ 3) and hence an arbitrary S_2 contains 6 of them. We find easily also the following results.

THEOREM. *The conics of the general L_1 lie on a surface of order eight.*

THEOREM. *There are four conics of the L_1 that touch an arbitrary plane.*

COROLLARY. *The conics of the L_1 cut a given plane in points of a curve of order* 8. *They determine on this curve an involution of points, in which four pairs of points are self-corresponding.*

The basal system of the L_1 consists of a linear ∞^6 of point-quadrics. We know (2) that such a system includes eight plane-doubles. But every envelope-quadric touches every plane that is a plane-double apolar to it; so that every conic of the L_1 touches these eight fixed planes. This leads to another definition of the L_1.

THEOREM. *The general L_1 consists of the totality of conics that touch eight fixed planes, whereof seven may be chosen arbitrarily.*

In other words: *Let seven planes be chosen arbitrarily in space. There are ∞^1 planes, generating a sextic developable, in each of which the seven lines cut by the fixed planes are all tangent to a conic. The conics thus determined lie on a surface of order eight, and all touch an eighth fixed plane.*

8. The L_0, or system of conics whose coördinates satisfy *eight* linear equations, may be defined by

$$w_{ij} = \lambda_1 a_{ij} + \lambda_2 b_{ij} \qquad \Delta(w) = 0. \tag{27}$$

We may state the result:

The general L_0 consists of four conics having common tangent planes that generate a quartic developable. A system of this kind is uniquely determined by any two quadrics; the planes of the four conics form the only tetrahedron that is self-conjugate with regard to all the quadrics of the series containing the L_0.

7. Complete Linear Systems

We now propose to determine all possible types of complete linear systems. First we will note that such a system may be given parametrically in the form

$$w_{ij} = \lambda_1 a_{ij}^{(1)} + \cdots + \lambda_7 a_{ij}^{(r)} \qquad (i, j = 1, \cdots, 4),$$

where $\Delta(w) \equiv 0$ for all values of $(\lambda_1, \cdots, \lambda_7)$.

Consider, in particular, two conics (a) and (b) of a complete linear system. Then every quadric

$$w_{ij} = \lambda a_{ij} + \mu b_{ij}$$

must be a conic of the system. We have then

$$\lambda^4 \Delta(a) + \lambda^3 \mu \Theta(a, b) + \lambda^2 \mu^2 \Phi(a, b) + \lambda \mu^3 \Theta(b, a) + \mu^4 \Delta(b) = 0.$$

We have already, since (a) and (b) are conics,

$$\Delta(a) = \Delta(b) = 0,$$

and the following further relations are necessary:

$$\Theta(a, b) = 0, \qquad \Theta(b, a) = 0, \qquad \Phi(a, b) = 0.$$

From the results of § 5, it follows that we must have one of the following situations:

(A) If neither (a) nor (b) is degenerate, then (1) they are coplanar; (2) each touches the plane of the other, and they cut the line of intersection of their planes in a harmonic set of points.

(B) When (a) is a proper conic, and (b) is degenerate, then (1) if (b) is a point-pair, its axis meets (a), and the plane of (a) touches (b); (2) if (b) is a point-double, the plane of (a) touches it.

(C) When both conics are degenerate, then (1) if both are point-pairs, their axes intersect; (2) if either is a point-double, the equation is satisfied.

Considering these cases, we see that except in $A2$ and $B1$, the conics are coplanar. In case $A2$ they are necessarily tangent; and in case $B1$, either they are coplanar, or the point-pair (b) has one member lying on (a).

We introduce now the following

Definition. *A point-pair shall be said to be tangent to a proper conic, when one of its points lies on the conic.*

And we have the result: *Every two conics of a complete linear system are either coplanar or tangent. Conversely, we see without difficulty that if two conics are coplanar or tangent, every quadric linearly dependent on them is a conic.*

Concerning the locus of these conics, we present without detailed proof the following statement:

Theorem. *If* $(w) = \lambda(a) + \mu(b)$ *be a one-parameter complete linear system, it is of one of the two types*:

(1) *the* SERIES (*Schaar*) *consisting of the conics in a plane that touch four fixed lines in the plane*;

(2) *the one-parameter* SHEAF, *consisting of the conics cut from a quadric cone by the planes of a pencil whose axis is a tangent line to the cone. The axis of the pencil, which is evidently tangent to all the conics of the sheaf, we shall call the axis of the sheaf*; *and the point of contact we shall call the vertex of the sheaf.*

We may divide complete linear systems in general into two classes, according as all their members lie in the same plane or not. The planar systems we shall not discuss further than to remark that they are the systems studied in plane geometry as linear systems in tangential coördinates.

Let us consider a system in which not all the conics are coplanar. By brief reasoning, which we here suppress, one arrives at this result:

THEOREM. *In any non-planar complete linear system, the members all touch a fixed line at a fixed point*; *and conversely, any number of conics tangent to a fixed line at a fixed point will determine a complete linear system.*

We have not considered degenerate conics. Evidently a point-pair can belong to a system of this type if and only if it has either (1) one member at the vertex, or (2) both members on the axis; and a point-double can belong to the system, if and only if it lies on the axis.

DEFINITION. *A non-planar linear complete system, of which the coördinates of the members are linearly dependent on those of* $r+1$ *linearly independent conics, shall be called an r-parameter sheaf.*

We now examine the most general sheaf and find readily the

THEOREM. *The most general sheaf of conics consists of the totality of conics that touch a fixed axis at a fixed point. It is projectively equivalent to the system whose coördinates are given by the matrix*

$$\begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & a_0 & a_3 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

for all sets of values $(a_0, a_1, a_2, a_3, a_4)$ *not all zero. It follows that such a sheaf is a four-parameter system.*

DEFINITION. *A four-parameter sheaf shall be called an entire sheaf.*

We note the following properties of the entire sheaf:

An entire sheaf is determined by a point and a line through it, hence there are in all ∞^5 of them.

Any proper conic is a member of ∞^1 entire sheaves, one with its vertex at each point of the conic.

If two entire sheaves have in common a proper conic, their axes are coplanar, and they have in common a pencil of conics having double contact.

A point-pair belongs to ∞^2 entire sheaves, viz., to any one whose vertex is at either point of the pair, or whose axis is the axis of the pair.

Every two entire sheaves have in common one point-pair, made up of the vertices of the sheaves. Two entire sheaves have more than one common point-pair, if they have the same vertex, or if the vertex of one passes through the axis of the other.

A point-double belongs to every entire sheaf whose axis passes through the point.

Translating these results in S_9, we will denote by Λ the singular domain S_6^{10} that represents point-pairs, and by Φ the linear S_4 on Σ that represents an entire sheaf.

Then Σ contains ∞^5 Φ's. Through an ordinary point of Σ there are ∞^1, of which any two intersect in a line. In general, two Φ's do not intersect at any ordinary point of Σ, but at a single point of Λ. If they intersect at any ordinary point, they have a common line.

Through an arbitrary point of Λ, there are in general ∞^2 Φ's. Through any point of the S_3^8 that represents point-doubles there are ∞^3 Φ's.

The point-pairs of the above entire sheaf are determined by either $a_0 = 0$, or $a_1 = a_2 = 0$, corresponding respectively to point-pairs having one member at the vertex, and to those having both members on the axis. The former are represented in Φ by a linear three-space Q, the latter by a plane R.

The point-doubles are determined by $a_1 = a_2 = 0$, $a_0 a_4 = a_3^2$, and are represented in Φ by a S_1^2, i. e., a *conic*.

The entire sheaf contains sheaves of one, two, and three parameters, which we denote by T_1, T_2, T_3 respectively, with reference to which we give here only summary statements.

1. The one parameter sheaf T_1 we have already considered. If (a) and (b) be any two conics of T_4, then a T_1 is given by

$$w_i = \lambda a_i + \mu b_i \qquad (i = 0, \cdots, 4).$$

This corresponds to a *line* in S_4. There are therefore ∞^6 T_1's in any T_4.

If (a) and (b) are not coplanar, the conics lie on a cone that touches the axis at the vertex of T. In this case the sheaf contains just one degenerate conic. There are, however, special cases not described here.

2. The T_2 is defined by the equations

$$w_i = \lambda_1 a_i + \lambda_2 b_i + \lambda_3 c_i \qquad (i = 0, 1, 2, 3, 4),$$

where (a), (b), and (c) are linearly independent members of a T_4. It is represented in S_4 by a plane (π).

The plane (π) cuts Q in a line k and R in a point K, and can be described by lines drawn from K to the points of k. Each of these lines represents a planar T_1, as may easily be seen. Hence the T_2 may be regarded as composed of ∞^1 planar T_1's. Moreover, the common tangents to the members of each planar T_1 meet the axis at the same points R_1, R_2, since each of the lines cuts R at the same point. We have then the following definition of the T_2:

THEOREM. *The totality of conics of a T_4 that touch two fixed planes, whose intersection does not meet the axis of the T_4, constitute a T_2 of the most general type.*

Special types of T_2 correspond to special relations of (π) to the space containing the line k and the plane containing the point K. For example, it may happen that K lies on k. In this case *a T_2 of the type under consideration consists of all the conics of a T_4 that have contact of the second order with one another at the vertex, and also touch a fixed plane not through the vertex.*

Two other special types are evidently those in which either k or K is indeterminate. The latter is a planar T_2, and we find the

THEOREM. *The general planar T_2 consists of all the conics of a T_4 that lie in a plane and touch any two lines so drawn from two fixed points on the axis of the T_4 as to intersect on a fixed line in the plane. It follows that the common tangents to any two conics of the system intersect on this fixed line.*

3. The three-parameter system T_3 can be treated in the same way. A T_3 is represented in S_4 by a *space* S_3; and in general this cuts Q in a plane N, and R in a line n. To the former correspond ∞^2 point-pairs of the T_3, consisting of the vertex P, taken with the ∞^2 points of a plane (σ). To the latter correspond ∞^1 point-pairs on the axis of the T.

The S_3 may be regarded as swept out by a pencil of planes through the line n. Each of these planes corresponds to a planar T_2 of the kind just studied. We have seen that such a system has the property that the common tangents to any two of its members meet on a fixed line l. In our pencil of T_2's, these fixed lines all lie in the plane (σ). Also the vertex of the cone of any T_1 in the T_3 lies on (σ). We may then define the T_3 as follows:

Let a plane (σ) and a point-pair (R_1, R_2) on the axis be chosen arbitrarily. The totality of conics of the T_4, which touch a pair of lines drawn from R_1 and R_2 to any point of the plane (σ), generate a T_3. The common tangents to any two coplanar members of the T_3 meet on (σ), and the vertex of the cone through any two other members lies on (σ).

We mention two special types. There is one special T_3, namely, $a_0 = 0$, consisting of all the point-pairs whereof the vertex is one member. Another type is represented by a S_3 through the plane containing the point K. It is obvious that such a S_3 represents a planar T_3.

CAMBRIDGE, MASS.,
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THE WEIERSTRASS E-FUNCTION FOR PROBLEMS OF THE CALCULUS OF VARIATIONS IN SPACE*

BY

GILBERT AMES BLISS

For the problem of minimizing the definite integral

$$\int F(x, y, z, p, q, r)\, dt,$$

$$(1) \qquad p = \frac{dx}{dt}, \qquad q = \frac{dy}{dt}, \qquad r = \frac{dz}{dt},$$

where the function F has the homogeneity property

$$(2) \qquad F(x, y, z, kp, kq, kr) = kF(x, y, z, p, q, r) \qquad (k > 0),$$

the Weierstrass E-function

$$E(x, y, z;\ p, q, r;\ p', q', r')$$

has the form

$$\begin{aligned} E = F(x, y, z, p', q', r') &- p' F_p(x, y, z, p, q, r) \\ &- q' F_q(x, y, z, p, q, r) \\ &- r' F_r(x, y, z, p, q, r). \end{aligned}$$

Behaghel† has deduced a very useful expression for the E-function in terms of the quadratic form

$$(3) \qquad \begin{aligned} &Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) \\ &\qquad = F_{pp}\,\xi^2 + F_{qq}\,\eta^2 + F_{rr}\,\zeta^2 + 2F_{qr}\,\eta\zeta + 2F_{rp}\,\zeta\xi + 2F_{pq}\,\xi\eta. \end{aligned}$$

In his proof he makes use of functions, in his notation A, B, C, which are analogous to one which is used for a similar problem in the plane. For the space problem, unfortunately, these functions may become infinite even for very simple integrals of the type (1), and in such cases his proof of the formula would not be valid.‡

* Presented to the Society, December 26, 1913.

† Mathematische Annalen, vol. 73 (1913), p. 596.

‡ For the function

$$F = \sqrt{p^2 + q^2 + (q + r)^2}$$

the expression for A is

$$A = -F_{qr}/qr = \left(1 - \frac{p^2}{qr}\right) / [p^2 - q^2 + (q + r)^2]^{3/2},$$

and this is infinite for any direction in the y-plane or z-plane not perpendicular to the x-axis.

In the following sections it is proposed to give a proof of a formula somewhat more elegant than that of Behaghel and which can be reduced to his by a simple transformation. The objections mentioned above are avoided, and the special case when the directions (p, q, r) and (p', q', r') are in the same line but opposite, treated separately in Behaghel's paper, is here quite unexceptional.* The method used applies at once to problems in space of any number of dimensions. In § 2 some useful consequences of the formula are deduced, and in § 3 some peculiarities of the extension of the results found to spaces of higher dimensions are explained.

1. The Formula for the E-function

It will be supposed that for a fixed set of values (x, y, z, p, q, r) the function F has continuous first and second derivatives for arguments (p', q', r') defining directions in a cone shaped region P_ϵ, shown in Fig. 1

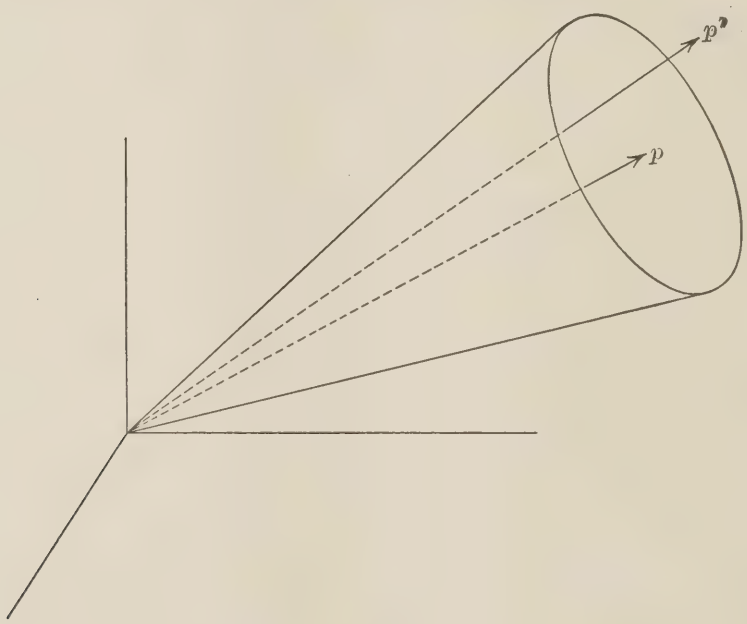

Fig. 1.

about an initial direction p, q, r. This and other directions will be referred to hereafter by their first letters only, as illustrated in the figure. Analytically the region is defined by an inequality of the form

$$|\omega| < \epsilon \tag{4}$$

* A formula given by Mason and Bliss, these Transactions, vol. 9 (1908), p. 459, is less effective than the one proved in the text for a similar reason. It does not necessarily hold when the two directions x', y', z' and x_u, y_u, z_u are opposite.

where ω is the angle between p and p' defined by the formula

$$\cos\omega = \frac{pp' + qq' + rr'}{\sqrt{p^2 + q^2 + r^2}\sqrt{p'^2 + q'^2 + r'^2}} \qquad (0 \leqq \omega \leqq \pi).$$

In as much as the properties of the E-function to be considered in this section do not depend upon the arguments x, y, z, the latter may be omitted in most of the equations. It is understood that for the direction p the quantities p, q, r are not all zero, and a similar remark applies for other directions.

The function F satisfies the relation

$$pF_p + qF_q + rF_r = F, \tag{5}$$

found by differentiating (2) for k and putting $k = 1$, and also the relations

$$\begin{aligned} pF_{pp} + qF_{pq} + rF_{pr} &= 0, \\ pF_{qp} + qF_{qq} + rF_{qr} &= 0, \\ pF_{rp} + qF_{rq} + rF_{rr} &= 0, \end{aligned} \tag{6}$$

which follow by differentiating (5) for p, q, r, respectively. Furthermore the first derivatives of F are positively homogeneous with order zero. For by differentiating (2) with respect to p, for example, it is found that

$$F_p(x, y, z, kp, kq, kr) = F_p(x, y, z, p, q, r) \qquad (k > 0).$$

From these results it follows readily that the E-function itself has the homogeneity property

$$E(x, y, z;\ kp, kq, kr;\ k'p', k'q', k'r') = k'E(x, y, z;\ p, q, r;\ p', q', r') \qquad (k > 0,\ k' > 0),$$

and the direction ratios which occur among its arguments may be taken without loss of generality to be direction cosines.

It will be helpful to use the notation

$$\begin{aligned} \Phi(p, \xi, \xi') = {} & \xi'(F_{pp}\xi + F_{pq}\eta + F_{pr}\zeta) \\ & + \eta'(F_{qp}\xi + F_{qq}\eta + F_{qr}\zeta) \\ & + \zeta'(F_{rp}\xi + F_{rq}\eta + F_{rr}\zeta). \end{aligned}$$

The form Φ has the following properties to be used later, the first of which gives an expression for the quadratic form Q in terms of Φ:

$$Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) = \Phi(p, \xi, \xi), \tag{7}$$

$$\Phi(p, \xi, \xi') = \Phi(p, \xi', \xi), \tag{8}$$

$$\Phi(p, \xi, u\xi' + v\xi'') = u\Phi(p, \xi, \xi') + v\Phi(p, \xi, \xi''), \tag{9}$$

$$\Phi(p, \xi, p) = 0, \tag{10}$$

where in the third formula $u\xi' + v\xi''$ is a symbol for the values

$$u\xi' + v\xi'', \qquad u\eta' + v\eta'', \qquad u\zeta' + v\zeta''.$$

The first three of these formulas follow readily from the definition of Φ, while the last is a consequence of the homogeneity properties (6).

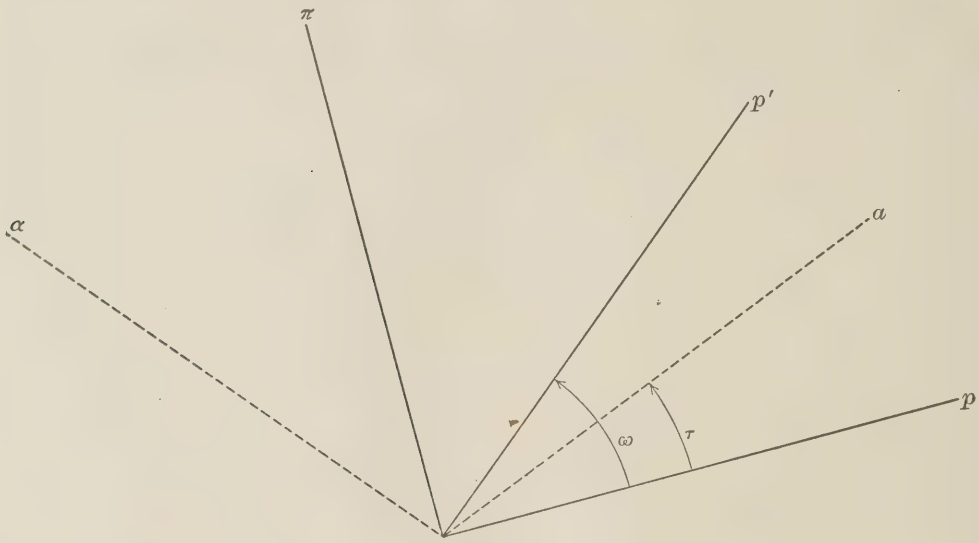

FIG. 2.

Consider now the two directions p and p' indicated in Fig. 2. There will always be a direction π which is orthogonal to p and co-planar with p and p'. Let a and α be two directions orthogonal to each other in the same plane with p, p', and π. Their direction cosines are expressible in the form

$$\begin{aligned} a &= p\cos\tau + \pi\sin\tau, & \alpha &= -p\sin\tau + \pi\cos\tau, \\ b &= q\cos\tau + \kappa\sin\tau, & \beta &= -q\sin\tau + \kappa\cos\tau, \\ c &= r\cos\tau + \rho\sin\tau, & \gamma &= -r\sin\tau + \rho\cos\tau, \end{aligned} \tag{11}$$

where τ is the angle between a and p. By changing the sense of the direction π, if necessary, it can be effected that for $\tau = \omega$ the direction a coincides with p'.

The E-function has then the expressions

$$\begin{aligned} E(p, p') &= \int_0^\omega \frac{d}{d\tau}\{p' F_p(a) + q' F_q(a) + r' F_r(a)\}\,d\tau \\ &= \int_0 \Phi(a, \alpha, p')\,d\tau, \end{aligned}$$

which follow readily from (5), (11), and the definition of Φ. From the relations

$$p' = a\cos(\omega - \tau) + \alpha\sin(\omega - \tau),$$
$$q' = b\cos(\omega - \tau) + \beta\sin(\omega - \tau),$$
$$r' = c\cos(\omega - \tau) + \gamma\sin(\omega - \tau),$$

and the formulas (9) and (10) it is seen that

$$E(p, p') = \int_0^\omega \sin(\omega - \tau)\Phi(a, \alpha, \alpha)\,d\tau,$$

or, by applying the mean value theorem for a definite integral,

$$E(p, p') = (1 - \cos\omega)\Phi(a^*, \alpha^*, \alpha^*), \tag{12}$$

where a^* is the direction a corresponding to a suitably chosen value τ^* between 0 and ω, and α^* is the corresponding direction perpendicular to a^*.

Hence the function

$$E(x, y, z;\ p, q, r;\ p', q', r')$$

is expressible in the form

$$E = (1 - \cos\omega)Q(x, y, z;\ a, b, c;\ \alpha, \beta, \gamma). \tag{13}$$

In this formula ω is the angle $(0 \leqq \omega \leqq \pi)$ between the directions (p, q, r) and (p', q', r'), while (a, b, c) and (α, β, γ) are directions orthogonal to each other in the plane of the other two and defined by the formulas (11) *for a suitably chosen intermediate value of τ satisfying the inequality $0 < \tau < \omega$.* The relations between these directions is clearly shown in Fig. 2.

It is easy to derive Behaghel's formula from this result. The relations (11) give

$$p = a\cos\tau - \alpha\sin\tau, \qquad q = b\cos\tau - \beta\sin\tau, \qquad r = c\cos\tau - \gamma\sin\tau,$$

and with the help of (9), (10), and (8) it follows that

$$\Phi(a, p, p) = \sin^2\tau\,\Phi(a, \alpha, \alpha).$$

Hence the expression (12) can be put into the form

$$E(p, p') = (1 - \cos\omega)\frac{\Phi(a^*, p, p)}{\sin^2\tau^*}$$
$$= (1 - \cos\omega)\frac{Q(x, y, z;\ a^*, b^*, c^*;\ p, q, r)}{\sin^2\tau^*},$$

which is the formula required.

2. Consequences of the Formula

Consider an arc C defined by the continuous functions

$$(C) \qquad x = x(t), \qquad y = y(t), \qquad z = z(t) \qquad (t_1 \leqq t \leqq t_2)$$

and having a continuously turning tangent whose direction cosines will be represented by $p(t)$, $q(t)$, $r(t)$. The neighborhood C_ϵ of order zero of this arc is the totality of values (x, y, z, p, q, r) for each of which the inequalities

$$(C_\epsilon) \qquad \begin{gathered} |x - x(t)| < \epsilon, \qquad |y - y(t)| < \epsilon, \qquad |z - z(t)| < \epsilon, \\ (p, q, r) \neq (0, 0, 0) \end{gathered}$$

are true for at least one value of t between t_1 and t_2. The neighborhood C'_ϵ of order one is similarly defined by the conditions

$$(C'_\epsilon) \qquad \begin{gathered} |x - x(t)| < \epsilon, \qquad |y - y(t)| < \epsilon, \qquad |z - z(t)| < \epsilon, \\ (p, q, r) \neq (0, 0, 0), \qquad |\theta| < \epsilon, \end{gathered}$$

where θ is the angle $(0 \leqq \theta \leqq \pi)$ between the directions p and $p(t)$. Besides the region P_ϵ of § 1 and the two neighborhoods just defined, there is a fourth region involved in the following theorems and defined by the conditions

(R) $\quad (x, y, z)$ *in a continuum of xyz-points*; $\qquad (p, q, r) \neq (0, 0, 0)$.

For each of the theorems the function F is supposed to be continuous and to have continuous first and second derivatives with respect to p, q, r in some region including the one involved in the statement of the theorem.

If for a fixed set of values (x, y, z, p, q, r) *the condition*

$$E(x, y, z;\ p, q, r;\ p', q', r') \geqq 0 \tag{14}$$

is satisfied for all directions (p', q', r') *in a neighborhood* P_ϵ *of the direction* (p, q, r), *then the condition*

$$Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) \geqq 0 \tag{15}$$

is also satisfied for any values whatsoever of ξ, η, ζ. *In other words the necessary condition of Weierstrass, for either a strong or a weak minimum, at any point of a given curve, implies the necessity of Legendre's condition also.*

If in the formula (13) the direction p' approaches the direction p while remaining in a fixed plane through the latter, the directions a and α approach, respectively, p and a direction π normal to p in the plane. Hence the inequality (15) must be true when the direction ξ coincides with π. But p' may be made to approach p in any plane through p, and consequently π may be any normal to p. If ξ lies in a plane with p and π and makes an angle τ with p, the formulas

$$\xi = p \cos \tau + \pi \sin \tau, \qquad \eta = q \cos \tau + \kappa \sin \tau, \qquad \zeta = r \cos \tau + \rho \sin \tau,$$

together with (9) and (10) show that

$$Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) = \sin^2\tau\, Q(x, y, z;\ p, q, r;\ \pi, \kappa, \rho).$$

Hence the inequality (15) is true for all values of ξ, η, ζ.

The form Q in the expression (13) is said to be regular in a region of values $(x, y, z;\ p, q, r)$ if it does not take opposite signs for different values of its arguments and vanishes only when ξ, η, ζ are all zero, or when the directions (ξ, η, ζ) and (p, q, r) are in the same line.

If the function Q is regular at all values (x, y, z, p, q, r) on an arc C, then there exists a neighborhood C'_δ in which it remains regular and does not take opposite signs.

For Q can always be transformed by a homogeneous orthogonal linear transformation of ξ, η, ζ* into a sum of squares whose coefficients $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$(16)\qquad \begin{vmatrix} F_{pp}+\lambda & F_{pq} & F_{pr} \\ F_{qp} & F_{qq}+\lambda & F_{qr} \\ F_{rp} & F_{rq} & F_{rr}+\lambda \end{vmatrix} = 0.$$

The term independent of λ in this equation is zero since it is the determinant of the coefficients of p, q, r in the equations (6). Hence one of the roots, say λ_3, is always zero; and the other two are different from zero and of the same sign for any set of values (x, y, z, p, q, r) corresponding to a point of the arc C, since for such values Q is regular and can vanish for one direction ξ, η, ζ only. Since the roots λ_1 and λ_2 are continuous functions of the arguments x, y, z, p, q, r, they will remain different from zero and will not change sign in a sufficiently small neighborhood C'_δ.

Suppose that Q is regular along the arc C and that the inequality

$$E[x(t), y(t), z(t);\ p(t), q(t), r(t);\ p', q', r'] > 0$$

is satisfied at every point of C whenever the direction (p', q', r') is distinct from that of C. Then there exists a neighborhood C'_ϵ such that in the region of values $(x, y, z;\ p, q, r;\ p', q', r')$ defined by the relations

$$(x, y, z, p, q, r) \text{ in } C'_\epsilon, \qquad (p', q', r') \neq (0, 0, 0)$$

the function E does not take opposite signs and vanishes only when the directions (p, q, r) and (p', q', r') coincide.

To prove this consider the function E_1 defined by the equations

$$(17)\qquad \begin{aligned} &E_1(x, y, z;\ p, q, r;\ \pi, \kappa, \rho;\ \omega) = E/(1-\cos\omega) \qquad (0 < \omega \leqq \pi), \\ &E_1(x, y, z;\ p, q, r;\ \pi, \kappa, \rho;\ 0) = Q(x, y, z;\ p, q, r;\ \pi, \kappa, \rho), \end{aligned}$$

* See Kowalewski, *Einfuhrung in die Determinantentheorie*, § 116.

the first of which is formed by replacing the values (p', q', r') in E by expressions similar to (11) with $\tau = \omega$. From its definition E_1 is continuous for all values of its arguments for which the point (x, y, z) is sufficiently near to C, and $0 < \omega \leqq \pi^r$,* while the other arguments are the direction cosines of two perpendicular directions. It is continuous also for $\omega = 0$, since the formula (13) shows that as ω approaches zero the values of the quotient in the first part of the definition of E_1 approach continuously the values defined by the second of equations (17). Along the arc C the quotient E_1 is different from zero for directions π normal to $p(t)$, and for $0 \leqq \omega \leqq \pi^r$, on account of the hypothesis of the theorem and the expressions (17). Since it is continuous it retains these properties when the values (x, y, z, p, q, r) are in a neighborhood C'_ϵ of the arc C. The function E has therefore the properties described in the theorem, since E is expressible as a non-vanishing factor times E_1 whenever p and p' are distinct.

Consider a problem which is regular in a region R of the kind described in the first paragraph of this section, that is, a problem such that the form Q is everywhere regular in R. Then the function $E(x, y, z; p, q, r; p', q', r')$ does not take opposite signs when (x, y, z, p, q, r) and (x, y, z, p', q', r') are in R, and vanishes only when the directions (p, q, r) and (p', q', r') are coincident.

At every point of the region R the quadratic form Q is regular, and hence the expression (13) for the E-function vanishes only when p and p' coincide. Further any two sets of values $(x, y, z; p, q, r; p', q', r')$ can be joined by a continuous path over which p and p' are nowhere coincident. Hence the sign of the E-function is always the same at points where it does not vanish.

The regularity or non-regularity of the quadratic form Q is controlled by the behavior of a function F_1 defined by the equations

$$p^2 F_1 = F_{qq} F_{rr} - F_{qr}^2, \qquad qrF_1 = F_{pq} F_{pr} - F_{pp} F_{qr},$$

$$q^2 F_1 = F_{rr} F_{pp} - F_{rp}, \qquad rpF_1 = F_{qr} F_{qp} - F_{qq} F_{rp},$$

$$r^2 F_1 = F_{pp} F_{qq} - F_{pq}^2, \qquad pqF_1 = F_{rp} F_{rq} - F_{rr} F_{pq}.$$

The existence and continuity of this function are consequences of the relations (6). The coefficient of the first power of λ in the equation (16) is precisely F_1, and consequently λ_1 and λ_2 will be different from zero and have the same sign if and only if F_1 is positive.

A necessary and sufficient condition that the quadratic form Q be regular at a set of values (x, y, z, p, q, r) is

$$F_1(x, y, z, p, q, r) > 0.$$

* The symbol π^r stands for π radians.

3. Extension of the Preceding Results to Higher Spaces

The formula (13) which was the goal of the discussion of § 1 has an analogue for an integral of the form

$$\int F(y, p)\, dt,$$

where y and p are symbols for sets of elements of the form

$$y = (y_1, y_2, \cdots, y_n),$$

$$p = (p_1, p_2, \cdots, p_n) = \left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \cdots, \frac{dy_n}{dt}\right).$$

The function F is supposed to have a homogeneity property with respect to the arguments p similar to (2), and relations analogous to (5) and (6) are readily derived.

The E-function and the quadratic form Q have the forms

$$E(y, p, p') = F(y, p') - \sum_{i=1}^{n} p_i F_i,$$

$$Q(y, p, \xi) = \sum_{i,j=1}^{n} F_{ij}(y, p)\, \xi_i \xi_j$$

where F_i and F_{ij} are symbols for the derivatives $\partial F/\partial p_i$ and $\partial^2 F/\partial p_i p_j$, respectively. The E-function has the homogeneity property

$$E(y, \kappa p, \kappa' p') = \kappa' E(y, p, p') \qquad (\kappa > 0,\ \kappa' > 0),$$

and as before there will be no loss of generality if all the directions p are supposed to be normed, that is, to satisfy the relation

$$p_1^2 + p_2^2 + \cdots + p_n^2 = 1.$$

The condition for the orthogonality of two directions, p and π, in n-dimensional space is

$$p_1 \pi_1 + p_2 \pi_2 + \cdots + p_n \pi_n = 0.$$

The auxiliary form Φ has the definition

$$\Phi(p, \xi, \xi') = \sum_{i,j=1}^{n} F_{ij}(y, p)\, \xi_i \xi_j'$$

and satisfies four relations analogous to (7)–(10) of § 1.

There will always be at least $n - 2$ independent directions $q', q'', \cdots, q^{(n-2)}$ orthogonal to both p and p', which with p itself form a system of $n - 1$ independent directions; and the last system of $n - 1$ directions determines a unique direction π orthogonal to each of them. The directions π and p are independent, and orthogonal to each of the directions $q', q'', \cdots, q^{(n-2)}$. Since every other direction having this property must be expressible linearly

in terms of p and π, it follows that p' must have the form

$$p_i' = p_i \cos \omega + \pi_i \sin \omega \quad (i = 1, 2, \cdots, n; 0 \leqq \omega \leqq \pi^r).$$

In order to satisfy the last restriction in the parenthesis it may be necessary to change the sign of the direction π.

If the directions a and α are defined by the equations

$$a_i = p_i \cos \tau + \pi_i \sin \tau, \qquad \alpha_i = -p_i \sin \tau + \pi_i \cos \tau \quad (i = 1, 2, \cdots, n),$$

the argument proceeds exactly as in § 1, and the formula found for the E-function is

$$E(y, p, p') = (1 - \cos \omega) Q(y, a, \alpha),$$

where a and α are defined by the equations just given for a suitably chosen intermediate value of τ between 0 and ω.

All of the results of § 2 except the last can be proved for n dimensions in a manner analogous to that which is effective for three dimensions. The analogue of F_1 is a function defined by relations of the form

$$p_i p_k F_1 = A_{ik} \qquad (i, k = 1, 2, \cdots, n),$$

where A_{ik} is the cofactor of F_{ik} in the matrix of these derivatives. In the equation corresponding to (16) for n dimensions the term independent of λ vanishes as before, and F_1 is the coefficient of the first power of λ. Hence if Q is regular at a set of values (y, p) it will remain so in any neighborhood of those values in which F_1 is different from zero. But more than the requirement that F_1 shall have a certain sign is needed in order to insure the regularity of Q.

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THE SUBGROUPS OF THE QUATERNARY ABELIAN LINEAR GROUP*

BY

HOWARD H. MITCHELL

1. Introduction

With the possible exception of the binary modular groups there is no finite linear group which is of more importance in analysis and geometry than what is known as the *abelian*† linear group. In the investigations of Hermite and Jordan the coefficients were taken as rational integers reduced modulo p (a prime), in which case it is the Galois group of the equation for the p-section of the periods of hyperelliptic functions.‡ It is also the group of isomorphisms of a certain type of commutative group and is isomorphic with certain systems of collineation groups.§

For the case of four variables and the modulus 3, it is the group of the equation for the 27 straight lines on a general cubic surface|| as well as the group for the problem of reducing a binary sextic to the canonical form $T^2 - U^3$. One of the collineation groups with which it is isomorphic in this case is the quaternary G_{25920}, which has received considerable attention.

If the coefficients of the transformations, instead of being integers reduced modulo p, are taken as marks of a general Galois field $GF(p^n)$, a more general group is obtained which has been investigated by Dickson.¶ This group requires an even number of variables and has a certain invariant bilinear function analogous to that for the ordinary continuous group in space which

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† This group is called abelian on account of its connection with abelian functions. It is not commutative.

‡ Cf. Jordan, *Traité des substitutions*, pp. 171–186; 354–369; Witting, Mathematische Annalen, vol. 29 (1886), p. 157; Dissertation (Göttingen, 1887); Burkhardt, Mathematische Annalen, vol. 38 (1891), p. 163.

§ Cf. Jordan, loc. cit., pp. 420–450; Witting, loc. cit.; Burkhardt, loc. cit.; Dickson, these Transactions, vol. 1 (1900), pp. 30–38; the author, Bulletin of the American Mathematical Society, vol. 20 (1913), pp. 134–138.

|| Cf. Jordan, loc. cit., pp. 316–319; Dickson, *Linear Groups*, pp. 303–307; Coble, Johns Hopkins University Circular (1908) no. 7, pp. 80–88.

¶ Quarterly Journal of Mathematics, vol. 29 (1897), pp. 169–178, vol. 31 (1899), pp. 383–4; *Linear Groups*, pp. 89–109.

has an invariant linear complex. It is called *general* or *special* according as this function is left relatively (to within a factor in the field) or absolutely invariant. The first has an invariant subgroup of order $p^n - 1$ consisting of all the transformations which multiply each variable by the same mark of the field, and the second has for $p > 2$ an invariant subgroup of order 2 which contains a transformation changing the sign of each variable. The quotient group of the second with respect to its invariant subgroup of order 2, Dickson has denoted by the symbol, $A(2m, p^n)$, $2m$ representing the number of variables. For $p = 2$ the *special* group is itself represented by this symbol. These groups are simple except for $m = 1$, $p^n = 2, 3$ and $m = 2$, $p^n = 2$.

The object of this paper is the determination of the maximal subgroups of the $A(2m, p^n)$ for $m = 2$ and $p > 2$, although incidentally some other subgroups are noticed. For this purpose we consider the collineation group in a modular space* having an invariant linear complex. The group of all such collineations is $(1, 1)$ isomorphic with the quotient group of the *general* linear group with respect to its invariant subgroup of order $p^n - 1$. For $p > 2$ it contains an invariant subgroup of index 2, each transformation in which multiplies the linear function of the plücker line coördinates by a square in the field. This multiplier may be made unity by multiplying each coefficient of the collineation by the same properly chosen factor. This group is thus $(1, 1)$ isomorphic with the quotient group of the *special* linear group with respect to its invariant subgroup of order 2, i. e. with Dickson's $A(4, p^n)$. We shall depart slightly from Dickson's notation and shall represent the whole collineation group by $A_\nu(p^n)$ and its invariant subgroup by $A_1(p^n)$.

Although considerable work has been done on the determination of the subgroups, the problem seems to have been completely solved only for $n = 1$, $p = 3$.† For the general case Dickson has found a complete list of conjugate operators,‡ and has discussed a number of the subgroups.§ He has also shown that for $n = 1$, $p > 3$ there is no subgroup of index less than

$$p^3 + p^2 + p + 1.\|$$

The author has obtained a complete set of maximal subgroups in the general case. The possible subgroups whose orders are prime to the modulus p were

* Cf. Veblen and Bussey, these Transactions, vol. 7 (1906), pp. 241–259.

† Dickson, these Transactions, vol. 5 (1904), pp. 126–166. Jordan had previously shown that there was no subgroup of index less than 27 (loc. cit., pp. 319–329).

‡ These Transactions, vol. 2 (1901), pp. 103–138.

§ These Transactions, vol. 1 (1900), pp. 91–96; vol. 4 (1903), pp. 371–386; vol. 5 (1904), pp. 1–38; Quarterly Journal of Mathematics, vol. 32 (1900), pp. 42–63; Bulletin of the American Mathematical Society, vol. 10 (1903), pp. 178–184.

‖ These Transactions, vol. 6 (1905), pp. 48–57. Cf. also Jordan, loc. cit., p. 666, note E.

determined in a previous paper.* They were found to be the same as the ordinary finite groups which have an invariant linear complex.

With the exception of a few groups which exist only for particular moduli the subgroups whose orders are divisible by p are found to be of two general types. Those of the first type are the $A_1(p^k)$, if k is a factor of n, and the $A_\nu(p^k)$, if $2k$ is a factor of n. Those of the second type are analogous to the ordinary continuous groups which have an invariant linear complex. They have an invariant point and plane, a congruence (three types), a quadric (two types), or a twisted cubic.

As a check on our list of maximal subgroups we find that they agree with Dickson's† for the special case $n = 1$, $p = 3$. For this case he found them to be of indices 27, 36, 40, 40, 45. We also find that for $p^n > 3$ there is no subgroup of index less than $p^{3n} + p^{2n} + p^n + 1$, a result due to him‡ for $n = 1$.

2. General Properties of the Group

There are three types of transformations of period p in $A_1(p^n)$ if $p > 3$ and two types if $p = 3$. The first, which we shall refer to as *central elations*, leave fixed all the points in a plane and all the planes and lines through a point in that plane. All the lines in the plane of fixed points which pass through the point of fixed planes belong to the invariant complex.

The transformations of the second type leave invariant all the points on a line and every line of a flat pencil through each point of the axis. We shall refer to them as *skew elations*. The axis belongs to the complex as do also two of the above mentioned pencils. The coördinates of these pencils need not be in the $GF(p^n)$ however.

The transformations of the third type have a single invariant point, an invariant line through that point and an invariant plane through that line. All the lines in the fixed plane which pass through the fixed point belong to the complex. The path curves are conics in the fixed plane and twisted cubics outside of that plane. If $p = 3$ these transformations are of period 9.

There are two types of transformations of period 2 in $A_1(p^n)$.§ Both of them leave invariant all the points on each of two skew lines, but in the one case these lines are lines of the complex and in the other they are axes of a congruence contained by the complex. In the former case the coördinates of these lines are or are not in the $GF(p^n)$ according as p^n has the form $4l + 1$ or $4l - 1$. In the latter case their coördinates are always in the $GF(p^n)$.

* These Transactions, vol. 14 (1913), pp. 123–142. This paper will be referred to as Transactions I.

† Loc. cit.

‡ Loc. cit.

§ For a more general result see Dickson, *Linear Groups*, § 289; Annals of Mathematics, ser. 2, vol. 6 (1905), p. 148.

3. Groups Containing Central Elations

We proceed now to prove

THEOREM 1. *The only subgroups of $A_1(p^n)$ which contain central elations are:*

(1) *groups having an invariant point and plane with coördinates in the $GF(p^n)$;*

(2) *groups having an invariant line of the complex with coördinates in the $GF(p^n)$;*

(3) *groups having an invariant congruence hyperbolic with respect to the $GF(p^n)$;*

(4) *groups of the same type and order as $A_1(p^k)$, where k is a factor of n;*

(5) *groups of the same type and order as $A_\nu(p^k)$, where $2k$ is a factor of n.*

We consider a particular subgroup of $A_1(p^n)$ and suppose that the largest subgroup it contains which consists of central elations with common center and axial plane is of order p^k. We denote such a group by (E_1). If all the other centers of elations are in the axial plane of (E_1), either that plane or a line in that plane through the center of (E_1) must remain invariant. If this is not the case we may suppose the existence of a second group of elations (E_2) with center not in the axial plane of (E_1).

These two additive groups leave invariant in common the line joining their centers and the line of intersection of their axial planes. The group generated cannot by hypothesis contain an additive group of elations of higher order than p^k. Hence its order on the line joining the centers must be

$$\tfrac{1}{2}(p^k+1)p^k(p^k-1)$$

or $60\,(p^k=3)$.* The group itself is of twice this order, since it contains an invariant reflection whose axes are axes of the invariant congruence. We shall first assume either that $p^k>3$ or that if $p^k=3$ no group of order $2\cdot 60$ of this type is present. The exceptional case will be considered later.

In the general case, if the invariant complex be $p_{14}+p_{23}=0$, every group of the sort described may be shown to be conjugate under $A_\nu(p^n)$ with that generated by

$$(E_1):[x_1,x_2+\lambda x_3,x_3,x_4],\qquad (E_2):[x_1,x_2,\lambda x_2+x_3,x_4],$$

where λ takes all values in the $GF(p^k)$.

Either the invariant congruence of this group must remain fixed or else there will be elations with centers not on either of the two axes. We denote one such group of elations by (E_3). The line through its center which is in the congruence is a line of the complex and hence must be in the axial plane of (E_3).

* The binary modular groups for the general Galois field were first determined by E. H. Moore and Wiman. For a new determination of these groups and for references to the earlier work the reader may consult a paper by the author in these Transactions, vol. 12 (1911), pp. 207–211.

The center of (E_3) must be in one of the $p^k + 1$ planes conjugate with the axial planes of (E_1) and (E_2) under the group generated by them. For the group (E_3) cannot be commutative with both (E_1) and (E_2) and hence we may suppose that it is not commutative with (E_2). Hence (E_2) and (E_3) must generate a group of the same sort as the above and this will contain an invariant reflection. This reflection and the one commutative with (E_1) and (E_2) must generate a dihedral group of order $2p$ under which the reflections are conjugate. If the center of (E_3) does not lie in one of the $p^k + 1$ planes mentioned above, its center will be conjugate under this group with a point on the line joining the centers of (E_1) and (E_2), which is distinct from any of the $p^k + 1$ centers on that line. But this is impossible. Hence the center of (E_3) must be in one of the $p^k + 1$ planes and we may suppose that it is in the axial plane of (E_1).

Since the group generated by (E_1) and (E_2) is invariant under all transformations of the form

$$[\alpha x_1 + \beta x_4, x_2, x_3, \gamma x_1 + \delta x_4],$$

where $\alpha, \beta, \gamma, \delta$ are any marks of the $GF(p^n)$ such that $\alpha\delta - \beta\gamma = 1$, it follows that we may choose the center of (E_3) as $(1\ 1\ 0\ 0)$. In order that (E_2) and (E_3) shall generate a group of order $(p^k + 1)p^k(p^k - 1)$ the latter must be given by

$$[x_1 + \lambda(x_3 + x_4), x_2 + \lambda(x_3 + x_4), x_3, x_4].$$

We then find that the group of the points in the invariant plane, $x_4 = 0$, is of order $(p^k + 1)p^k(p^k - 1) \cdot p^{2k}$. There is in addition a group of elations of order p^k leaving fixed all the points in $x_4 = 0$. For in the group generated by (E_1) and (E_3) are contained all the transformations

$$[x_1 + \lambda(x_3 + x_4), x_2 - \lambda(x_3 - x_4), x_3, x_4].$$

The group conjugate with (E_3) under the reflection commutative with (E_1) and (E_2) is

$$[x_1 - \lambda(x_3 - x_4), x_2 + \lambda(x_3 - x_4), x_3, x_4].$$

As products of these we obtain a group of elations of order p^k having $(1\ 0\ 0\ 0)$ and $x_4 = 0$ for center and axial plane. Hence the group generated by (E_1), (E_2), and (E_3) is of order $p^{4k}(p^{2k} - 1)$.

If now $(1\ 0\ 0\ 0)$ and $x_4 = 0$ do not remain invariant there must be a group of elations (E_4) with center not in $x_4 = 0$. Its center must lie in at least one of each set of $p^k + 1$ planes similar to those considered above. Hence it must lie in one of the lines through $(1\ 0\ 0\ 0)$ in which $p^k + 1$ of these planes meet. There are p^{2k} of these lines and they are all conjugate under the group generated by (E_1), (E_2), and (E_3). Hence we may take

a center on $x_2 = 0$, $x_3 = 0$, and since that group is invariant under any elation with center $(1\ 0\ 0\ 0)$ and axial plane $x_4 = 0$, we may suppose it to be $(0\ 0\ 0\ 1)$. The group (E_4) must then be given by

$$[x_1, x_2, x_3, \lambda x_1 + x_4].$$

Under the group now generated there are $p^{3k} + p^{2k} + p^k + 1$ conjugate groups of elations, their centers being the points with coördinates in the $GF(p^k)$. The subgroup which leaves $(1\ 0\ 0\ 0)$ and $x_4 = 0$ invariant must be of order at least $1/2(p^k - 1) \cdot p^{4k}(p^{2k} - 1)$. For the group generated by (E_4) and the additive group having $(1\ 0\ 0\ 0)$ and $x_4 = 0$ for center and axial plane must contain the transformation $[\epsilon x_1, x_2, x_3, \epsilon^{-1} x_4]$, where ϵ is a primitive root in the $GF(p^k)$. In $x_4 = 0$ this is an homology of period $p^k - 1$, whereas in the group generated by (E_1), (E_2), and (E_3) there are no homologies in that plane of higher period than 2.

The order of the whole group generated must then be at least

$$\tfrac{1}{2}(p^{4k} - 1)p^{4k}(p^{2k} - 1).$$

But this is exactly the order of $A_1(p^k)$ and all four groups of elations are contained by that group.

If $A_1(p^k)$ is a subgroup of $A_1(p^n)$, k must be a factor of n, and if $A_\nu(p^k)$ is a subgroup of $A_1(p^n)$, $2k$ must be a factor of n. The groups of either of these types form a single conjugate set under $A_\nu(p^n)$. If n/k is odd the groups of the type of $A_1(p^k)$ also form a single conjugate set under $A_1(p^n)$. If n/k is even, the groups of the type of $A_1(p^k)$ or $A_\nu(p^k)$ each form two conjugate sets under $A_1(p^n)$.

The subgroup of all transformations in $A_1(p^n)$ which leave invariant a point and a plane through it whose coördinates are in the $GF(p^n)$ is of order $1/2\, p^{4n}(p^{2n} - 1)(p^n - 1)$. The subgroup of all such transformations which leave invariant a line of the complex with coördinates in the $GF(p^n)$ is of the same order. The subgroup of all such transformations which leave invariant a congruence which is hyperbolic with respect to the $GF(p^n)$ is of order $p^{2n}(p^{2n} - 1)^2$. The groups of each of these three types form a single conjugate set under $A_1(p^n)$.

We still have to consider the case where $p^k = 3$ and the group generated by two elations is of order $2 \cdot 60$. Such a group is generated by

$$(E_1) : [x_1, x_2 + x_3, x_3, x_4]; \qquad (E_2) : [x_1, x_2, ix_2 + x_3, x_4],$$

where $i^2 = -1$. It may be proved just as in the general case that the centers of all elations which can exist must be in one of the axial planes of this group. We consider therefore a third elation, whose center and axial plane we may take to be $(1\ 1\ 0\ 0)$ and $x_3 + x_4 = 0$. It must generate with (E_2) a group

of order $2\cdot 12$ or $2\cdot 60$. The reflection invariant under this group is

$$(R):[-x_1+2x_2,\, x_2,\, x_3+2x_4,\, -x_4].$$

The group generated by (E_1) and (E_2) contains the transformation

$$(S):[x_1,\, ix_2,\, -ix_3,\, x_4].$$

It is readily found that the group generated by the elations which are conjugate with (E_1) under the group generated by (R) and (S) contains a group of elations of order 3^2 having $(1\ 0\ 0\ 0)$ and $x_4 = 0$ for center and axial plane. But this is contrary to our assumption that $p^k = 3$.

4. Groups Containing Skew Elations but no Central Elations

We establish the following theorems:

Theorem 2. *Any subgroup of $A_1(p^n)$ which does not contain central elations cannot contain two skew elations with different axes if the axis of either is a fixed line of the other.*

If E_1 and E_2 are two skew elations with the same invariant complex and such that each leaves fixed the axis of the other, then it is easy to prove that $E_1^{-1}E_2^{-1}E_1E_2$ is a central elation. If E_1 leaves fixed the axis of E_2, but E_2 does not leave fixed the axis of E_1, then $E_2^{-1}E_1E_2$ and E_1 will have different axes but each will leave fixed the axis of the other. Hence a central elation will appear in this case also.

Theorem 3. *In any subgroup of $A_1(p^n)$ which does not contain central elations any two skew elations with different axes must leave invariant in common either all the lines of a regulus or the two axes of a congruence.*

Two skew elations with different axes must leave invariant the two axes of a congruence if their axes intersect, one of which is in the plane of their axes and the other of which passes through the point of intersection of their axes. If their axes do not intersect they must leave invariant at least one line in common, since their product can leave fixed no points which are not on a line left invariant by both. If they leave invariant only one line in common there must be in all cases among the products of their powers transformations with two fixed points on that line. Some power of such a transformation must be a skew elation having the invariant line for axis. But this is impossible, since its axis would be left invariant by the skew elations which generate the group. Hence the two generating skew elations must leave invariant either the two axes of a congruence or all the lines of a regulus.

Theorem 4. *Any subgroup of $A_1(p^n)$ which contains skew elations, but no central elations, and which leaves invariant the two axes of a congruence, must be $(1,1)$ or $(2,1)$ isomorphic with the groups of the points on the two axes, with the exception of three groups of order $2\cdot 4\cdot 3$, $2\cdot 4\cdot 12$, $2\cdot 4\cdot 24$, which exist for $p = 3$.*

We notice that if there are any transformations which leave invariant all the points on one axis, the group generated by them must form an invariant subgroup. If there are any such transformations which are of period p on the second axis either they are central elations or some of their powers are central elations. The group on an axis cannot contain an invariant subgroup of order prime to p except for $p = 3$, in which case it must be of order 12 or 24. This readily leads to the exceptional groups noted in the theorem.

THEOREM 5. *Any subgroup of $A_1(p^n)$, which contains skew elations but no central elations, and which leaves invariant the two axes of a congruence, must contain subgroups of order $1/2\, p(p^2 - 1)$ which leave invariant all the lines of a regulus, provided that no point on either axis remains fixed.*

Any group of the sort described in the theorem must contain one or more subgroups which are of order $1/2p(p^2 - 1)$ on each axis of the congruence. In such a subgroup there will be skew elations whose product is of period 2 on each axis. But any such transformation in $A_1(p^n)$ may easily be shown to be a skew perspectivity leaving fixed all the points on each of two lines in the congruence. Since the product of the two skew elations can leave invariant no point which is not on a line left invariant by both, it follows that they must leave invariant in common all the lines of a regulus.

THEOREM 6. *Any subgroup of $A_1(p^n)$ containing an additive group of skew elations of higher order than 3, but no central elations, must leave invariant a point and plane, a congruence, or a quadric.*

An additive group of skew elations with a common axis will leave invariant either a pencil of lines (i. e., all the lines of the pencil) through each point of the axis, two pencils neither of which belongs to the complex, a single pencil belonging to the complex, or no pencils whatever. We denote the largest subgroup of the whole additive group which leaves invariant a pencil through each point of the axis by $H^{(R)}$ and suppose that its order is p^k.

If no point or line remains invariant under the group, $H^{(R)}$ must be contained either by a group leaving invariant all the lines of a regulus or by a group leaving invariant the two axes of a congruence. The latter must however contain subgroups of the former type and some of these will contain $H^{(R)}$. The largest group containing $H^{(R)}$ which leaves invariant all the lines of a regulus must be of order $1/2p^k(p^{2k} - 1)$, $p^k(p^{2k} - 1)$, or 60 $(p^k = 3)$. We denote one such group by $G^{(R)}$.

If there is another skew elation having the same axis as $H^{(R)}$, it must generate with $G^{(R)}$ a group having an invariant congruence. This congruence cannot be parabolic, since in that case there would be skew elations present having but one invariant line in common. The group generated must be of order $1/2p^m(p^{2m} - 1)$ or $p^m(p^{2m} - 1)$ on each axis of the congruence, m being a multiple of k. We denote this group by $G^{(C)}$ and its additive subgroup of order p^m, which is commutative with $H^{(R)}$, by $H^{(C)}$.

The group $G^{(C)}$ must contain subgroups of order $1/2(p^m - 1)$ or $p^m - 1$ on each axis of the congruence, which permute the operators of $H^{(C)}$ among themselves. Some of these will contain cyclic subgroups of order $1/2(p^k - 1)$ or $p^k - 1$ which belong to $G^{(R)}$. These are skew perspectivities which leave invariant all the points on each of two axes of elations in $G^{(R)}$. A group of order $1/2(p^m - 1)$ or $p^m - 1$ which leaves two such axes invariant must be of period on each of them greater than or equal to $p^k + 1$.* The period will clearly be greater than this unless $m = 2k$, in which case it may conceivably be $1/2(p^k + 1)$. In this case however $G^{(R)}$ would contain a skew perspectivity of order $p^k - 1$ and hence would be of order $p^k(p^{2k} - 1)$. It therefore contains also skew perspectivities of order $p^k + 1$. Under $G^{(C)}$ one of these would be commutative with the one of order $p^k - 1$. Hence the period on the two axes of elations could not in this case be less than $p^k + 1$.

If now there are other skew elations which have the same axis as $H^{(C)}$, there must be other groups of the same sort as $G^{(C)}$ containing $G^{(R)}$. There must then be more than one cyclic group leaving fixed two particular axes of elations in $G^{(R)}$ and the period of neither on these axes can be less than $p^k + 1$. The group generated by these cyclic groups must in all cases contain transformations which are of period p on each axis. Either such a transformation is a skew elation or else a power of it is a skew elation. But this is impossible since it would leave fixed the two axes of elations in $G^{(R)}$.

Hence in all cases the whole additive group of skew elations having a common axis must leave invariant either a pencil through each point of the axis or else two pencils. We will show that in the latter case a congruence (the invariant congruence of $G^{(C)}$) must remain invariant. Suppose this is not the case. Then there must be an elation with axis not in the congruence. It must generate with $H^{(R)}$ a group having either an invariant congruence or else an invariant regulus. In the former case there will be in the group generated subgroups of the latter type and not more than one of these can be in $G^{(C)}$. Any one of these must be of order $1/2p^k(p^{2k} - 1)$, and one of them must generate with $H^{(C)}$ a group of order $1/2p^m(p^{2m} - 1)$. The axes of the invariant congruence of this group will meet the axis of $H^{(C)}$ in the same points as the axes of the invariant congruence of $G^{(C)}$, i. e., the two points through which pass the two invariant pencils.

Hence $H^{(C)}$ will be left invariant by more than one metacyclic group, for the whole metacyclic group in a $G^{(C)}$ which leaves $H^{(C)}$ invariant can leave invariant but one congruence, as may be readily shown. The group generated by two transformations which permute the operators of $H^{(C)}$ in the same way, but which have different invariant congruences, will contain skew elations

* $G^{(R)}$ cannot now be of order 60, since $G^{(C)}$ would in that case contain a subgroup of order 360 permuting the same ten axes of elations and hence leaving the same regulus invariant.

with the same axis as $H^{(C)}$, which permute the lines of the two pencils. But this is impossible, as shown above.

We suppose finally that the entire additive group of skew elations is identical with $H^{(R)}$, i. e., leaves invariant a pencil of lines through each point of the axis. We will show that if $p^k > 3$ a regulus (the invariant regulus of $G^{(R)}$) must remain invariant. For if there be a skew elation not in $G^{(R)}$ it must generate with $H^{(R)}$ either a $G^{(C)}$ or another $G^{(R)}$. In the latter case the two $G^{(R)}$ must generate a $G^{(C)}$. If $p^k > 3$ every $G^{(C)}$ contains an additive group of order p^m, where m is a multiple of k. If $p^k = 3$ there are exceptional $G^{(C)}$ of order 60, $2\cdot 60$, $2\cdot 4\cdot 3$, $2\cdot 4\cdot 12$, $2\cdot 4\cdot 24$. That of order $2\cdot 4\cdot 3$ may be generated by two skew elations whose axes intersect and contains no $G^{(R)}$.

It is perhaps worth while here to discuss briefly the groups which have an invariant congruence or an invariant regulus. If the congruence is parabolic, or if it is hyperbolic with respect to the $GF(p^n)$, such groups are subgroups of subgroups which contain central elations. If the congruence is elliptic with respect to the $GF(p^n)$, there are no central elations in $A_1(p^n)$ which leave it invariant. The order of the largest subgroup of $A_1(p^n)$ which leaves one of these congruences invariant is $p^{2n}(p^{4n}-1)$. The totality of all such subgroups is a single conjugate set under $A_1(p^n)$.

There are two types of reguli, all the lines of which belong to the complex. In the one case there are two real lines of the conjugate regulus which belong to the complex and in the other these two lines are conjugate imaginary with respect to the $GF(p^n)$. The largest subgroup of $A_1(p^n)$ which leaves a regulus of the former type invariant is $(p^n+1)p^n(p^n-1)^2$. In the other case the largest subgroup is of order $(p^n+1)^2 p^n(p^n-1)$. The groups of each sort form a single conjugate set under $A_1(p^n)$. Neither of them is a maximal subgroup for $p^n = 3$. In that case the former leaves invariant both a hyperbolic and an elliptic congruence and the latter is a subgroup of a group of order 960 to be discussed later.

THEOREM 7. *If $p = 3$ every subgroup of $A_1(p^n)$ which contains skew elations but no central elations must leave invariant a point and plane, a congruence, or a regulus with the exception of two groups of order* $16\cdot 60$ *and* $16\cdot 120$. *The latter is a subgroup of $A_1(3^n)$ only if n is even.*

We first consider those subgroups which contain $G^{(C)}$ of order $2\cdot 4\cdot 12$ or $2\cdot 4\cdot 24$. Each of these contains C_4 which are the identity on one axis of the congruence and are of period 2 on the other. We will show that any reflection which is conjugate with the invariant reflection of this subgroup must be commutative with it, i. e., must interchange its axes.

Suppose first that there are two such reflections whose axes intersect. Their product will then be a skew elation. Consider a C_4 of the sort described above which is commutative with one of these reflections. It will either

transform the skew elation into another one having the same axis or will transform it into another one whose axis is left fixed by the first. In the first case there would be an additive group of skew elations present of higher order than 3. The second case is impossible by a previous theorem.

Suppose now that there are two such reflections whose axes do not intersect. If they do not interchange each other's axes their product cannot be of period less than 4. They must leave invariant in common all the lines of a regulus, since their axes are axes of congruences contained by the complex. A C_4 having one of them for its square must generate with the other a group which leaves two of the lines of this regulus invariant. If no additive group of higher order than 3 is present the group on each of these lines must be of order 24. But there must then be present reflections of the type considered whose products are skew elations.

Hence in any case two reflections which are squares of such C_4 must interchange each other's axes. The only groups which can contain $G^{(C)}$ of order $2\cdot 4\cdot 12$ or $2\cdot 4\cdot 24$ may then be shown* to be two groups of order $16\cdot 60$ and $16\cdot 120$ having an invariant G_{16}.

We suppose finally that the only $G^{(C)}$ which can be present are of order 60 or $2\cdot 60$. We may then prove much as in Transactions I (under Theorem 13) that there must be commutative C_3. The proof that the R, A, and B of that discussion must leave invariant two lines in common is different, but may be made easily geometrically. But we have considered here the case where commutative C_3 are present.

5. Groups which Contain no Central Elations and no Skew Elations, but which Contain Skew Perspectivities of Higher Period than 2

The determination of the groups which contain no central elations or skew elations may be made much as in Transactions I, pp. 125–134. This is especially the case when the groups contain skew perspectivities of higher period than 2. Most of the general properties of the groups as stated there, p. 125, still hold if transformations of period p with a single fixed point are present. There are however the exceptions that if $p = 5$ two skew perspectivities of period 3 which generate a group of order 60, in which the operators of period 5 have each a single fixed point, leave invariant but one line in common. Also two reflections whose product is of period p can have but one fixed line in common. Much of the discussion is simplified on account of the limitation of the present discussion to those groups which have an invariant complex. Aside from this the differences are very slight. The results may be summarized as follows:

Theorem 8. *Any subgroup of $A_1(p^n)$ which does not contain central*

* Cf. Transactions I, p. 128.

elations or skew elations, but which contains skew perspectivities of higher period than 2, *has either an invariant regulus or congruence or else is a group of one of the following orders*: $16\cdot 120$, $16\cdot 60$, 360, 720, 2520, 5040. *Those of order* $16\cdot 60$ *and* 360 *exist in every* $A_1(p^n)$, *that of order* $16\cdot 120$ *only for* $p^n = 8h \pm 1$, *that of order* 720 *only for* $p^n = 12f \pm 1$, *that of order* 2520 *only for* $p = 7$, *and that of order* 5040 *only for* $p = 7$ *and* n *even.*

The groups occurring here which have invariant reguli or congruences are subgroups of those previously discussed. In the case of the groups of order $16\cdot 120$ and $16\cdot 60$, the axes of the five reflections contained by the invariant G_{16} which are axes of congruences must have coördinates in the $GF(p^n)$. The axes of the other ten are lines of the complex and must be real or conjugate imaginary with respect to the $GF(p^n)$ according as p^n has the form $4l+1$ or $4l-1$. These ten reflections are in the $G_{16\cdot 120}$ the squares of skew perspectivities of period 4, which exist in $A_1(p^n)$ only if $p^n = 8h \pm 1$.

A $G_{16\cdot 120}$ is generated by

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & d & 0 \\ b & 0 & 0 & d \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

where $a + d = 2$, $a^2 + 2b^2 + d^2 = 0$. These groups always exist in $A_\nu(p^n)$ and form a single conjugate set under that group, and (if they exist) two conjugate sets under $A_1(p^n)$. The subgroups of order $16\cdot 60$ always exist in A_1, and if $p^n = 8h \pm 3$, they form a single conjugate set under A_1.

To determine when the other groups are subgroups of A_1 we distinguish two cases according as the axes of the skew perspectivities of period 3 are real or conjugate imaginary with respect to the $GF(p^n)$, i. e., according as $p^n = 3k+1$ or $3k-1$. The G_{360} may be generated by three such C_3 of which two are commutative and the third generates a G_{12} with one and a G_{60} with the other. If their axes are real we find that (if we consider conjugacy under A_ν) they may be taken to be

$$(1\,0\,0\,0)(0\,1\,0\,0), \qquad (0\,0\,1\,0)(0\,0\,0\,1);$$

$$(1\,0\,0\,0)(0\,0\,1\,0), \qquad (0\,1\,0\,0)(0\,0\,0\,1);$$

$$(0\,1\,1\,1)(1\,0\,1\,-1), \qquad (0\,-2\,1\,1)(-2\,0\,1\,-1).$$

A G_{720} containing this G_{360} must also contain the reflection whose axes are $(1\,i\,0\,0)(0\,0\,1\,i)$, $(1\,-i\,0\,0)(0\,0\,1\,-i)$, where $i^2+1=0$. These are axes of a congruence and hence the G_{720} are subgroups of A_1 only if they are real, i. e., if $p^n = 4l+1$, and hence $p^n = 12f+1$.

If the axes of the C_3 are conjugate imaginary with respect to the field, i. e.,

if $p^n = 3k - 1$, we may take the G_{360} to be generated by the three C_3, C_3', C_3'', C_3''' where one axis of C_3' is

$$(1\ 0\ 0\ \omega)(0\ 1\ \omega\ 0),$$

one axis of C_3'' is

$$(0\ 1\ \omega\ 0)(1\ 0\ 0\ \omega^2)$$

and one axis of C_3''' is

$$(1 - \lambda\alpha, -\alpha, -\alpha\omega^2, \omega^2 - \lambda\alpha\omega)(1, \alpha + \lambda, \alpha\omega^2 + \lambda\omega, \omega^2)$$

where $\omega^2 + \omega + 1 = 0$, $\alpha\alpha' = 1$, $\lambda\lambda' = -1$, α' and λ' being the conjugates of α and λ. The G_{720} must also contain the reflection whose axes are

$$(i\alpha, 1, \omega, i\alpha\omega)(1, i\alpha, i\alpha\omega^2, \omega^2)$$

and

$$(-i\alpha, 1, \omega, -i\alpha\omega)(1, -i\alpha, -i\alpha\omega^2, \omega^2).$$

These must be real and this is the case if i does not exist in the field, i. e., if $p^n = 4l - 1$ and hence also $p^n = 12f - 1$.

Hence A_1 has primitive subgroups of order 360 in all cases $(p > 3)$, whereas it has primitive subgroups of order 720 only if $p^n = 12f \pm 1$. These groups form each a single conjugate set under A_ν, whereas the G_{720} (if they exist) form two conjugate sets under A_1, and the G_{360} (if the G_{720} do not exist) form a single conjugate set under A_1.

The G_{2520} has an invariant complex only if $p = 7$ and the G_{5040} exists only in this case. The axes of the C_3 are then real and we may take the first choice of coördinates for a subgroup of order 360. The G_{2520} must also contain the C_3 whose axes are $(1\ 2\ 0\ 0)(0\ 0\ 3\ 1)$ and $(1\ 3\ 0\ 0)(0\ 0\ 2\ 1)$. The G_{5040} is generated by the G_{720} and this C_3. It appears in A_1 only if n is even. In this case A_1 contains two conjugate sets of G_{5040}, whereas if n is odd it contains a single conjugate set of G_{2520}.

6. Groups which Contain no Central Elations, no Skew Elations, and no Skew Perspectivities of Higher Period than 2

The results to be obtained in this section may be summarized in the following theorem.

THEOREM 9. *The only subgroups of $A_1(p^n)$ which contain no central elations, no skew elations, and no skew perspectivities of higher period than 2, are subgroups either of groups previously considered or of a group of order $1/2p^n(p^{2n} - 1)$ having an invariant twisted cubic.*

The groups of this type whose orders are prime to p and which do not have an invariant congruence or regulus are of order $16\cdot5$, $16\cdot10$, $16\cdot20$, 60, 120. These are subgroups respectively of groups of order $16\cdot60$, $16\cdot60$, $16\cdot120$, 360, 720. It is only necessary to consider the groups which contain trans-

formations of period p. We may also in consequence of the discussion in Transactions I, p. 132, assume that these groups do not contain reflections whose axes are axes of congruences. We may also suppose that no transformation other than the identity can leave fixed the axes of more than one reflection. It may also easily be shown that two cyclic groups can have no transformations (other than the identity) in common unless both are contained by a larger cyclic group.

We consider an additive group of order p^m and suppose a transformation in this additive group to be

$$S : [x_1 + ax_2 + bx_3 + cx_4, x_2 + dx_3 + ex_4, x_3 + fx_4, x_4],$$

where $a \neq 0$, $d \neq 0$, $f \neq 0$. If $p_{14} + p_{23} = 0$ is invariant we must have $a + f = 0$, $b - e + df = 0$. If the additive group is invariant under a metacyclic group there must be present transformations such as

$$T : [\alpha x_1, \beta x_2, \beta^{-1} x_3, \alpha^{-1} x_4],$$

the period of which is a divisor of $p^m - 1$. The transformation $T^{-1} ST$ must be commutative with S. For otherwise if we denote $T^{-1} ST$ by S_1, then $S^{-1} S_1^{-1} SS_1$ would be either a central elation or a skew elation. This leads to the conditions $\alpha = \beta^3$, $b + e = 0$.

We inquire next under what conditions the cyclic group of order d_1 generated by T can be invariant under a group of order $4d_1$ which contains transformations making a cyclic permutation of period 4 on the vertices of its invariant tetrahedron. This leads to $\beta^{10} = 1$ and hence $d_1 = 5$.

If now we denote the order of the whole group by Ω, it is clear that there must be in the group $(p^m - 1)\Omega/d_1 p^m$ additive transformations. If $d_1 > 3$ any maximal cyclic group of order d_1 can be invariant only under groups of order d_1, $2d_1$, or $4d_1$, and in the last case we must have $d_1 = 5$. In the same way any other maximal cyclic group of order d_i can be invariant only under groups of order d_i, $2d_i$, or $4d_i$, and in the last case it may easily be shown that d_i can be divisible only by primes of the form $4l + 1$. There must then be $(d_i - 1)\Omega/f_i d_i$ transformations in each conjugate set of cyclic groups, where $f_i = 1, 2, 4$.

If we attempt to enumerate the transformations which the group must contain we are led to the diophantine equation

$$\Omega = 1 + (p^m - 1)\Omega/d_1 p^m + \sum_{i=1}^{r} (d_i - 1)\Omega/f_i d_i.$$

This equation is slightly different if the metacyclic group is of order $2p^m$ or $3p^m$, for transformations of period 2 or 3 which leave the additive group invariant may be contained by cyclic groups of higher order which do not

leave it invariant. Hence if the first denominator were $2p^m$ or $3p^m$ we could conclude only that d_1 is divisible by 2 or 3 respectively. The first would give an equation of essentially the same type as that considered in Transactions I under Theorem 14 and no solutions exist which correspond to groups. The second leads to either $r = 2, f_1 = f_2 = 2$ or $r = 2, f_1 = 2, f_2 = 4$.

Since p^m must have the form $6f + 1$, and since d_1 and d_2 can have no odd factor in common, it follows that in the first of these two cases $d_2 = 2$. If $d_1 = 6$, we obtain $\Omega = 3p^m$, which is impossible. Hence $d_1 \geqq 9$. This leads to $p^m < 12$ and hence $p^m = 7$. We then have $d_1 < 14$ and consequently $d_1 = 9, 12$. For $d_1 = 9$ there is no solution, whereas for $d_1 = 12$ we obtain $\Omega = 168$. This solution however can correspond to no group, since there would be seven cyclic groups of order 12 and hence the invariant reflection in any one of these groups would have to interchange the axes of each of the other six reflections of the same sort. This is easily seen to be impossible.

In the second case, since $f_2 = 4$, it follows that d_1 must be divisible by 12. Since Ω must be the least common multiple of $3p^m$, $2d_1$, and $4d_2$, we have $\Omega = 2p^m d_1 d_2$. This leads to

$$4d_1 d_2 + 6p^m d_2 + 3p^m d_1 = 6 + p^m d_1 d_2.$$

If $4d_1 d_2$ is the largest term on the left, it follows that $p^m < 12$ and hence $p^m = 7$. We then conclude that either $d_1 < 28$ or $d_2 < 14$, and hence that $d_1 = 12, 24$ or $d_2 = 5, 13$. None of these leads to a solution. The same result is obtained if $6p^m d_2$ or $3p^m d_1$ is supposed the largest term on the left.

We now take up the discussion of the general case. If no f_i has the value 4, the equation may be solved exactly as in a previous paper by the author.* The only solutions which need concern us here are:

$$r = 2, \quad f_1 = f_2 = 2, \quad d_1 = \tfrac{1}{2}(p^m - 1), \quad d_2 = \tfrac{1}{2}(p^m + 1),$$
$$\Omega = \tfrac{1}{2}p^m(p^{2m} - 1);$$
$$r = 2, \quad f_1 = f_2 = 2, \quad d_1 = p^m - 1, \quad d_2 = p^m + 1,$$
$$\Omega = p^m(p^{2m} - 1).$$

It will be shown later that there are groups which correspond to these two solutions.

We now suppose that at least one f_i has the value 4. If $f_1 = 4$, then (as shown above) $d_1 = 5$. Then one of the d's, say d_2, must be divisible by 4 and hence $f_2 = 1, 2$. It is evident from the equation that $f_2 = 2$ is the only possibility, since otherwise the sum of the coefficients of Ω on the right would exceed unity. It may then be seen from the equation that $r = 3$,

* These Transactions, vol. 12 (1911), p. 210.

$f_3 = 4$. Hence d_3 can be divisible only by primes of the form $4l + 1$, and since $d_1 = 5$, it follows that $d_3 \nless 13$. Since $p^m \nless 11$, we find that $d_2 = 4$. The order, Ω, of the group must be the least common multiple of the denominators, $5p^m$, 20, 8, $4d_3$, and hence must be $40d_3\, p^m$. This leads to

$$10p^m + 8d_3 = 1 + d_3\, p^m,$$

which may readily be shown to be impossible.

We now suppose that $f_1 = 1, 2$. If now any other d_i, say d_2, is even, we must have $f_2 = 1, 2$. This leads to $r = 2$, so that no f_i can be 4. Hence d_1 must be divisible by 4 and we then find $r = 3, f_1 = 2, f_2 = f_3 = 4$.

We observe that in this case the order of the group must be given by each of the two expressions, $\Omega = d_1\, p^m(1 + kp^m) = 2d_1\, d_2\, d_3\, p^m$. We may also obtain another expression for Ω by multiplying the number of reflections by the order of the group leaving a particular reflection invariant. There are d_1 reflections commutative with a particular reflection and $d_1 - 2$ other reflections commutative with each of these. The given reflection must be contained by two metacyclic groups of order $d_1\, p^m$, each of which contains $p^m - 1$ other reflections. It must also be contained by $d_1/2$ groups of order $4d_2$ and $d_1/2$ groups of order $4d_3$ which contain respectively $d_2 - 1$ and $d_3 - 1$ other reflections. Hence we must have

$$\Omega = 2d_1\,[1 + d_1(d_1 - 1) + 2(p^m - 1) + \tfrac{1}{2}d_1(d_2 - 1) + \tfrac{1}{2}d_1(d_3 - 1)].$$

Comparing this with one of the orders given above we find

$$2d_1(d_1 - 2) + (4p^m - 2) + d_1\, d_2 + d_1\, d_3 = p^m(1 + kp^m).$$

Also $2d_2\, d_3 = 1 + kp^m$. Since $d_1 < p^m$ and since we may suppose that $d_2 \nless 5$, $d_3 \nless 13$, it follows that

$$(4p^m - 2) + d_1\, d_2 + d_1\, d_3 < \tfrac{22}{130}\, p^m(1 + kp^m).$$

Hence

$$2d_1(d_1 - 2) > \tfrac{108}{130}\, p^m(1 + kp^m).$$

This leads to $k = 1, 2$; $d_1 = p^m - 1$. We then find from the equation given above that $d_2 + d_3 - 4$ must be divisible by p^m. Hence we may suppose that $d_3 > p^m/2$. But this is impossible since $2d_2\, d_3 \ngtr 1 + 2p^m$. Hence no solution exists if any f_i has the value 4.

We proceed to construct the groups of order $1/2p^m(p^{2m} - 1)$ and $p^m(p^{2m} - 1)$. We may suppose that the former contains the transformations S and T written above, where $a + f = 0$, $b - e + df = 0$, $b + e = 0$, $\alpha = \beta^3$. We then obtain, by transforming S by the powers of T, the transformations

$$[x_1 + a\gamma x_2 + b\gamma^2 x_3 + c\gamma^3 x_4,\ x_2 + d\gamma x_3 + e\gamma^2 x_4,\ x_3 + f\gamma x_4,\ x_4],$$

where γ denotes any power of β^2. There are $1/2(p^m - 1)$ such transformations and they must generate the group of order p^m. There must be among these transformations some whose product is also in the set unless $p = 5$, $m = 1$. This leads to the condition, $3c = -ab$. [This may also be proved in the special case by a consideration of the other additive group left invariant by T.]

If for the present we consider conjugacy only under $A_\nu(p^n)$, we may choose a and d to be arbitrary marks in the $GF(p^n)$ different from 0. We suppose therefore that $a = d = 1$. The additive group then contains the transformations:

$$\left[x_1 + tx_2 + \frac{t^2}{2}x_3 - \frac{t^3}{6}x_4,\ x_2 + tx_3 - \frac{t^2}{2}x_4,\ x_3 - tx_4,\ x_4\right],$$

where t takes all values in the $GF(p^m)$.

The cyclic group generated by T must also leave another additive group invariant. Since the fixed points of the $p^m + 1$ additive groups must be $(1\ 0\ 0\ 0)$, $(0\ 0\ 0\ 1)$, and the $p^m - 1$ points $(-t^3/6,\ -t^2/2,\ -t,\ 1)$, we find this additive group to be

$$[x_1,\ 3tx_1 + x_2,\ 6t^2x_1 + 4tx_2 + x_3,\ -6t^3x_1 - 6t^2x_2 - 3tx_3 + x_4].$$

The largest group of this type which belongs to $A_1(p^n)$ is that of order $1/2p^n(p^{2n} - 1)$ obtained for $m = n$. They form a single conjugate set under $A_1(p^n)$ and may be said to correspond to the ordinary three-parameter continuous groups which have invariant twisted cubics.

7. Summary

We are now in a position to summarize our results. In the statement which follows we have, in a few places, denoted the order of $A_1(p^n)$ by $\Omega(p^n)$.

Theorem 10. *The maximal subgroups of $A_1(p^n)$ are as follows:*

(1) *groups of index $\Omega(p^n)/\Omega(p^k)$, each of which is conjugate under $A_1(p^n)$ with $A_1(p^k)$, where n/k is an odd prime;*

(2) *groups of index $\Omega(p^n)/2\Omega(p^k)$, each of which is conjugate under $A_\nu(p^n)$ with $A_\nu(p^k)$, where n is even and $n/k = 2$;*

(3) *a single conjugate set of groups of index $p^{3n} + p^{2n} + p^n + 1$, each of which has an invariant point and plane;*

(4) *a single conjugate set of groups of index $p^{3n} + p^{2n} + p^n + 1$, each of which has an invariant parabolic congruence;*

(5) *a single conjugate set of groups of index $p^{2n}(p^{2n} + 1)/2$, each of which has an invariant hyperbolic congruence;*

(6) *a single conjugate set of groups of index $p^{2n}(p^{2n} - 1)/2$, each of which has an invariant elliptic congruence;*

(7) *a single conjugate set of groups of index* $p^{3n}(p^{2n}+1)(p^n+1)/2\,[p^n>3]$, *each of which has an invariant quadric such that all the lines of one regulus and two real lines of the other belong to the complex*;

(8) *a single conjugate set of groups of index* $p^{3n}(p^{2n}+1)(p^n-1)/2\,[p^n>3]$, *each of which has an invariant quadric such that all the lines of one regulus and two imaginary lines of the other belong to the complex*;

(9) *a single conjugate set of groups of index* $p^{3n}(p^{4n}-1)\,[p>3,\ p^n>7]$, *each of which has an invariant twisted cubic*;

(10) *two conjugate sets of groups of index* $\Omega(p^n)/16\cdot 120$ *if* $n=1, p=8h\pm 1$, *and a single conjugate set of groups of index* $\Omega(p^n)/16\cdot 60$ *if* $n=1$, $p=3$ *or* $8h\pm 3$.

(11) *two conjugate sets of groups of index* $\Omega(p^n)/720$ *if* $n=1, p=12f\pm 1$, *a single conjugate set of groups of index* $\Omega(p^n)/360$ *if* $n=1, p=5$ *or* $12f\pm 5$, $p\neq 7$, *and a single conjugate set of groups of index* $\Omega(p^n)/2520$ *if* $n=1$, $p=7$.

The G_{5040} is not a maximal subgroup of $A_1(7^n)$ for any n, but is a maximal subgroup of $A_\nu(7)$. Similarly for $p=12f\pm 5$, $p\neq 7$, the G_{720} is a maximal subgroup of $A_\nu(p)$, and if $p=3$ or $8h\pm 3$ the $G_{16\cdot 120}$ is a maximal subgroup of $A_\nu(p)$. For $p^n=5, 7$ the groups of index $p^{3n}(p^{4n}-1)$ are subgroups respectively of the G_{360} and the G_{2520}.

We observe that for $p^n=3$ the maximal subgroups are of index 40, 40, 45, 36, 27, which agrees with Dickson's result (loc. cit.). Also we find that the smallest index of any subgroup is $p^{3n}+p^{2n}+p^n+1$ except for $p^n=3$, in which case it is 27. Hence we have

THEOREM 11. *The smallest number of letters on which* $A_1(p^n)$ $[p$ *odd*$]$ *may be represented as a permutation group is* $p^{3n}+p^{2n}+p^n+1$ *except for* $p^n=3$, *in which case it is* 27.

This result is due to Jordan for $n=1$, $p=3$ and to Dickson* for $n=1$, $p>3$.

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* Loc. cit.

TRANSFORMATIONS OF CONJUGATE SYSTEMS WITH EQUAL POINT INVARIANTS*

BY

LUTHER PFAHLER EISENHART

In a recent memoir† we have established a transformation of a surface S into a surface S_1 such that the lines joining corresponding points form a congruence whose developables meet S and S_1 in conjugate systems with equal tangential invariants; we have called it a *transformation* Ω. In the present paper we consider a transformation of a surface S into a surface S_1 such that the developables of the congruence of joins of corresponding points meet S and S_1 in conjugate systems with equal point invariants. In both cases the fundamental operation is the transformation of Moutard‡ of an equation of the Laplace type with equal invariants, just as this transformation lies at the base of the various types of transformations in which the two surfaces are the focal sheets of a W-congruence. We identify the transformation under discussion with one which is involved in part in a theorem established from a different point of view by Koenigs, who at the time was considering an entirely different problem.§ We make a detailed study of this transformation,‖ because it seems to be the foundation of certain important transformations which have been established in other ways and it points the way to other important transformations. The transformations K admit a theorem of permutability just as the other two general types of transformations referred to above. So far as we know every class of transformations of surfaces of a particular kind into surfaces of the same kind is reducible to one of these three types. Consequently we not only know the basis for the existence of a theorem of permutability in the known particular transformations, but in subsequent investigations we shall have a more immediate method of establishing such a theorem for transformations under consideration. An example of this is given at the close of the memoir.

* Presented to the Society, April 25, 1914.

† *Conjugate systems with equal tangential invariants and the transformation of Moutard*, Rendiconti del circolo matematico di Palermo, vol. 38.

‡ Journal de l'école polytechnique, cahier 45 (1878), p. 1; cf. also, Darboux, *Leçons*, vol. 2, p. 145.

§ Comptes Rendus, vol. 113 (1891), p. 1022.

‖ Referred to in the following pages as a transformation K.

We show that transformations K are commutative with certain *radial transformations*, under which the points of a surface are transformed along lines which are concurrent. These transformations enable us to deduce the general theorem of permutability for transformations K from a special case in which the four surfaces consist of two pairs of associate surfaces, that is, two surfaces corresponding with parallelism of tangent planes and such that to asymptotic lines on either correspond a conjugate system on the other.

Although the treatment of §§ 1–5 is stated in terms of ordinary space, the results are equally true in n-space, if we substitute for a surface referred to a conjugate system a *reseau*, as defined by Guichard* and use the term *congruence* in the restricted sense given to it by this author.

The remainder of the memoir deals with ordinary space. It is shown that each pair of surfaces in the relation of a transformation K gives rise to four surfaces Σ, Σ_1, Σ_2, Σ', such that the pairs of surfaces $\Sigma\Sigma_1$, $\Sigma\Sigma_2$, $\Sigma_1\Sigma'$, $\Sigma_2\Sigma'$ are the focal surfaces of W-congruences, and conversely, when four surfaces are so related there exist four pairs of surfaces each pair of which are in the relation of a transformation K.

In order to apply the transformations K to particular types of surfaces it is desirable to put the equations in the form developed in § 9. The memoir contains one application, namely the solution of the problem of determining when S and S_1 envelop a two-parameter family of spheres. It is shown that in this case S and S_1 are isothermic surfaces in the relation of the transformation discovered by Darboux† and studied at length by Bianchi.‡ The theorem of permutability of these transformations is an immediate consequence of our general theorem.

The closing section of the memoir contains the solution of the problem under what conditions does the congruence of joins of corresponding points on two surfaces in the relation of a transformation K possess other pairs of points which describe surfaces similarly related. It is shown that, if a congruence possesses more than one pair of such points, it possesses an infinity of pairs.

In a subsequent memoir we shall consider the case where the joins of corresponding points on two surfaces S and S_1 form a normal congruence and we shall show that the surfaces orthogonal to such a congruence admit a transformation of the Ribaucour type.

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*Annales de l'école normale supérieure, sér. 3, vol. 14 (1897), pp. 467–516.

†Annales de l'école normale supérieure, sér. 3, vol. 16 (1899), pp. 491–508.

‡Annali di Matematica, ser. 3, vol. 11 (1905), pp. 93–158.

1. Fundamental transformation

By definition if x, y, z are the cartesian coördinates of a surface S referred to a conjugate system with equal point invariants, these functions are solutions of an equation of the form

$$\frac{\partial^2 \theta}{\partial u\, \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \theta}{\partial v} = 0, \tag{1}$$

where ρ is in general a function of u and v.

In accordance with the transformation of Moutard*, if θ is any solution of equation (1), the function ξ defined by the equations

$$\frac{\partial \xi}{\partial u} = -\rho\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial \xi}{\partial v} = \rho\left(x\frac{\partial \theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right) \tag{2}$$

is the solution of another equation with equal invariants. In fact, if we differentiate the first of (2) with respect to v, we find readily that ξ is a solution of the equation

$$\frac{\partial^2 \theta'}{\partial u\, \partial v} - \frac{\partial \log \sqrt{\rho}\,\theta}{\partial v}\frac{\partial \theta'}{\partial u} - \frac{\partial \log \sqrt{\rho}\,\theta}{\partial u}\frac{\partial \theta'}{\partial v} = 0. \tag{3}$$

If in (2) we replace ξ and x by the quantities η and y respectively, and also by ζ and z, the functions η and ζ so defined satisfy equation (3).

When the functions ξ, η, ζ are replaced by λx_1, λy_1, λz_1, the necessary and sufficient condition that the surface S_1, whose coördinates are x_1, y_1, z_1, is referred to a conjugate system is that λ be a solution of equation (3). We assume that λ is such a function and write equations (2) in the form

$$\frac{\partial}{\partial u}(\lambda x_1) = -\rho\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial}{\partial v}(\lambda x_1) = \rho\left(x\frac{\partial \theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right). \tag{4}$$

It is readily shown that x_1, y_1, z_1 are solutions of

$$\frac{\partial^2 \phi}{\partial u\, \partial v} + \frac{\partial}{\partial v}\log\frac{\lambda}{\sqrt{\rho}\,\theta}\cdot\frac{\partial \phi}{\partial u} + \frac{\partial}{\partial u}\log\frac{\lambda}{\sqrt{\rho}\,\theta}\cdot\frac{\partial \phi}{\partial v} = 0. \tag{5}$$

Hence the parametric conjugate system on S_1 has equal point invariants.

We denote by M and M_1 corresponding points on S and S_1 and by x_0, y_0, z_0 the coördinates of a point M_0 on the line joining M and M_1. Evidently

$$x_0 = x + t(x_1 - x), \qquad y_0 = y + t(y_1 - y), \qquad z_0 = z + t(z_1 - z), \tag{6}$$

where t is a function of u and v.

By means of equations (4) the first derivatives of x_0 assume the form

* Loc. cit.

$$
(7)\qquad
\begin{aligned}
\frac{\partial x_0}{\partial u} &= \left[t\left(\frac{\rho\theta}{\lambda}-1\right)+1\right]\frac{\partial x}{\partial u}+(x_1-x)\left(\frac{\partial t}{\partial u}-t\frac{\partial \log\lambda}{\partial u}\right)\\
&\qquad -\frac{xt}{\lambda}\left(\frac{\partial\lambda}{\partial u}+\rho\frac{\partial\theta}{\partial u}\right),\\
\frac{\partial x_0}{\partial v} &= \left[1-t\left(1+\frac{\rho\theta}{\lambda}\right)\right]\frac{\partial x}{\partial v}+(x_1-x)\left(\frac{\partial t}{\partial v}-t\frac{\partial \log\lambda}{\partial v}\right)\\
&\qquad -\frac{xt}{\lambda}\left(\frac{\partial\lambda}{\partial v}-\rho\frac{\partial\theta}{\partial v}\right).
\end{aligned}
$$

It is our purpose to show that λ can be so chosen that the developables of the congruence (G) of lines joining corresponding points M and M_1 meet the surfaces S and S_1 in the parametric conjugate systems with equal point invariants.

In the first place we observe that 1 is a solution of equation (1) and consequently it follows from (2), on replacing x by 1, that the equations

$$
(8)\qquad \partial\lambda/\partial u = -\rho\,\partial\theta/\partial u, \qquad \partial\lambda/\partial v = \rho\,\partial\theta/\partial v
$$

are consistent and that the function λ so defined is a solution of equation (3).

If we substitute this value of λ in equations (7) and give to t either of the values

$$
(9)\qquad t_1 = \lambda/(\lambda-\rho\theta), \qquad t_2 = \lambda/(\lambda+\rho\theta),
$$

we obtain respectively

$$
\frac{\partial x_0/\partial u}{x_1-x} = \frac{\partial y_0/\partial u}{y_1-y} = \frac{\partial z_0/\partial u}{z_1-z}, \qquad \frac{\partial x_0/\partial v}{x_1-x} = \frac{\partial y_0/\partial v}{y_1-y} = \frac{\partial z_0/\partial v}{z_1-z}.
$$

Hence the parametric ruled surfaces of the congruence (G) are developable.

Furthermore, if F_1 and F_2 denote the focal points whose coördinates are given by (6) when t is replaced by t_1 and t_2 from (9), it is readily shown that M and M_1 are harmonic to F_1 and F_2.

For the function λ given by equations (8) the equations (4) assume the form

$$
(10)\qquad \frac{\partial x_1}{\partial u} = \frac{\rho}{\lambda}\left[(x_1-x)\frac{\partial\theta}{\partial u}+\theta\frac{\partial x}{\partial u}\right], \qquad \frac{\partial x_1}{\partial v} = -\frac{\rho}{\lambda}\left[(x_1-x)\frac{\partial\theta}{\partial v}+\theta\frac{\partial x}{\partial v}\right].
$$

The foregoing results are stated in

THEOREM 1. *When a surface S referred to a conjugate system with equal point invariants is known, and also any solution θ of the point equation of S, the coördinates of a surface S_1 can be found by quadratures* (10), *and this surface S_1 is such that the developables of the congruence of lines joining corresponding points on S and S_1 meet these surfaces in conjugate systems with equal point invariants. Moreover, the focal points of the congruence are harmonic to the corresponding points on S and S_1.*

From (10) it follows that the necessary and sufficient condition that the tangents to the parametric curves on S and S_1 at corresponding points be parallel is that θ be constant. From equations (8) it follows that λ also must be constant. If we take $\theta = -\lambda = 1$, equations (10) become

$$\frac{\partial x_1}{\partial u} = -\rho \frac{\partial x}{\partial u}, \qquad \frac{\partial x_1}{\partial v} = \rho \frac{\partial x}{\partial v}. \tag{11}$$

These equations show that in this case S and S_1 are associate surfaces.* Moreover, it is known that the corresponding conjugate systems on any two associate surfaces have equal point invariants and their equations are of the form (11). Hence the problem of associate surfaces is a special case of the general problem which we are considering.

2. The theorem of Koenigs. The transformations K

In this section we show that by means of equations (10) we obtain the most general surface S_1 such that the congruence of lines joining corresponding points of S and S_1 possess the properties indicated in Theorem 1. We begin by looking upon the congruence (G) as composed of the tangents to the curves $v =$ const. on a surface S_0, whose coördinates are x_0, y_0, z_0 and on which the parametric system is conjugate.

Lucien Levy† has shown that the cartesian coördinates x, y, z of any point M of a line of the congruence (G) which describes a surface S cut in a conjugate system by the developables of (G) are expressible in the form

$$x = x_0 - \frac{\sigma}{\dfrac{\partial \sigma}{\partial u}} \frac{\partial x_0}{\partial u}, \qquad y = y_0 - \frac{\sigma}{\dfrac{\partial \sigma}{\partial u}} \frac{\partial y_0}{\partial u}, \qquad z = z_0 - \frac{\sigma}{\dfrac{\partial \sigma}{\partial u}} \frac{\partial z_0}{\partial u}, \tag{12}$$

where σ is a solution of the point equation of S_0, namely

$$\frac{\partial^2 \sigma}{\partial u\, \partial v} + a \frac{\partial \sigma}{\partial u} + b \frac{\partial \sigma}{\partial v} = 0. \tag{13}$$

Moreover, each solution of this equation leads to such a surface S.

We consider a second surface S_1 with coördinates x_1, y_1, z_1 given by

$$x_1 = x_0 - \frac{\sigma_1}{\dfrac{\partial \sigma_1}{\partial u}} \frac{\partial x_0}{\partial u}, \qquad y_1 = y_0 - \frac{\sigma_1}{\dfrac{\partial \sigma_1}{\partial u}} \frac{\partial y_0}{\partial u}, \qquad z_1 = z_0 - \frac{\sigma_1}{\dfrac{\partial \sigma_1}{\partial u}} \frac{\partial z_0}{\partial u}, \tag{14}$$

σ_1 being another solution of (13).

* E. p. 380. A reference of this kind is to the author's *Differential Geometry*, Ginn and Company, Boston (1909).

† *Sur quelques équations linéaires aux derivées partielles*, Journal de l'école polytechnique, cahier 56 (1886), p. 77.

For the sake of brevity we put

$$(14') \qquad \begin{aligned} M &= a + \frac{1}{\sigma}\frac{\partial \sigma}{\partial v}, & N &= b + \frac{1}{\sigma}\frac{\partial \sigma}{\partial u}, \\ h &= \frac{\partial a}{\partial u} + ab, & k &= \frac{\partial b}{\partial v} + ab. \end{aligned}$$

Thus h and k are the invariants of equation (13). It follows from (13) and (14′) that

$$(15) \qquad \frac{\partial M}{\partial u} = h - MN, \qquad \frac{\partial N}{\partial v} = k - MN.$$

In accordance with the theory of the Laplace transformation* of equation (13) the coördinates of the other focal point of the congruence (G) are of the form

$$x_0' = x_0 + \frac{1}{b}\frac{\partial x_0}{\partial u}.$$

The condition that the focal points be harmonic with respect to the corresponding points on S and S_1, whose coördinates are given by (12) and (14) is

$$(16) \qquad N_1 + N = 0,$$

where N_1 is given by (14′) when σ is replaced by σ_1. If we express the condition that N_1 satisfies an equation analogous to the second of (15), we have

$$(17) \qquad M_1 = M - 2k/N,$$

where M_1 is given by (14′) when σ is replaced by σ_1. If we require that M_1 satisfies an equation analogous to the first of (15) we get

$$(18) \qquad \frac{\partial N}{\partial u} = N\frac{\partial \log k}{\partial u} + \frac{N^2}{k}(MN - k).$$

Whenever equations (13) and (18) admit a common solution σ, this function and σ_1, which follows by quadratures from (16) and (17), determine two surfaces S and S_1 in the relation of a transformation K.

Later (§ 11) we shall consider the conditions under which equations (13) and (18) possess a common solution. Now we are interested in showing that when these conditions are satisfied the coördinates of the two surfaces are in the relations expressed by equations (10).

From (12) we obtain by differentiation

$$(19) \qquad \begin{aligned} \frac{\partial x}{\partial u} &= \sigma\left(\frac{\partial \sigma}{\partial u}\right)^{-2}\left(\frac{\partial^2 \sigma}{\partial u^2}\frac{\partial x_0}{\partial u} - \frac{\partial \sigma}{\partial u}\frac{\partial x_0}{\partial v}\right), \\ \frac{\partial x}{\partial v} &= \sigma\left(\frac{\partial \sigma}{\partial u}\right)^{-2} N\left(-\frac{\partial \sigma}{\partial v}\frac{\partial x_0}{\partial u} + \frac{\partial \sigma}{\partial u}\frac{\partial x_0}{\partial v}\right). \end{aligned}$$

* E. p. 404.

The expression for $\partial^2 x/\partial u\,\partial v$ obtained from either of these equations can be put in the form

$$\frac{\partial^2 x}{\partial u\,\partial v} = \left(\frac{\partial \sigma}{\partial u}\right)^{-1}\left(N\frac{\partial \sigma}{\partial v}\frac{\partial x}{\partial u} + \frac{\frac{\partial \sigma}{\partial u}\left(\frac{\partial b}{\partial u} - b^2\right) - b\frac{\partial^2 \sigma}{\partial u^2}}{N}\frac{\partial x}{\partial v}\right). \tag{20}$$

A necessary and sufficient condition that this equation be of the form (1) is that there exist a function ρ defined by

$$\begin{aligned} \frac{\partial \log \sqrt{\rho}}{\partial u} &= -\left(\frac{\partial \sigma}{\partial u}\right)^{-1} N\frac{\partial \sigma}{\partial v}, \\ \frac{\partial \log \sqrt{\rho}}{\partial v} &= -\frac{\frac{\partial b}{\partial u} - b^2 - b\left(\frac{\partial \sigma}{\partial u}\right)^{-1}\frac{\partial^2 \sigma}{\partial u^2}}{N}. \end{aligned} \tag{21}$$

It is readily shown that these equations are consistent if equation (18) is satisfied. Owing to the fact that the relation between S and S_1 is reciprocal, it follows that the parametric system on S_1 also is conjugate with equal point invariants. Hence we have the theorem of Koenigs:*

THEOREM 2. *If two surfaces S and S_1 are so related that the focal points of the congruence of lines joining corresponding points M and M_1 on these surfaces are harmonic with respect to these points, and also if the developables of the congruence meet S and S_1 in conjugate systems, the latter have equal point invariants.*

Now we show that coördinates the of S and S_1 are in the relation of equations (10).

Since σ_1 determined by quadratures from (16) and (17) is a solution of equation (13), it is evident from (12) that the function θ given by

$$\theta = \sigma_1 - \frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u} = 2\sigma_1\left(1 + b\frac{\sigma}{\frac{\partial \sigma}{\partial u}}\right) \tag{22}$$

is a solution of equation (20).

The first derivatives of θ are given by equations obtained from equations (19) on replacing x_0 by σ_1; and from these same equations on substituting σ_1 for σ we get the first derivatives of x_1. When these values are substituted in equations (10), they are satisfied, provided that

$$\lambda = \rho\left(\sigma_1 - \frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u}\right)\frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u}\Big/\sigma_1 = \rho\frac{\theta}{\sigma_1}(\sigma_1 - \theta). \tag{23}$$

When this expression for λ is substituted in equations (8), we are led to equations (21). Hence we have

* Loc. cit.

THEOREM 3. *The transformation* (10) *of a surface S into a surface S_1 is the most general case of pairs of surfaces in the relation of Koenigs.*

In view of this result we shall hereafter say that S_1 given by (10) is obtained from S by a *transformation K*.

3. THEOREM OF PERMUTABILITY

Suppose that we have two solutions θ_1 and θ_2 of equation (1) and consider the two surfaces S_1 and S_2 which arise from S by transformations K_1 and K_2 determined by θ_1 and θ_2 respectively. The coördinates x_1, y_1, z_1 and x_2, y_2, z_2 are given by equations of the form (10), namely

$$\begin{aligned} \frac{\partial x_i}{\partial u} &= \frac{\rho}{\lambda_i}\left[(x_i - x)\frac{\partial \theta_i}{\partial u} + \theta_i \frac{\partial x}{\partial u}\right], \\ \frac{\partial x_i}{\partial v} &= -\frac{\rho}{\lambda_i}\left[(x_i - x)\frac{\partial \theta_i}{\partial v} + \theta_i \frac{\partial x}{\partial v}\right] \end{aligned} \qquad (i = 1, 2), \tag{24}$$

where, in accordance with (8),

$$\frac{\partial \lambda_i}{\partial u} = -\rho\frac{\partial \theta_i}{\partial u}, \qquad \frac{\partial \lambda_i}{\partial v} = \rho\frac{\partial \theta_i}{\partial v} \qquad (i = 1, 2). \tag{25}$$

From (5) it follows that the functions x_i, y_i, z_i, defined by (24) and similar equations in y_i and z_i, satisfy the corresponding equation

$$\frac{\partial^2 \phi_i}{\partial u\, \partial v} + \frac{\partial}{\partial v}\log\frac{\lambda_i}{\sqrt{\rho}\,\theta_i}\frac{\partial \phi_i}{\partial u} + \frac{\partial}{\partial u}\log\frac{\lambda_i}{\sqrt{\rho}\,\theta_i}\frac{\partial \phi_i}{\partial v} = 0 \qquad (i = 1, 2). \tag{26}$$

It is evident from (4) that these equations are satisfied also by the functions θ_1' and θ_2' respectively, defined by

$$\begin{aligned} \frac{\partial}{\partial u}(\theta_i' \lambda_i) &= -\rho\left(\theta_i \frac{\partial \theta_j}{\partial u} - \theta_j \frac{\partial \theta_i}{\partial u}\right), \\ \frac{\partial}{\partial v}(\theta_i' \lambda_i) &= \rho\left(\theta_i \frac{\partial \theta_j}{\partial v} - \theta_j \frac{\partial \theta_i}{\partial v}\right) \end{aligned} \qquad \left(\begin{matrix} i = 1, 2, \\ j = 1, 2; \end{matrix} \quad i \neq j\right). \tag{27}$$

As thus defined each of the functions θ_i' is determined only to within the additive function c_i/λ_i, where c_i is an arbitrary constant. Hereafter in using pairs of functions θ_1' and θ_2' simultaneously we understand that the constants c_i are so determined that

$$\theta_1' \lambda_1 + \theta_2' \lambda_2 = 0, \tag{28}$$

which evidently is possible because of the form of (27).

It is our purpose to show that, if we use the functions θ_1' and θ_2' to determine transformations K_1' and K_2' of S_1 and S_2 respectively, the resulting surfaces coincide, giving a surface S' with coördinates x', y', z'.

In the first place we observe that if we put

$$\rho_i = \lambda_i^2/\rho\theta_i^2, \tag{29}$$

equations (26) may be given a form similar to (1). Hence for the transformations K_1' and K_2' the equations analogous to (25) are

$$\frac{\partial\lambda_i'}{\partial u} = -\frac{\lambda_i^2}{\rho\theta_i^2}\frac{\partial\theta_i'}{\partial u}, \qquad \frac{\partial\lambda_i'}{\partial v} = \frac{\lambda_i^2}{\rho\theta_i^2}\frac{\partial\theta_i'}{\partial v} \qquad (i = 1, 2). \tag{30}$$

In order that the transforms of S_1 and S_2 may coincide, equations analogous to (24) necessitate the conditions

$$\begin{aligned} \frac{\partial x'}{\partial u} &= \frac{\rho_1}{\lambda_1'}\left[(x' - x_1)\frac{\partial\theta_1'}{\partial u} + \theta_1'\frac{\partial x_1}{\partial u}\right] = \frac{\rho_2}{\lambda_2'}\left[(x' - x_2)\frac{\partial\theta_2'}{\partial u} + \theta_2'\frac{\partial x_2}{\partial u}\right], \\ \frac{\partial x'}{\partial v} &= -\frac{\rho_1}{\lambda_1'}\left[(x' - x_1)\frac{\partial\theta_1'}{\partial v} + \theta_1'\frac{\partial x_1}{\partial v}\right] = -\frac{\rho_2}{\lambda_2'}\left[(x' - x_2)\frac{\partial\theta_2'}{\partial v} + \theta_2'\frac{\partial x_2'}{\partial v}\right]. \end{aligned} \tag{31}$$

If we substitute in the second equation of the first row the expressions for $\partial\theta_1'/\partial u$ and $\partial x_i/\partial u$ given by (24) and (27), we obtain

$$\left(\theta_2\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial\theta_2}{\partial u}\right)P + \frac{\lambda_1\theta_1'}{\theta_2}\left(\frac{1}{\lambda_1'\theta_1} + \frac{1}{\lambda_2'\theta_2}\right)\left[(x' - x)\frac{\partial\theta_2}{\partial u} + \theta_2\frac{\partial x}{\partial u}\right] = 0, \tag{32}$$

where for the sake of brevity we have put

$$P = x'\left[\frac{\lambda_1}{\lambda_1'\theta_1^2\theta_2}(\theta_1' + \theta_2) + \frac{\lambda_2}{\lambda_2'\theta_2'}\right] - \left(\frac{x_1\lambda_1}{\lambda_1'\theta_1^2} + \frac{x_2\lambda_2}{\lambda_2'\theta_2^2}\right) - \frac{\lambda_1\theta_1' x}{\theta_1^2\theta_2\lambda_1'}. \tag{33}$$

In a similar manner we obtain from the second row of (31) the equation

$$\left(\theta_2\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial\theta_2}{\partial v}\right)P + \frac{\lambda_1\theta_1'}{\theta_2}\left(\frac{1}{\lambda_1'\theta_1} + \frac{1}{\lambda_2'\theta_2}\right)\left[(x' - x)\frac{\partial\theta_2}{\partial v} + \theta_2\frac{\partial x}{\partial v}\right] = 0. \tag{34}$$

The equations of condition (32) and (34) are satisfied if $P = 0$ and if

$$\lambda_1'\theta_1 + \lambda_2'\theta_2 = 0. \tag{35}$$

We shall show that these conditions are satisfied and lead to the desired result.

If we differentiate equation (35) with respect to u and v separately and require that the functions λ_i' shall satisfy (30), we find that we must have

$$\lambda_1'\theta_1 = \lambda_1(\theta_1' + \theta_2) - \lambda_2\theta_1, \qquad \lambda_2'\theta_2 = \lambda_2(\theta_2' + \theta_1) - \lambda_1\theta_2. \tag{36}$$

In consequence of (28) these equations are consistent with (35) and furthermore these expressions for λ_i' satisfy equations (30).

Because of (36) the equation obtained by equating P to zero is reducible to

$$\lambda_1'\theta_1 x' = \theta_1'\lambda_1 x + \lambda_1\theta_2 x_1 - \lambda_2\theta_1 x_2, \tag{37}$$

which, in consequence of (28) and (35), is equivalent to

$$\lambda_2' \theta_2 x' = \theta_2' \lambda_2 x + \lambda_2 \theta_1 x_2 - \lambda_1 \theta_2 x_1 . \tag{38}$$

With the aid of equations (36) it is shown that the value of x' given by (37) satisfies equations (31). Hence not only does S' exist satisfying the above requirements, but its coördinates can be found without quadrature after θ_1' and θ_2' have been obtained by the quadratures (27).

We shall say that four surfaces S, S_1, S_2, S' in the above relation form a *quatern*. From (37) it is evident that the corresponding points on the four surfaces lie in a plane.

From the form of (27) it is seen that θ_1' is determined to within the additive quantity c/λ_1, where c is an arbitrary constant. The first of equations (36) shows that the corresponding change in λ_1' is the addition of c/θ_1. Hence if θ_1' is a function giving a surface S' and if we denote by S_c' the surface determined by $\theta_1' + c/\lambda_1$, the coördinates x_c', y_c', z_c' of S_c' are expressible in the form

$$(\lambda_1' \theta_1 + c) x_c' = (\theta_1' \lambda_1 + c) x + \lambda_1 \theta_2 x_1 - \lambda_2 \theta_1 x_2 ,$$

which may be written also

$$(\lambda_1' \theta_1 + c) x_c' = cx + \lambda_1' \theta_1 x' . \tag{39}$$

Hence all the corresponding points on the family of surfaces S_c' lie on the line joining M and M', and each point divides the segment MM' in the ratio $c/\lambda_1' \theta_1$.

The foregoing results may be assembled into

THEOREM 4. *If S_1 and S_2 are two surfaces arising from S by transformations K, there can be found by a quadrature an infinity of surfaces S' each of which is in the relation of transformations K with both S_1 and S_2. The corresponding points on the suite of surfaces S' lie on a line which passes through the corresponding point on S and in the plane determined by the latter point and the corresponding points on S_1 and S_2.*

We consider in particular the case when S_1 is an associate surface of S. As shown in § 1 this arises when we take $\theta_1 = -\lambda_1 = 1$. Now equations (27) for $i = 1$ become

$$\frac{\partial \theta_1'}{\partial u} = \rho \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_1'}{\partial v} = -\rho \frac{\partial \theta_2}{\partial v} .$$

Comparing these with (25) and making use of (28) we see that

$$\theta_1' = -\lambda_2 + c, \qquad \theta_2' = -1 + c/\lambda_2 .$$

Hence if we take $c = 0$ the surfaces S_2 and S' are associate. In this case

equation (37) can be written

$$\theta_2 (x' - x_1) = \lambda_2 (x_2 - x). \tag{40}$$

Therefore we have

THEOREM 5. *If S and S_1 are two associate surfaces, and if S_2 is any surface arising from S by a transformation K, there exists a surface S' associate to S_2 which is in the relation of a transformation K with S_1; moreover, the lines joining corresponding points of S_1 and S' and of S and S_2 are parallel.*

4. Envelope of the planes of a quatern

We have just seen that the four corresponding points M, M_1, M_2, M' of the surfaces of a quatern S, S_1, S_2, S' lie in a plane. Since this plane contains the lines MM_1 and MM_2 which generate congruences whose developables meet S in a conjugate system, it envelopes a surface upon which the curves corresponding to these developables form a conjugate system, as follows from the general theory of conjugate systems and congruences.* Moreover, if Π is the point of the envelope corresponding to the point M on S, the tangent at Π to one of these curves passes through the focal points F_1' and F_2' of the lines MM_1 and MM_2 respectively, and the tangent to the other curve passes through the other focal points F_1'' and F_2''. Since the set of points M, M_1, F_1', F_1'' are harmonic and likewise the points M, M_2, F_2', F_2'', it follows that the line joining M_1 and M_2 passes through Π. As the surfaces S and S' bear to S_1 a relation similar to that of S_1 and S_2 to S, it follows that the point Π lies also on the line MM'.

If X, Y, Z denote the coördinates of Π, in order that it be the intersection of the lines MM' and $M_1 M_2$, there must exist functions σ and τ such that

$$X = x + \sigma (x' - x) = x_1 + \tau (x_2 - x_1), \tag{41}$$

and similar expressions for Y and Z. If we eliminate x' from equation (37) and from the equation formed by the last two members of (41), we obtain an equation of the form $ax + bx_1 + cx_2 = 0$. Since the same equation is satisfied by the y's and z's, the expressions a, b, and c must be zero. These equations are consistent in consequence of (36) and give the expressions

$$\sigma = \frac{\lambda_1' \theta_1}{\lambda_1' \theta_1 - \lambda_1 \theta_1'}, \qquad \tau = -\frac{\lambda_2 \theta_1}{\lambda_1 \theta_2 - \lambda_2 \theta_1}.$$

When these values are substituted in (41), we obtain

$$X = \frac{\lambda_1' \theta_1 x' - \theta_1' \lambda_1 x}{\lambda_1' \theta_1 - \theta_1' \lambda_1} = \frac{\lambda_1 \theta_2 x_1 - \lambda_2 \theta_1 x_2}{\lambda_1 \theta_2 - \lambda_2 \theta_1}, \tag{42}$$

which are consistent because of (37).

* Cf. Guichard, loc. cit., p. 491.

From (6) and (9) it follows that the coördinates of the focal points, namely x_i', y_i', z_i' and x_i'', y_i'', z_i'' for $i = 1, 2$, are given by expressions of the form

$$x_i' = \frac{\lambda_i x_i - \rho\theta_i x}{\lambda_i - \rho\theta_i}, \qquad x_i'' = \frac{\lambda_i x_i + \rho\theta_i x}{\lambda_i + \rho\theta_i} \qquad (i = 1, 2). \tag{43}$$

Hence the direction-parameters of the lines $F_1' F_2'$, $F_1'' F_2''$ are proportional to expressions of the form

$$\begin{aligned} &x_1 \lambda_1 (\lambda_2 - \rho\theta_2) - x_2 \lambda_2 (\lambda_1 - \rho\theta_1) + \rho x (\lambda_1 \theta_2 - \lambda_2 \theta_1), \\ &x_1 \lambda_1 (\lambda_2 + \rho\theta_2) - x_2 \lambda_2 (\lambda_1 + \rho\theta_1) - \rho x (\lambda_1 \theta_2 - \lambda_2 \theta_1), \end{aligned} \tag{44}$$

respectively. With the aid of these values it is readily verified that the point whose coördinates are given by (42) is the intersection of the lines $F_1' F_2'$ and $F_1'' F_2''$.

From the second expression (42) for X we obtain by differentiation

$$\begin{aligned} \frac{\partial X}{\partial u} &= [x_1 \lambda_1 (\lambda_2 + \rho\theta_2) - x_2 \lambda_2 (\lambda_1 + \rho\theta_1) + \rho x (\theta_1 \lambda_2 - \theta_2 \lambda_1)] K_1, \\ \frac{\partial X}{\partial v} &= [x_1 \lambda_1 (\lambda_2 - \rho\theta_2) - x_2 \lambda_2 (\lambda_1 - \rho\theta_1) - \rho x (\theta_1 \lambda_2 - \theta_2 \lambda_1)] K_2, \end{aligned} \tag{45}$$

where

$$K_1 = \frac{\theta_2 \dfrac{\partial\theta_1}{\partial u} - \theta_1 \dfrac{\partial\theta_2}{\partial u}}{(\theta_1 \lambda_2 - \theta_2 \lambda_1)^2}, \qquad K_2 = \frac{\theta_2 \dfrac{\partial\theta_1}{\partial v} - \theta_1 \dfrac{\partial\theta_2}{\partial v}}{(\theta_1 \lambda_2 - \theta_2 \lambda_1)^2}. \tag{46}$$

Comparing equations (44) and (45), we note that the curves $v =$ const. on the envelope are tangent to the corresponding lines $F_1'' F_2''$ and the curves $u =$ const. to the lines $F_1' F_2'$, as previously remarked from the general theory. As the congruences of lines $M' M_1$ and $M' M_2$ bear to the surface S' a relation similar to that of the lines MM_1 and MM_2 to S, the focal points on the lines $M' M_1$ and $M' M_2$ lie on the tangents to the curves $v =$ const. or $u =$ const. It follows that the two tangents at a point Π are harmonic with respect to the lines MM' and $M_1 M_2$ through the point.

In order to find the equation satisfied by the coördinates of Π, we differentiate the first of equations (45) with respect to v. After direct reductions we find that

$$\begin{aligned} \frac{\partial^2 X}{\partial u\, \partial v} &+ \frac{K_2}{K_1} \left[\frac{\partial \log \sqrt{\rho}}{\partial u} + \frac{\lambda_2 \dfrac{\partial\theta_1}{\partial u} - \lambda_1 \dfrac{\partial\theta_2}{\partial u}}{\theta_1 \lambda_2 - \theta_2 \lambda_1} \right] \frac{\partial X}{\partial u} \\ &+ \frac{K_1}{K_2} \left[\frac{\partial \log \sqrt{\rho}}{\partial v} + \frac{\lambda_2 \dfrac{\partial\theta_1}{\partial v} - \lambda_1 \dfrac{\partial\theta_2}{\partial v}}{\theta_1 \lambda_2 - \theta_2 \lambda_1} \right] \frac{\partial X}{\partial v} = 0. \end{aligned} \tag{47}$$

It is evident that ordinarily the invariants of this equation are not equal.

We inquire whether the point Π ever coincides with one of the suite defined by equation (39) for all values of u and v. If we equate this value of x'_c to the second expression (42) for X, we are led to the condition $\theta'_1 \lambda_1 = -c$, which by (27) necessitates $\theta_2/\theta_1 = \text{const}$. In this case S_1 and S_2 are homothetic.

Gathering together the above results we have

THEOREM 6. *If S, S_1, S_2, S' are four surfaces of a quatern, the plane Π of four corresponding points M, M_1, M_2, M' touches its envelope in the intersection Π of the lines MM' and $M_1 M_2$; the parametric lines on the envelope form a conjugate system (with invariants ordinarily unequal) whose tangents are harmonic to the lines MM' and $M_1 M_2$ and contain the focal points of the lines MM_1, MM_2, $M' M_1$, $M' M_2$.*

5. Relations between transformations K and radial transformations

If ω is a solution of equation (1) which is linearly independent of x, y, z, the surface $\bar{S}$ whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ are given by

$$\bar{x} = x/\omega, \qquad \bar{y} = y/\omega, \qquad \bar{z} = z/\omega \tag{48}$$

is referred to a conjugate system with equal point invariants. In fact, it is easily shown that the coördinates of $\bar{S}$ satisfy the equation

$$\frac{\partial^2 \bar{\theta}}{\partial u\, \partial v} + \frac{\partial}{\partial v} \log \sqrt{\rho}\, \omega \frac{\partial \bar{\theta}}{\partial u} + \frac{\partial}{\partial u} \log \sqrt{\rho}\, \omega \frac{\partial \bar{\theta}}{\partial v} = 0. \tag{49}$$

We shall say that $\bar{S}$ is obtained from S by a *radial transformation.*

If θ is a function linearly independent of x, y, z, and ω, we shall find that the surfaces S_1 and $\bar{S}_1$ resulting from S and $\bar{S}$ by transformations K, determined by θ and $\bar{\theta}$ respectively, where

$$\bar{\theta} = \theta/\omega, \tag{50}$$

are in the relation of a radial transformation. In fact, we shall show that the coördinates x_1, y_1, z_1 and $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ of S_1 and $\bar{S}_1$ respectively are in the relations

$$\bar{x}_1 = x_1/\omega_1, \qquad \bar{y}_1 = y_1/\omega_1, \qquad \bar{z}_1 = z_1/\omega_1, \tag{51}$$

where ω_1 is given by the quadratures

$$\frac{\partial \omega_1}{\partial u} = \frac{\rho}{\lambda}\left[(\omega_1 - \omega)\frac{\partial \theta}{\partial u} + \theta \frac{\partial \omega}{\partial u}\right], \qquad \frac{\partial \omega_1}{\partial v} = -\frac{\rho}{\lambda}\left[(\omega_1 - \omega)\frac{\partial \theta}{\partial v} + \theta \frac{\partial \omega}{\partial v}\right]. \tag{52}$$

As equations (52) are of the form of (10), it is evident that they are consistent and that ω_1 is a solution of the point equation of S_1, namely (5).

The equations of the transformation from $\bar{S}$ to $\bar{S}_1$, analogous to (10), are

$$
(53)\qquad
\begin{aligned}
\frac{\partial \bar{x}_1}{\partial u} &= \frac{\rho\omega^2}{\bar{\lambda}}\left[\left(\frac{x_1}{\omega_1}-\frac{x}{\omega}\right)\frac{\partial}{\partial u}\left(\frac{\theta}{\omega}\right)+\frac{\theta}{\omega}\frac{\partial}{\partial u}\left(\frac{x}{\omega}\right)\right],\\
\frac{\partial \bar{x}_1}{\partial v} &= -\frac{\rho\omega^2}{\bar{\lambda}}\left[\left(\frac{x_1}{\omega_1}-\frac{x}{\omega}\right)\frac{\partial}{\partial v}\left(\frac{\theta}{\omega}\right)+\frac{\theta}{\omega}\frac{\partial}{\partial v}\left(\frac{x}{\omega}\right)\right].
\end{aligned}
$$

When the value of $\bar{x}_1$ from (51) is substituted in (53) and in the reduction equations (52) are used, it is found that the necessary and sufficient condition that these equations be satisfied is that

$$
(54)\qquad \bar{\lambda} = \omega_1 \lambda .
$$

With the aid of equations (52) it is readily shown that this value of $\bar{\lambda}$ satisfies the equations

$$
(55)\qquad \frac{\partial \bar{\lambda}}{\partial u} = -\rho\omega^2 \frac{\partial}{\partial u}\left(\frac{\theta}{\omega}\right), \qquad \frac{\partial \bar{\lambda}}{\partial v} = \rho\omega^2 \frac{\partial}{\partial v}\left(\frac{\theta}{\omega}\right),
$$

which are analogous to equations (8). Hence we have

THEOREM 7. *If two surfaces S and $\bar{S}$ are in the relation of a radial transformation and a transformation K of S is known, it is possible to find by quadratures a transformation K of $\bar{S}$ such that the new surfaces S_1 and $\bar{S}_1$ are in the relation of a radial transformation.*

If in particular $\theta = \omega$, it follows from (55) that $\bar{\lambda}$ is constant. When we take $\bar{\lambda} = -1$, equation (54) becomes

$$
(56)\qquad \omega_1 = -1/\lambda ,
$$

which we have seen in § 3 is a solution of equation (5). With the aid of equations (8) it can be shown that this value of ω_1 satisfies equations (52). Now $\bar{\theta} = 1$, consequently $\bar{S}_1$ is associate to $\bar{S}$, as shown in § 1. From these results we have

THEOREM 8. *If a surface S is subjected to a radial transformation determined by a function θ, and if the associate of the resulting surface, determined by the parametric conjugate system, undergoes a radial transformation determined by $-\lambda$, where λ is given by the quadratures* (8), *the new surface S_1 and S are in the relation of the transformation K determined by θ.*

Suppose now that we have a quatern of surfaces S, S_1, S_2, S', expressed in terms of the same functions appearing in § 3. If θ_2 determines a radial transformation of S, a comparison of equations (52) and (27) reveals the fact that the function $\omega_1 = -\theta_1'$. On the other hand $\omega_2 = -1/\lambda_2$. Again, since the radial transformation function of S_1 differs only in sign from the function θ_1' of the transformation of S_1 into S', it follows that $\omega' = 1/\lambda_1'$.

Hence the four surfaces $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}'$, whose x-coördinates are respectively of the form

$$(57) \qquad \bar{x} = x/\theta_2, \qquad \bar{x}_1 = -x_1/\theta_1', \qquad \bar{x}_2 = -\lambda_2 x_2, \qquad \bar{x}' = \lambda_1' x',$$

are radial transforms of the surfaces of the original quatern; and these surfaces themselves form a quatern of the special type of Theorem 5, the surfaces $\bar{S}$ and $\bar{S}_2$ being associate and likewise $\bar{S}_1$ and $\bar{S}'$. The transformation functions of this new quatern are

$$(58) \qquad \begin{gathered} \bar{\theta}_1 = \theta_1/\theta_2, \qquad \bar{\lambda}_1 = -\lambda_1 \theta_1'; \qquad \bar{\theta}_2 = -\bar{\lambda}_2 = 1; \qquad \bar{\theta}_1' = -\bar{\lambda}_1' = -1; \\ \bar{\theta}_2' = \lambda_1 \theta_1', \qquad \bar{\lambda}_2' = -\theta_1/\theta_2. \end{gathered}$$

It is readily shown that these functions and the coördinates (57) satisfy equations analogous to (37) and (38).

In like manner the four surfaces whose coördinates are of the form

$$x/\theta_1, \qquad -\lambda_1 x_1, \qquad -x_2/\theta_2', \qquad \lambda_2' x'$$

form a quatern in radial relation to the corresponding surfaces of the other two quaterns.

Accordingly we have

THEOREM 9. *When a quatern of surfaces is known, two other quaterns can be found without quadratures, each of which consists of two pairs of associate surfaces.*

6. W-CONGRUENCES ASSOCIATED WITH A TRANSFORMATION K

We consider a surface S and the associate surface S_0 whose coördinates x_0, y_0, z_0 are given by quadratures of the form

$$(59) \qquad \frac{\partial x_0}{\partial u} = -\rho \frac{\partial x}{\partial u}, \qquad \frac{\partial x_0}{\partial v} = \rho \frac{\partial x}{\partial v},$$

as follows from (11). These coördinates are solutions of

$$(60) \qquad \frac{\partial^2 \theta_0}{\partial u\, \partial v} = \frac{\partial \log \sqrt{\rho}}{\partial v} \frac{\partial \theta_0}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u} \frac{\partial \theta_0}{\partial v},$$

as is readily shown, if we note that x is a solution of equation (1). If we put

$$(61) \qquad \sigma = \theta_0/\sqrt{\rho},$$

the function σ is a solution of the equation

$$(62) \qquad \frac{\partial^2 \sigma}{\partial u\, \partial v} = \sqrt{\rho} \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sqrt{\rho}}\right)\sigma.$$

In accordance with the theory of infinitesimal deformation of surfaces the surface Σ, whose coördinates ξ, η, ζ are given by quadratures of the form

$$\frac{\partial \xi}{\partial u} = z_0 \frac{\partial y}{\partial u} - y_0 \frac{\partial z}{\partial u}, \qquad \frac{\partial \xi}{\partial v} = z_0 \frac{\partial y}{\partial v} - y_0 \frac{\partial z}{\partial v}, \tag{63}$$

corresponds to S with orthogonality of linear elements.* For the sake of brevity we say that Σ is the *ortho-surface* of S determined by S_0.

By means of (59), equations (63) are reducible to

$$\frac{\partial \xi}{\partial u} = -\frac{1}{\rho}\left(z_0 \frac{\partial y_0}{\partial u} - y_0 \frac{\partial z_0}{\partial u}\right), \qquad \frac{\partial \xi}{\partial v} = \frac{1}{\rho}\left(z_0 \frac{\partial y_0}{\partial v} - y_0 \frac{\partial z_0}{\partial v}\right). \tag{64}$$

If we put

$$x_0 = \sqrt{\rho}\,\alpha, \qquad y_0 = \sqrt{\rho}\,\beta, \qquad z_0 = \sqrt{\rho}\,\gamma, \tag{65}$$

the functions α, β, γ are solutions of equation (62) and equations (64) assume the Lelieuvre form

$$\frac{\partial \xi}{\partial u} = \left(\beta \frac{\partial \gamma}{\partial u} - \gamma \frac{\partial \beta}{\partial u}\right), \qquad \frac{\partial \xi}{\partial v} = -\left(\beta \frac{\partial \gamma}{\partial v} - \gamma \frac{\partial \beta}{\partial v}\right). \tag{66}$$

In like manner there is a surface S_{10} associate to a surface S_1 resulting from S by a transformation K and its coördinates, x_{10}, y_{10}, z_{10}, are given by

$$\frac{\partial x_{10}}{\partial u} = -\rho_1 \frac{\partial x_1}{\partial u}, \qquad \frac{\partial x_{10}}{\partial v} = \rho_1 \frac{\partial x_1}{\partial v}, \tag{67}$$

where, as follows from (29),

$$\rho_1 = \lambda^2/\rho\theta^2. \tag{68}$$

It can readily be shown that the four surfaces S, S_0, S_1, S_{10} satisfy the requirements of Theorem 5, so that S_0 and S_{10} are in the relation of a transformation K. In fact, in this case equation (40) becomes

$$\theta(x_{10} - x_0) = \lambda(x_1 - x). \tag{69}$$

If the expressions (10) for $\partial x_1/\partial u$ and $\partial x_1/\partial v$ be substituted in (67) and $(x_1 - x)$ be replaced by its value from (69), we have

$$\frac{\partial}{\partial u}(\theta x_{10}) = x_0 \frac{\partial \theta}{\partial u} + \frac{\lambda}{\rho}\frac{\partial x_0}{\partial u}, \qquad \frac{\partial}{\partial v}(\theta x_{10}) = x_0 \frac{\partial \theta}{\partial v} - \frac{\lambda}{\rho}\frac{\partial x_0}{\partial v}. \tag{70}$$

In a manner analogous to (65) we put

$$x_{10} = \sqrt{\rho_1}\,\alpha_1 = -\frac{\lambda}{\sqrt{\rho}\,\theta}\alpha_1. \tag{71}$$

As defined by equations (8), the function λ is a solution of equation (60).

* E., p. 382.

Consequently the function σ_1, defined by

$$\sigma_1 = \lambda / \sqrt{\rho}, \tag{72}$$

is a solution of (62). In consequence of equations (71), (72), and (8), equations (70) can be written in the form

$$\frac{\partial}{\partial u}(\sigma_1 \alpha_1) = -\left(\sigma_1 \frac{\partial \alpha}{\partial u} - \alpha \frac{\partial \sigma_1}{\partial u}\right), \qquad \frac{\partial}{\partial v}(\sigma_1 \alpha_1) = \left(\sigma_1 \frac{\partial \alpha}{\partial v} - \alpha \frac{\partial \sigma_1}{\partial v}\right). \tag{73}$$

Moreover, the coördinates ξ_1, η_1, ζ, of Σ_1, the ortho-surface to S_1 determined by S_{10}, are given by the quadratures

$$\frac{\partial \xi_1}{\partial u} = \left(\beta_1 \frac{\partial \gamma_1}{\partial u} - \gamma_1 \frac{\partial \beta_1}{\partial u}\right), \qquad \frac{\partial \xi_1}{\partial v} = -\left(\beta_1 \frac{\partial \gamma_1}{\partial v} - \gamma_1 \frac{\partial \beta_1}{\partial v}\right). \tag{74}$$

From these results we know that Σ and Σ_1 can be so placed in space that they are the focal surfaces of a W-congruence, and that the following relations hold:

$$\begin{gathered} \xi_1 - \xi = \beta\gamma_1 - \beta_1 \gamma, \qquad \eta_1 - \eta = \gamma\alpha_1 - \gamma_1 \alpha, \\ \zeta_1 - \zeta = \alpha\beta_1 - \alpha_1 \beta.^* \end{gathered} \tag{75}$$

Each solution of equation (62) determines a surface associate to Σ. We denote by Σ_{01} the associate surface corresponding to the solution σ_1 given by (72), and by Σ_{02} the associate surface determined by the solution

$$\sigma_2 = 1 / \sqrt{\rho}. \tag{76}$$

The first of these gives rise to the W-congruence whose focal surfaces are Σ and Σ_1. The surface Σ_{01} determines an ortho-surface to Σ, namely $\bar{S}$, whose coördinates $\bar{x}, \bar{y}, \bar{z}$ are

$$\bar{x} = \sigma_1 \alpha_1, \qquad \bar{y} = \sigma_1 \beta_1, \qquad \bar{z} = \sigma_1 \gamma_1, \tag{77}$$

where $\alpha_1, \beta_1, \gamma_1$ are given by (71). The surface $\bar{S}_0$ whose coördinates are

$$\bar{x}_0 = \alpha / \sigma_1, \qquad \bar{y}_0 = \beta / \sigma_1, \qquad \bar{z}_0 = \gamma / \sigma_1, \tag{78}$$

is associate to $\bar{S}$ and is an ortho-surface of Σ_1.†

The equation for Σ_1 analogous to (62), namely

$$\frac{\partial^2 \bar{\sigma}}{\partial u\, \partial v} = \sqrt{\rho_1}\, \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sqrt{\rho_1}}\right) \bar{\sigma}, \tag{79}$$

admits the solution $1/\sigma_1$. This solution determines an associate surface of Σ_1 which leads to the ortho-surface $\bar{S}_0$ of Σ_1. Moreover, the joins of corresponding points on this associate surface and Σ_{01} are concurrent.‡

* Cf. E., pp. 417–420.

† Cf. E., p. 420.

‡ Cf. Darboux, *Leçons*, vol. 4, p. 69.

We observe that $1/\sqrt{\rho_1}$ also is a solution of equation (79). We denote by Σ_{10} the surface associate to Σ_1 determined by this solution. If we apply the Moutard transformation of the form (73) to $1/\sqrt{\rho}$, we obtain $1/\sqrt{\rho_1}$. In fact, it is readily shown that the equations

$$
(80)\qquad
\begin{aligned}
\frac{\partial}{\partial u}\left(\sigma_1 \frac{1}{\sqrt{\rho_1}}\right) &= -\left(\sigma_1 \frac{\partial}{\partial u}\frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}}\frac{\partial \sigma_1}{\partial u}\right),\\
\frac{\partial}{\partial v}\left(\sigma_1 \frac{1}{\sqrt{\rho_1}}\right) &= \left(\sigma_1 \frac{\partial}{\partial v}\frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}}\frac{\partial \sigma_1}{\partial v}\right),
\end{aligned}
$$

reduce to equations (8). Making use of results which we have established elsewhere,* we have

THEOREM 10. *The surfaces Σ_{02} and Σ_{10}, associate to Σ and Σ_1, and determined by $1/\sqrt{\rho}$ and $1/\sqrt{\rho_1}$ respectively, are such that the lines joining corresponding points on these surfaces form a congruence whose developables meet them in the conjugate parametric system, which has equal tangential invariants; and the lines of intersection of the tangent planes to Σ_{02} and Σ_{10} form a congruence whose developables are parametric and whose focal planes are harmonic to these tangent planes.*

We say that two surfaces, related as Σ_{02} and Σ_{10} are in this theorem, are in the relation of a *transformation* Ω.

We turn to the consideration of the W-congruence, determined by the solution $\sigma_2 = 1/\sqrt{\rho}$ of equation (62), of which Σ is one of the focal surfaces. Since S is the corresponding ortho-surface of Σ, the functions α_2, β_2, γ_2 for the second sheet Σ_2 are given by the equations

$$
(81)\qquad \alpha_2 = \sqrt{\rho}\, x, \qquad \beta_2 = \sqrt{\rho}\, y, \qquad \gamma_2 = \sqrt{\rho}\, z,
$$

which are analogous to equations (77). These functions are solutions of the equation

$$
(82)\qquad \frac{\partial^2 \sigma'}{\partial u\, \partial v} = \frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial u\, \partial v}\cdot \sigma',
$$

as is readily shown. Moreover, from (59), (65), and (81) it follows that

$$
(83)\qquad
\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\alpha_2}{\sqrt{\rho}}\right) &= -\left(\frac{1}{\sqrt{\rho}}\frac{\partial \alpha}{\partial u} - \alpha \frac{\partial}{\partial u}\frac{1}{\sqrt{\rho}}\right),\\
\frac{\partial}{\partial v}\left(\frac{\alpha_2}{\sqrt{\rho}}\right) &= \left(\frac{1}{\sqrt{\rho}}\frac{\partial \alpha}{\partial v} - \alpha \frac{\partial}{\partial v}\frac{1}{\sqrt{\rho}}\right),
\end{aligned}
$$

which are of the same form as (73).

In like manner the function $\bar{\sigma}_1 = 1/\sqrt{\rho_1}$ determines a W-congruence for

* Rendiconti, loc. cit.

which Σ_1 is one of the focal surfaces. Since S_1 is the corresponding orthosurface to Σ_1, the functions α', β', γ' for the second focal sheet Σ' are given by

$$(84)\qquad \alpha' = \sqrt{\rho_1}\, x_1 = -\frac{\lambda}{\sqrt{\rho}\,\theta} x_1, \qquad \beta' = -\frac{\lambda}{\sqrt{\rho}\,\theta} y_1, \qquad \gamma' = -\frac{\lambda}{\sqrt{\rho}\,\theta} z_1.$$

It is easily found that the function

$$(85)\qquad \sigma_2' = \sqrt{\rho}\,\theta$$

is a solution of equation (82). If we substitute in equations (10) the expressions for x_1 and x which follow from equations (81) and (84), the resulting equations may be given the form

$$(86)\qquad \begin{aligned} \frac{\partial}{\partial u}(\sigma_2'\,\alpha') &= -\left(\sigma_2'\frac{\partial \alpha_2}{\partial u} - \alpha_2\frac{\partial \sigma_2'}{\partial u}\right), \\ \frac{\partial}{\partial v}(\sigma_2'\,\alpha') &= \left(\sigma_2'\frac{\partial \alpha_2}{\partial v} - \alpha_2\frac{\partial \sigma_2'}{\partial v}\right). \end{aligned}$$

Comparing these equations with (73), we observe that the lines joining corresponding points on the surfaces Σ_2 and Σ' form a W-congruence, for which these are the focal surfaces. It remains to find the relations between the coördinates of these surfaces.

From equations (65), (69), (71), (81), and (84), we obtain the relation

$$(87)\qquad \alpha' = \alpha + \frac{\lambda}{\rho\theta}(\alpha_1 - \alpha_2).$$

Analogous to equations (75) are the following:

$$(88)\qquad \begin{aligned} \xi_2 - \xi = \beta\gamma_2 - \gamma\beta_2, \qquad \xi' - \xi_1 &= \beta_1\gamma' - \gamma_1\beta', \\ \xi' - \xi_2 &= \beta_2\gamma' - \gamma_2\beta', \end{aligned}$$

which are consistent in view of (87). From these follows also

$$(89)\qquad \xi' = \xi + \frac{\lambda}{\rho\theta}(\beta_2\gamma_1 - \beta_1\gamma_2).$$

From the foregoing results we have

THEOREM 11. *When a pair of surfaces S and S_1 in the relation of a transformation K are known, it is possible to find by quadratures four surfaces Σ, Σ_1, Σ_2, Σ' such that the pairs $\Sigma\Sigma_1$, $\Sigma\Sigma_2$, $\Sigma_1\Sigma'$, $\Sigma_2\Sigma'$ are the focal surfaces of four W-congruences and the asymptotic lines on these surfaces correspond.*

We say that four such W-congruences form a *quatern*, which we indicate by $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$.

From this theorem we derive

THEOREM 12. *Four surfaces S, S_1, S_2, S' forming a quatern under transformations K determine twelve W-congruences forming six quaterns of W-congruences.*

Let $W(\Sigma, \Sigma_1, \Sigma_3, \Sigma')$ and $W(\Sigma, \Sigma_2, \Sigma_3, \Sigma'')$ be the quaterns determined by S and S_1 and by S and S_2 respectively in accordance with Theorem 11. The pair of surfaces S_1 and S' determine a quatern $W(\Sigma_1, \Sigma_4, \Sigma', \Sigma''')$. But from the nature of the theorem of permutability it follows that there is a quatern $W(\Sigma_2, \Sigma_4, \Sigma'', \Sigma''')$ determined by S_2 and S'. Furthermore, it is readily seen that these focal surfaces are so placed that there exist also the quaterns $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma_4)$ and $W(\Sigma_3, \Sigma', \Sigma'', \Sigma''')$.

7. TRANSFORMATIONS K DETERMINED BY A QUATERN OF W-CONGRUENCES

In this section we shall show that every quatern of W-congruences gives rise to four pairs of surfaces in the relation of transformations K. In doing so we make use of the results of Bianchi* concerning the determination of such quaterns.

Let Σ be a surface referred to its asymptotic lines and defined by equations (66), where α, β, γ are linearly independent solutions of an equation of the form

$$\frac{\partial^2 \sigma}{\partial u\, \partial v} = M\sigma, \tag{90}$$

M being a function of u and v in general. If σ_1 and σ_2 are two independent solutions of this equation, the functions α_1, β_1, γ_1 and α_2, β_2, γ_2, given by quadratures of the form (73), determine two surfaces Σ_1 and Σ_2 whose cartesian coördinates ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 are expressible in the form (75). Moreover, the functions α_i, β_i, γ_i $(i = 1, 2)$ satisfy the equations

$$\frac{\partial^2 \bar{\sigma}_i}{\partial u\, \partial v} = \sigma_i \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_i}\right)\bar{\sigma}_i \qquad (i = 1, 2). \tag{91}$$

In accordance with the theorem of Moutard these equations are satisfied also by the functions σ_1' and σ_2', defined by

$$\frac{\partial}{\partial u}(\sigma_1 \sigma_1') = -\sigma_1^2 \frac{\partial}{\partial u}\left(\frac{\sigma_2}{\sigma_1}\right), \qquad \frac{\partial}{\partial v}(\sigma_1 \sigma_1') = \sigma_1^2 \frac{\partial}{\partial v}\left(\frac{\sigma_2}{\sigma_1}\right), \tag{92}$$

$$\frac{\partial}{\partial u}(\sigma_2 \sigma_2') = -\sigma_2^2 \frac{\partial}{\partial u}\left(\frac{\sigma_1}{\sigma_2}\right), \qquad \frac{\partial}{\partial v}(\sigma_2 \sigma_2') = \sigma_2^2 \frac{\partial}{\partial v}\left(\frac{\sigma_1}{\sigma_2}\right). \tag{93}$$

In the subsequent discussion we assume that the constants of integration are so chosen that

$$\sigma_2 \sigma_2' = -\sigma_1 \sigma_1', \tag{94}$$

* *Lezioni di Geometria Differenziale*, Pisa (1903), vol. 2, pp. 71–74.

in which case we have

$$\sigma_1' \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_1'}\right) = \sigma_2' \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_2'}\right). \tag{95}$$

Bianchi has shown that the functions α', β', γ', defined by

$$\begin{gathered} \alpha' = \alpha + t(\alpha_1 - \alpha_2), \qquad \beta' = \beta + t(\beta_1 - \beta_2), \\ \gamma' = \gamma + t(\gamma_1 - \gamma_2), \end{gathered} \tag{96}$$

where

$$t = -\frac{\sigma_2}{\sigma_1'} = \frac{\sigma_1}{\sigma_2'}, \tag{97}$$

satisfy equations (86), also the equations

$$\begin{gathered} \frac{\partial}{\partial u}(\sigma_1' \alpha') = -\left(\sigma_1' \frac{\partial \alpha_1}{\partial u} - \alpha_1 \frac{\partial \sigma_1'}{\partial u}\right), \\ \frac{\partial}{\partial v}(\sigma_1' \alpha') = \left(\sigma_1' \frac{\partial \alpha_1}{\partial v} - \alpha_1 \frac{\partial \sigma_1'}{\partial v}\right), \end{gathered} \tag{98}$$

and analogous ones for β' and γ', and finally also

$$\frac{\partial^2 \theta'}{\partial u\, \partial v} = M' \theta', \tag{99}$$

where M' is the function equal to either member of (95).

These functions α', β', γ' determine a surface Σ', whose coördinates are given by

$$\begin{gathered} \xi' = \xi + t(\beta_2 \gamma_1 - \beta_1 \gamma_2), \qquad \eta' = \eta + t(\gamma_2 \alpha_1 - \gamma_1 \alpha_2), \\ \zeta' = \zeta + t(\alpha_2 \beta_1 - \alpha_1 \beta_2), \end{gathered} \tag{100}$$

such that Σ_1 and Σ' are the focal surfaces of a W-congruence and likewise Σ_2 and Σ' of another W-congruence. Hence we have a quatern of W-congruences $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$.

We suppose that we have a quatern $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$ defined in this general way and we introduce three functions ρ, λ, θ, defined by the equations

$$\sigma_1 = \lambda/\sqrt{\rho}, \qquad \sigma_2 = 1/\sqrt{\rho}, \qquad \sigma_2' = \sqrt{\rho}\,\theta. \tag{101}$$

In terms of these functions equation (90) can be given the form (62) and equations (93) reduce to (8). In accordance with (94) we have

$$\sigma_1' = -\theta\sqrt{\rho}/\lambda = 1/\sqrt{\rho_1}, \tag{102}$$

the function $\sqrt{\rho_1}$ being defined by this equation.

In accordance with the general theory of W-congruences and infinitesimal

deformation of surfaces, the surfaces S and S_0 whose coördinates x, y, z and x_0, y_0, z_0 are given by (81) and (65) are ortho-surfaces of Σ and Σ_2 respectively. In like manner the surfaces $\bar{S}$ and $\bar{S}_0$ whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$ are of the form (77) and (78) are ortho-surfaces of Σ and Σ_1; and S_1 and S_{10}, whose coördinates x_1, y_1, z_1 and x_{10}, y_{10}, z_{10} are given by (84) and (71), are ortho-surfaces of Σ_1 and Σ'. Also the quantities $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ and $\bar{x}_{10}$, $\bar{y}_{10}$, $\bar{z}_{10}$ which are of the form

$$\bar{x}_1 = \sqrt{\rho}\,\theta\alpha', \qquad \bar{x}_{10} = \alpha_2/\sqrt{\rho}\,\theta \tag{103}$$

are the coördinates of ortho-surfaces $\bar{S}_1$ and $\bar{S}_{10}$ of Σ_2 and Σ' respectively.

When the values of α_2 and α' given by (81) and (84) are substituted in (86), we obtain equations (4). Hence the surfaces S and S_1 are in the relation of a transformation K. In like manner it can be shown that each pair of surfaces $S\bar{S}_1$, $\bar{S}_0\,\bar{S}_{10}$, $S_0\,S_{10}$ are in the relation of a transformation K. Moreover, it follows from Theorem 10 that each of the above four pairs determines a pair of surfaces referred to a conjugate system with equal tangential invariants which are in the relation of a transformation Ω, and the spherical representation of this conjugate system on any one of these surfaces is the same as of the asymptotic lines on one of the surfaces Σ, Σ_1, Σ_2, Σ'. Hence we have

THEOREM 13. *If Σ, Σ_1, Σ_2, Σ' are the focal surfaces of a quatern of W-congruences, the eight associated ortho-surfaces of the former can be arranged into four pairs such that the surfaces of each pair are in the relation of a transformation K; moreover, each pair determines the quatern of W-congruences; also each pair determines two other surfaces which are in the relation of a transformation Ω.*

8. EQUATIONS OF A TRANSFORMATION K IN ANOTHER FORM

In applying the transformations K to particular types of surfaces whose transforms are to be of the same type, it is frequently advisable to have the equations of the transformation in another form, which we will now establish.

Let S be a surface referred to a conjugate system with equal point invariants, and let X, Y, Z; X_1, Y_1, Z_1 and X_2, Y_2, Z_2, denote the direction-cosines of the normal to S and of the bisectors of the angles between the coördinate lines. If one of these angles be denoted by 2ω, we have

$$\frac{\partial x}{\partial u} = \sqrt{E}\,(\cos\omega X_1 - \sin\omega X_2), \qquad \frac{\partial x}{\partial v} = \sqrt{G}\,(\cos\omega X_1 + \sin\omega X_2), \tag{104}$$

where E, F, G denote the coefficients of the linear element of S. Since

$F = \sqrt{EG}\cos 2\omega$, we have

$$
(105)\qquad
\begin{aligned}
\frac{\partial X}{\partial u} &= -\frac{D}{\sqrt{E}}\frac{1}{\sin 2\omega}(\sin\omega\, X_1 - \cos\omega\, X_2),\\
\frac{\partial X}{\partial v} &= \frac{D''}{\sqrt{G}}\frac{1}{\sin 2\omega}(\sin\omega\, X_1 + \cos\omega\, X_2),
\end{aligned}
$$

where D and D'' denote the fundamental coefficients of the second order for S.*

If we compare equation (1) with the Gauss equations for S†, we observe that

$$
(106)\qquad \left\{{12 \atop 1}\right\} = -\frac{\partial \log\sqrt{\rho}}{\partial v}, \qquad \left\{{12 \atop 2}\right\} = -\frac{\partial \log\sqrt{\rho}}{\partial u},
$$

the Christoffel symbols $\left\{{rs \atop t}\right\}$ being formed with respect to the linear element of S. From the definition of these symbols‡ it follows that equations (106) lead to the identities

$$
(107)\qquad
\begin{aligned}
\frac{\partial\sqrt{E}}{\partial v} &= -\sqrt{E}\frac{\partial\log\sqrt{\rho}}{\partial v} - \sqrt{G}\cos 2\omega\frac{\partial\log\sqrt{\rho}}{\partial u},\\
\frac{\partial\sqrt{G}}{\partial u} &= -\sqrt{G}\frac{\partial\log\sqrt{\rho}}{\partial u} - \sqrt{E}\cos 2\omega\frac{\partial\log\sqrt{\rho}}{\partial v}.
\end{aligned}
$$

The expressions for the other Christoffel symbols are as follows:

$$
(108)\qquad
\begin{aligned}
\left\{{11 \atop 1}\right\} &= \frac{\partial\log\sqrt{E}}{\partial u} + 2\cot 2\omega\frac{\partial\omega}{\partial u} - \sqrt{\frac{E}{G}}\cos 2\omega\frac{\partial\log\sqrt{\rho}}{\partial v},\\
\left\{{11 \atop 2}\right\} &= \frac{E}{G}\frac{\partial\log\sqrt{\rho}}{\partial v} - 2\sqrt{\frac{E}{G}}\frac{1}{\sin 2\omega}\frac{\partial\omega}{\partial u},\\
\left\{{22 \atop 1}\right\} &= \frac{G}{E}\frac{\partial\log\sqrt{\rho}}{\partial u} - 2\sqrt{\frac{G}{E}}\frac{1}{\sin 2\omega}\frac{\partial\omega}{\partial v},\\
\left\{{22 \atop 2}\right\} &= \frac{\partial\log\sqrt{G}}{\partial v} + 2\cot 2\omega\frac{\partial\omega}{\partial v} - \sqrt{\frac{G}{E}}\cos 2\omega\frac{\partial\log\sqrt{\rho}}{\partial u}.
\end{aligned}
$$

If equations (104) be differentiated and use be made of equation (1) and

* E., p. 116.
† E., p. 154.
‡ E., p. 153.

of the Gauss equations*

$$
(109)\qquad
\begin{aligned}
\frac{\partial^2 x}{\partial u^2} &= \left\{ {11 \atop 1} \right\} \frac{\partial x}{\partial u} + \left\{ {11 \atop 2} \right\} \frac{\partial x}{\partial v} + DX, \\
\frac{\partial^2 x}{\partial v^2} &= \left\{ {22 \atop 1} \right\} \frac{\partial x}{\partial u} + \left\{ {22 \atop 2} \right\} \frac{\partial x}{\partial v} + D'' X,
\end{aligned}
$$

we obtain

$$
(110)\qquad
\begin{aligned}
\frac{\partial X_1}{\partial u} &= \frac{D}{2\sqrt{E}\cos\omega} X + \left(\sqrt{\frac{E}{G}} \frac{\partial \log \sqrt{\rho}}{\partial v} \sin 2\omega - \frac{\partial \omega}{\partial u} \right) X_2, \\
\frac{\partial X_1}{\partial v} &= \frac{D''}{2\sqrt{G}\cos\omega} X - \left(\sqrt{\frac{G}{E}} \frac{\partial \log \sqrt{\rho}}{\partial u} \sin 2\omega - \frac{\partial \omega}{\partial v} \right) X_2, \\
\frac{\partial X_2}{\partial u} &= \frac{-D}{2\sqrt{E}\sin\omega} X - \left(\sqrt{\frac{E}{G}} \frac{\partial \log \sqrt{\rho}}{\partial v} \sin 2\omega - \frac{\partial \omega}{\partial u} \right) X_1, \\
\frac{\partial X_2}{\partial v} &= \frac{D''}{2\sqrt{G}\sin\omega} X + \left(\sqrt{\frac{G}{E}} \frac{\partial \log \sqrt{\rho}}{\partial u} \sin 2\omega - \frac{\partial \omega}{\partial v} \right) X_1.
\end{aligned}
$$

The conditions of integrability of these equations are satisfied in consequence of the Gauss and Codazzi equations† which in the present case can be given the form

$$
(111)\qquad
\begin{aligned}
\frac{DD''}{\sqrt{EG}} \frac{1}{\sin 2\omega} = \frac{\partial}{\partial v} &\left(\sqrt{\frac{E}{G}} \sin 2\omega \frac{\partial \log \sqrt{\rho}}{\partial v} \right) \\
&+ \frac{\partial}{\partial u} \left(\sqrt{\frac{G}{E}} \sin 2\omega \frac{\partial \log \sqrt{\rho}}{\partial u} \right),
\end{aligned}
$$

and

$$
(112)\qquad
\begin{aligned}
\frac{\partial D}{\partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v} D + \left(\frac{E}{G} \frac{\partial \log \sqrt{\rho}}{\partial v} - 2\sqrt{\frac{E}{G}} \frac{1}{\sin 2\omega} \frac{\partial \omega}{\partial u} \right) D'' &= 0, \\
\frac{\partial D''}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u} D'' + \left(\frac{G}{E} \frac{\partial \log \sqrt{\rho}}{\partial u} - 2\sqrt{\frac{G}{E}} \frac{1}{\sin 2\omega} \frac{\partial \omega}{\partial v} \right) D &= 0.
\end{aligned}
$$

We introduce three functions a, b, w by means of equations of the form

$$
(113)\qquad x_1 - x = \frac{1}{\lambda} (aX_1 + bX_2 + wX),
$$

where x_1, y_1, z_1 are the cartesian coördinates of a surface S_1 in the relation of a transformation K with S and λ is the corresponding function defined by

* E., p. 154.

† E., p. 155.

equations (8). In consequence of (113), the fundamental equations (10) can be written

$$
(114)\quad
\begin{aligned}
\frac{\partial x_1}{\partial u} &= \frac{\rho}{\lambda^2}\left[\left(a\frac{\partial\theta}{\partial u} + \theta\lambda\cos\omega\sqrt{E}\right)X_1 \right.\\
&\qquad\qquad \left. + \left(b\frac{\partial\theta}{\partial u} - \theta\lambda\sin\omega\sqrt{E}\right)X_2 + w\frac{\partial\theta}{\partial u}X\right],\\
\frac{\partial x_1}{\partial v} &= -\frac{\rho}{\lambda^2}\left[\left(a\frac{\partial\theta}{\partial v} + \theta\lambda\cos\omega\sqrt{G}\right)X_1 \right.\\
&\qquad\qquad \left. + \left(b\frac{\partial\theta}{\partial v} + \theta\lambda\sin\omega\sqrt{G}\right)X_2 + w\frac{\partial\theta}{\partial v}X\right].
\end{aligned}
$$

When the expression (113) for x_1 is substituted in these equations, we obtain two equations of the form

$$A_1 X_1 + B_1 X_2 + C_1 X = 0, \qquad A_2 X_1 + B_2 X_2 + C_2 X = 0,$$

where $A_1, B_1, \cdots, C_2$ are determinate functions which must be zero, since these equations are satisfied also by the Y's and Z's. This gives the following six equations of condition:

$$
(115)\quad
\begin{aligned}
\frac{\partial a}{\partial u} &= (\rho\theta - \lambda)\sqrt{E}\cos\omega + b\left(\sqrt{\frac{E}{G}}\,\frac{\partial\log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right)\\
&\qquad\qquad + \frac{wD}{2\sqrt{E}\cos\omega},\\
\frac{\partial a}{\partial v} &= -(\rho\theta + \lambda)\sqrt{G}\cos\omega - b\left(\sqrt{\frac{G}{E}}\,\frac{\partial\log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right)\\
&\qquad\qquad + \frac{wD''}{2\sqrt{G}\cos\omega},\\
\frac{\partial b}{\partial u} &= -(\rho\theta - \lambda)\sqrt{E}\sin\omega - a\left(\sqrt{\frac{E}{G}}\,\frac{\partial\log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right)\\
&\qquad\qquad - \frac{wD}{2\sqrt{E}\sin\omega},\\
\frac{\partial b}{\partial v} &= -(\rho\theta + \lambda)\sqrt{G}\sin\omega + a\left(\sqrt{\frac{G}{E}}\,\frac{\partial\log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right)\\
&\qquad\qquad + \frac{wD''}{2\sqrt{G}\sin\omega},\\
\frac{\partial w}{\partial u} &= -\frac{D}{2\sqrt{E}}\left(\frac{a}{\cos\omega} - \frac{b}{\sin\omega}\right), \qquad \frac{\partial w}{\partial v} = -\frac{D''}{2\sqrt{G}}\left(\frac{a}{\cos\omega} + \frac{b}{\sin\omega}\right).
\end{aligned}
$$

With the aid of equations (8), (107), (111), and (112) it can be shown that the conditions of integrability of these equations are satisfied.

If we put for the sake of brevity

$$T^2 = a^2 + b^2 + w^2, \tag{116}$$

it follows from (115) that

$$\begin{aligned} T\frac{\partial T}{\partial u} &= (\rho\theta - \lambda)\sqrt{E}\,(a\cos\omega - b\sin\omega), \\ T\frac{\partial T}{\partial v} &= -(\rho\theta + \lambda)\sqrt{G}\,(a\cos\omega + b\sin\omega). \end{aligned} \tag{117}$$

If X', Y', Z' denote the direction-cosines of the normal to S_1, we have from (114),

$$\begin{aligned} \Delta X' = &- w\sin\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v} + \sqrt{G}\frac{\partial\theta}{\partial u}\right)X_1 \\ &+ w\cos\omega\left(-\sqrt{E}\frac{\partial\theta}{\partial v} + \sqrt{G}\frac{\partial\theta}{\partial u}\right)X_2 \\ &+ X\left[a\sin\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v} + \sqrt{G}\frac{\partial\theta}{\partial u}\right)\right. \\ &\left. + b\cos\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v} - \sqrt{G}\frac{\partial\theta}{\partial u}\right) + \theta\lambda\sqrt{EG}\sin 2\omega\right], \end{aligned} \tag{118}$$

where Δ^2 is equal to the sum of the squares of the coefficients of X_1, X_2, and X.

9. When S and S_1 envelop a two-parameter family of spheres. Transformations D_m of an isothermic surface

As an example of the foregoing method we consider the case where S and S_1 constitute the envelope of a two parameter family of spheres. In this case we must have

$$\begin{aligned} x_1 - x = \frac{R\Delta}{\lambda}(X' - X), \qquad y_1 - y = \frac{R\Delta}{\lambda}(Y' - Y), \\ z_1 - z = \frac{R\Delta}{\lambda}(Z' - Z), \end{aligned} \tag{119}$$

where $R\Delta/\lambda$ is the radius of the spheres.

When the expressions for x_1 and X' given by (113) and (118) are substituted in the first of equations (119), and the coefficients of X_1, X_2, and X on

the two sides of the resulting identity are equated, we get

$$
(120)\qquad
\begin{aligned}
a &= -w \sin\omega\, R\left(\sqrt{E}\frac{\partial\theta}{\partial v} + \sqrt{G}\frac{\partial\theta}{\partial u}\right),\\
b &= -w \cos\omega\, R\left(\sqrt{E}\frac{\partial\theta}{\partial v} - \sqrt{G}\frac{\partial\theta}{\partial u}\right),\\
w &= \left[a \sin\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v} + \sqrt{G}\frac{\partial\theta}{\partial u}\right) + b\cos\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v} - \sqrt{G}\frac{\partial\theta}{\partial u}\right)\right.\\
&\qquad\qquad \left. + \theta\lambda\sqrt{EG}\sin 2\omega - \Delta\right] R.
\end{aligned}
$$

By means of the first two of these equations the last is equivalent to

$$
(121)\qquad a^2 + b^2 + w^2 = Rw\,(\theta\lambda\sqrt{EG}\sin 2\omega - \Delta).
$$

In consequence of the first two of equations (120) equation (118) can be written

$$
(122)\qquad \Delta X' = \frac{a}{R}X_1 + \frac{b}{R}X_2 - X\left[\frac{a^2+b^2}{Rw} - \theta\lambda\sqrt{EG}\sin 2\omega\right],
$$

whence it follows that

$$
(123)\qquad \Delta^2 = \frac{a^2}{R^2} + \frac{b^2}{R^2} + \left(\frac{a^2+b^2}{Rw} - \theta\lambda\sqrt{EG}\sin 2\omega\right)^2.
$$

Eliminating Δ from (121) and (123), we obtain

$$
(124)\qquad a^2 + b^2 + w^2 = 2w\theta\lambda\sqrt{EG}\sin 2w\, R,
$$

and consequently from (121) and (124) we have

$$
(125)\qquad \Delta = -\theta\lambda\sqrt{EG}\sin 2\omega.
$$

By means of (120), (121), (125) we obtain the following equations:

$$
(126)\qquad
\begin{aligned}
\sqrt{E}\,(a\cos\omega - b\sin\omega) &= -w\sin 2\omega\sqrt{EG}\frac{\partial\theta}{\partial u}R = -\frac{T^2}{2\theta\lambda}\frac{\partial\theta}{\partial u},\\
\sqrt{G}\,(a\cos\omega + b\sin\omega) &= -w\sin 2\omega\sqrt{EG}\frac{\partial\theta}{\partial v}R = -\frac{T^2}{2\theta\lambda}\frac{\partial\theta}{\partial v}.
\end{aligned}
$$

With the aid of these expressions and (8) equations (117) are reducible to

$$
\begin{aligned}
\frac{\partial \log T^2}{\partial u} &= \left(\frac{1}{\theta} - \frac{\rho}{\lambda}\right)\frac{\partial\theta}{\partial u} = \frac{\partial}{\partial u}\log(\lambda\theta),\\
\frac{\partial \log T^2}{\partial v} &= \left(\frac{1}{\theta} + \frac{\rho}{\lambda}\right)\frac{\partial\theta}{\partial v} = \frac{\partial}{\partial v}\log(\lambda\theta).
\end{aligned}
$$

Hence on integration we have

$$T^2 = 2c\lambda\theta, \tag{127}$$

where c denotes an arbitrary constant.

Substituting in (126) the value (127) of T^2, we get

$$\begin{aligned} \sqrt{E}\,(a\cos\omega - b\sin\omega) &= -c\frac{\partial\theta}{\partial u}, \\ \sqrt{G}\,(a\cos\omega + b\sin\omega) &= -c\frac{\partial\theta}{\partial v}. \end{aligned} \tag{128}$$

If these equations be differentiated with respect to v and u respectively, and in the reduction use be made of the preceding equations, we find

$$(\lambda + \rho\theta)\sqrt{G}\cos 2\omega = 0, \qquad (\lambda - \rho\theta)\sqrt{E}\cos 2\omega = 0.$$

Hence we must have $2\omega = \pi/2$, that is, the parametric curves are the lines of curvature. From (107) it follows that

$$\sqrt{E} = V/\sqrt{\rho}, \qquad \sqrt{G} = U/\sqrt{\rho}, \tag{129}$$

where U and V are functions of u and v alone respectively, and consequently S is an isothermic surface. As the relation between S and S_1 is reciprocal, it follows that S_1 also is isothermic. We shall show that there exist pairs of isothermic surfaces satisfying these equations.

Suppose that S is an isothermic surface. The parameters of its lines of curvature can be chosen so that

$$\sqrt{E} = \sqrt{G} = 1/\sqrt{\rho} = e^{\phi}, \tag{130}$$

where ϕ is a function thus defined. Now in the above equations

$$\sin\omega = \cos\omega = 1/\sqrt{2}, \qquad D = e^{2\phi}/\rho_1, \qquad D'' = e^{2\phi}/\rho_2,$$

where ρ_1 and ρ_2 are the principal radii of curvature of S. If we put

$$\alpha = \frac{a-b}{\sqrt{2}}, \qquad \beta = \frac{a+b}{\sqrt{2}}, \qquad c = -\frac{1}{m}, \tag{131}$$

and replace θ and λ by $m\theta$ and $m\lambda$, which evidently is consistent with equations (8), equations (115), (128), and (8) are equivalent to

$$\frac{\partial\alpha}{\partial u} = -m\lambda e^{\phi} + me^{-\phi}\theta - \frac{\partial\phi}{\partial v}\beta + \frac{we^{\phi}}{\rho_1}, \qquad \frac{\partial\alpha}{\partial v} = \beta\frac{\partial\phi}{\partial u},$$

$$
(132)\qquad
\begin{aligned}
&\frac{\partial\beta}{\partial u} = \alpha\frac{\partial\phi}{\partial v}, \qquad \frac{\partial\beta}{\partial v} = -m\lambda e^{\phi} - me^{-\phi}\,\theta - \frac{\partial\phi}{\partial u}\alpha + \frac{we^{\phi}}{\rho_2},\\
&\frac{\partial w}{\partial u} = -\alpha\frac{e^{\phi}}{\rho_1}, \qquad \frac{\partial w}{\partial v} = -\beta\frac{e^{\phi}}{\rho_2},\\
&\frac{\partial\theta}{\partial u} = e^{\phi}\,\alpha, \qquad \frac{\partial\theta}{\partial v} = e^{\phi}\,\beta, \qquad \frac{\partial\lambda}{\partial u} = -e^{-\phi}\,\alpha, \qquad \frac{\partial\lambda}{\partial v} = e^{-\phi}\,\beta.
\end{aligned}
$$

Furthermore equation (127) becomes

$$
(133)\qquad \alpha^2 + \beta^2 + w^2 + 2\lambda\theta m = 0.
$$

It is readily shown that the first derivatives of the left hand member of equation (133) are zero, and consequently by choosing the initial values properly we can find solutions of equations (132) satisfying (133). It should be mentioned that all the conditions of integrability of (132) are easily shown to be satisfied.

If in the subsequent discussion X_1, Y_1, Z_1 and X_2, Y_2, Z_2 denote the direction-cosines of the tangents to the curves $v =$ const. and $u =$ const. respectively, it is necessary and sufficient to replace X_1 and X_2 in the foregoing formulas by $(X_1 + X_2)/\sqrt{2}$ and $(-X_1 + X_2)/\sqrt{2}$ respectively, as is evident from (104).

With these changes equations (113) and (114) become

$$
(134)\qquad x_1 - x = \frac{1}{\lambda m}(\alpha X_1 + \beta X_2 + wX)
$$

and

$$
(135)\qquad
\begin{aligned}
\frac{\partial x_1}{\partial u} &= \frac{e^{-\phi}}{\lambda^2 m}[(\alpha^2 + \theta\lambda m)X_1 + \alpha\beta X_2 + \alpha wX],\\
\frac{\partial x_1}{\partial v} &= -\frac{e^{-\phi}}{\lambda^2 m}[\alpha\beta X_1 + (\beta^2 + \theta\lambda m)X_2 + \beta wX].
\end{aligned}
$$

From these equations we have for the fundamental coefficients E_1, F_1, G_1, of the surface S_1 the expressions

$$
(136)\qquad E_1 = G_1 = \frac{e^{-2\phi}\,\theta^2}{\lambda^2}, \qquad F_1 = 0,
$$

which proves that S_1 also is isothermic.

For this case the direction-cosines X', Y', Z' of the normal to S_1, as given for the general case by (118), are of the form

$$
(137)\qquad X' = \frac{1}{\theta\lambda m}[(w^2 + \theta\lambda m)X + \alpha wX_1 + \beta wX_2].
$$

With the aid of this expression equation (134) may be written

$$x_1 + \frac{\theta}{w} X' = x + \frac{\theta}{w} X, \tag{138}$$

which shows that θ/w is the radius of the sphere tangent to S and S_1.

However, equations (132)–(138) are equivalent to those found by entirely different processes by Darboux* for the transformation of isothermic surfaces which he discovered and to which Bianchi has given the name *transformations* D_m.

In consequence of the foregoing results we have

THEOREM 14. *If two surfaces S and S_1 in the relation of a transformation K constitute the envelope of a two-parameter family of spheres, the surfaces are isothermic and S_1 is obtained from S by a transformation D_m.*

In accordance with (136) we define a function ϕ_1 by

$$e^{\phi_1} = \frac{e^{-\phi}\,\theta}{\lambda}. \tag{139}$$

If X_1', Y_1', Z_1' and X_2', Y_2', Z_2' denote the direction-cosines of the tangents to the parametric curves on S_1, it follows from (135) and (137) that

$$\begin{aligned} X_1' &= -\frac{1}{m\lambda\theta}[(\alpha^2 + \theta\lambda m) X_1 + \alpha\beta X_2 + \alpha w X], \\ X_2' &= \frac{1}{m\lambda\theta}[\alpha\beta X_1 + (\beta^2 + \theta\lambda m) X_2 + \beta w X]. \end{aligned} \tag{140}$$

On replacing X_1 and X_2 in (105) and (110) by $(X_1 + X_2)/\sqrt{2}$ and $(-X_1 + X_2)/\sqrt{2}$ respectively, we have the first derivatives of X, X_1, and X_2. With the aid of these formulas we obtain from (140)

$$\begin{aligned} \frac{\partial X'}{\partial u} &= \left[\frac{e^{\phi}}{\rho_1} + \frac{w}{\theta\lambda}(\lambda e^{\phi} - \theta e^{-\phi})\right] X_1', \\ \frac{\partial X'}{\partial v} &= -\left[\frac{e^{\phi}}{\rho_2} + \frac{w}{\theta\lambda}(\lambda e^{\phi} + \theta e^{-\phi})\right] X_2'. \end{aligned} \tag{141}$$

From these expressions and (135) it follows that the principal radii ρ_1' and ρ_2' of S_1 are given by

$$\begin{aligned} \frac{e^{\phi_1}}{\rho_1'} &= -\left[\frac{e^{\phi}}{\rho_1} + \frac{w}{\theta\lambda}(\lambda e^{\phi} - \theta e^{-\phi})\right], \\ \frac{e^{\phi_1}}{\rho_2'} &= \left[\frac{e^{\phi}}{\rho_2} + \frac{w}{\theta\lambda}(\lambda e^{\phi} + \theta e^{-\phi})\right]. \end{aligned} \tag{142}$$

* Loc. cit., p. 502.

10. Theorem of permutability of transformations D_m

In this section we apply the results of § 3 to an immediate proof of the theorem of permutability of transformations D_m which Bianchi* established at length by direct processes. We are thus able to appreciate the underlying reason for the existence of this theorem.

In the first place we seek solutions α_1', β_1', w_1', θ_1', λ_1' of equations for S_1 analogous to (132) and (133), it being understood that two sets of solutions of the latter equations are known, which we denote by α_i, β_i, w_i, θ_i, λ_i for $i = 1, 2$. The functions θ_1' and λ_1' are given by equations (27) and (36). Hence if α_1' and β_1' are to satisfy

$$\frac{\partial \theta_1'}{\partial u} = e^{\phi_1}\alpha_1' = -\frac{e^{-\phi}\theta_1}{\lambda_1}\alpha_1', \qquad \frac{\partial \theta_1'}{\partial v} = -\frac{e^{-\phi}\theta_1}{\lambda_1}\beta_1', \tag{143}$$

we must have

$$\begin{gathered} \alpha_1' = -\frac{\alpha_1}{\theta_1}(\theta_1' + \theta_2) + \alpha_2, \qquad \beta_1' = \frac{\beta_1}{\theta_1}(\theta_1' + \theta_2) - \beta_2, \\ \lambda_1' = \frac{\lambda_1}{\theta_1}(\theta_1' + \theta_2) - \lambda_2. \end{gathered} \tag{144}$$

These values of α_1' and β_1' satisfy identically the equations

$$\frac{\partial \alpha_1'}{\partial v} = \beta_1'\frac{\partial \phi_1}{\partial u}, \qquad \frac{\partial \beta_1'}{\partial u} = \alpha_1'\frac{\partial \phi_1}{\partial v}, \tag{145}$$

and when substituted in

$$\frac{\partial \alpha_1'}{\partial u} = -m_1'\lambda_1' e^{\phi_1} + m_1' e^{-\phi_1}\theta_1' - \frac{\partial \phi_1}{\partial v}\beta_1' + w_1'\frac{e^{\phi_1}}{\rho_1'},$$

$$\frac{\partial \beta_1'}{\partial v} = -m_1'\lambda_1' e^{\phi_1} - m_1' e^{-\phi_1}\theta_1' - \frac{\partial \phi_1}{\partial u}\alpha_1' + w_1'\frac{e^{\phi_1}}{\rho_2'},$$

and use is made of the foregoing equations, we get

$$A\frac{e^{\phi_1}}{\rho_1'} + (\theta_1 e^{-\phi} - \lambda_1 e^{\phi})B + \lambda_1 e^{\phi}\left(m_2\frac{\lambda_2}{\lambda_1} + m_1'\frac{\theta_2}{\theta_1}\right) - \theta_1 e^{-\phi}\left(m_1'\frac{\lambda_2}{\lambda_1} + m_2\frac{\theta_2}{\theta_1}\right) = 0,$$

$$A\frac{e^{\phi_1}}{\rho_2'} + (\theta_1 e^{-\phi} + \lambda_1 e^{\phi})B - \lambda_1 e^{\phi}\left(m_2\frac{\lambda_2}{\lambda_1} + m_1'\frac{\theta_2}{\theta_1}\right) - \theta_1 e^{-\phi}\left(m_1'\frac{\lambda_2}{\lambda_1} + m_2\frac{\theta_2}{\theta_1}\right) = 0,$$

* Loc. cit., pp. 109–125.

where

$$
(146)\qquad
\begin{aligned}
A &= w_1' - \frac{w_1}{\theta_1}(\theta_1' + \theta_2) + w_2, \\
B &= \frac{1}{\theta_1}\left[(\theta_1' + \theta_2)(m_1' - m_1) - \frac{1}{\lambda_1}(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2)\right].
\end{aligned}
$$

If we put $m_1' = m_2$, these equations become

$$
(147)\qquad A\frac{e^{\phi_1}}{\rho_1'} + \frac{C}{\theta_1}(\theta_1 e^{-\phi} - \lambda_1 e^{\phi}) = 0, \qquad A\frac{e^{\phi_1}}{\rho_2'} + \frac{C}{\theta_1}(\theta_1 e^{-\phi} + \lambda_1 e^{\phi}) = 0,
$$

where

$$
(148)\qquad
\begin{aligned}
C = (\theta_1' + \theta_2)(m_2 - m_1) - \frac{1}{\lambda_1}(\alpha_1\alpha_2 + \beta_1\beta_2 + w_1 w_2) \\
- \frac{m_2}{\lambda_1}(\lambda_2\theta_1 + \lambda_1\theta_2).
\end{aligned}
$$

Equations (147) are satisfied when $A = C = 0$. From these equations and (144), we have

$$
(149)\qquad
\begin{aligned}
-\alpha_1' &= \kappa\left[\frac{\Phi_1\alpha_1}{\theta_1\lambda_1} + (m_1 - m_2)\alpha_2\right], \\
\beta_1' &= \kappa\left[\frac{\Phi_1\beta_1}{\theta_1\lambda_1} + (m_1 - m_2)\beta_2\right], \\
w_1' &= \kappa\left[\frac{\Phi_1 w_1}{\theta_1\lambda_1} + (m_1 - m_2)w_2\right], \\
\lambda_1' &= \kappa\left[\frac{\Phi_1}{\theta_1} + (m_1 - m_2)\lambda_2\right], \\
\theta_1' &= \kappa\left[\frac{\Phi_1}{\lambda_1} + (m_1 - m_2)\theta_2\right], \qquad \kappa = \frac{1}{m_2 - m_1},
\end{aligned}
$$

where we have put

$$
(150)\qquad \Phi_1 = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 + m_2(\lambda_2\theta_1 + \lambda_1\theta_2).
$$

It is readily shown that these values of θ_1' and w_1' satisfy equations (143) and also

$$
\frac{\partial w_1'}{\partial u} = -\alpha_1'\frac{e^{\phi_1}}{\rho_1'}, \qquad \frac{\partial w_1'}{\partial v} = -\beta_1'\frac{e^{\phi_1}}{\rho_2'},
$$

and consequently the functions (149) satisfy all the conditions necessary to the problem. Hence we have established with Bianchi

THEOREM 15. *If S is an isothermic surface, and two isothermic surfaces S_1 and S_2 are obtained from S by transformations D_{m1} and D_{m2}, there exists a*

transformation D'_{m2} of S_1 into a surface S' which can be obtained from S_2 also by a transformation D'_{m1}; moreover, S' can be found without quadratures.

When S is isothermic, the associate surface defined by equations (11) is the Christoffel transform. Hence, as an immediate consequence of Theorem 5, we have that transformations D_m are commutative with transformations of Christoffel, a result which Bianchi established by direct calculation.

11. Congruences (G) with more than one pair of surfaces in the relation of a transformation K

In this section we consider the consistency of equations (13) and (18). We write the former in the form of the second of equations (15) and express the condition of integrability. This gives

$$\frac{\partial M}{\partial v} = \frac{k}{N^2} k_{-1} + M \frac{\partial \log k}{\partial v} + 2M^2 - \frac{3Mk}{N}, \tag{151}$$

where k_{-1}, given by

$$k_{-1} = 2k - h - \frac{\partial^2 \log k}{\partial u \, \partial v}, \tag{152}$$

is an invariant of the point equation of the second focal surface of tangents to the curves $u =$ const. on S_0.

The condition of integrability of (151) and of the first of (15) can be reduced to

$$N^2 h \frac{\partial}{\partial v} \log \frac{h}{k} + \left(k_{-1} \frac{\partial k}{\partial u} - k \frac{\partial k_{-1}}{\partial u} \right) + 3N(MN - k)(k_{-1} - h) = 0. \tag{153}$$

Hence the necessary and sufficient condition that (151) and (15) be consistent identically is that

$$k_{-1} = h = (c + 1)k, \tag{154}$$

where c denotes an arbitrary constant. In this case it follows from (152) and (154) that

$$\frac{\partial^2 \log k}{\partial u \, \partial v} + 2ck = 0. \tag{155}$$

When $c = 0$, we have

$$h = k = UV, \tag{156}$$

where U and V are arbitrary functions of u and v respectively. When $c \neq 0$, the general integral of (155) is

$$k = \frac{U' V'}{c(1 + UV)^2}, \qquad h = (c + 1)k, \tag{157}$$

U and V being arbitrary functions of u and v respectively and a prime indicating differentiation with respect to the argument.

When $c = -1$, $h = 0$ and it is readily shown that in this case S and S_1 coincide. Consequently we exclude this value.

If we take the expressions (156) or (157) for h and k, and determine a and b in accordance with (14′) two arbitrary functions are introduced. Then each of the functions M and N as determined by equations (15), (18), and (151) involves an arbitrary constant, and σ and σ_1 are found by quadratures. In view of these results we have

THEOREM 16. *Whenever the point equation of a surface S_0 has invariants of the form* (156) *or* (157), *there exist on the tangents to the curves v = const. of S_0 an infinity of pairs of points which generate pairs of surfaces in the relations of transformations K; the same is true of the tangents to the curves u = const. of S_0, if $h = k = UV$.*

Returning to the case when equation (153) is not satisfied identically, we differentiate this equation respectively with respect to u and v, and replace the derivatives of M and N by their expressions from (15), (18), and (151). By means of (153) the two resulting equations are reducible to the forms

$$PN^2 + Q = 0, \qquad RN^2 + S = 0, \tag{158}$$

where P, Q, R, S are determinate functions involving h, k, k_{-1} and their derivatives. When we express the conditions that these equations be consistent, and that the expressions for N thus given and of M from (153) shall satisfy equations (15), (18), (151), we obtain the equations for h and k which determine the surfaces S_0. We are not interested in these equations but in the fact that from the form of equations (158) and (16) it follows that there is only one pair of surfaces S and S_1 associated with the given congruence. Hence we have

THEOREM 17. *If a congruence (G) possesses more than one pair of points which generate surfaces in the relation of a transformation K, it possesses an infinity of such pairs.*

PRINCETON UNIVERSITY,
April 14, 1914.

PROOF OF THE FINITENESS OF THE MODULAR COVARIANTS OF A SYSTEM OF BINARY FORMS AND COGREDIENT POINTS*

BY

FORBES BAGLEY WILEY

Introduction

1. **Relation to the literature.** The question of the finiteness of the modular covariants of a system of forms and cogredient points is one that arises naturally at this time in view of the fact that the modular covariants of a system of forms and no cogredient points have been shown recently to be finite,† where the term finite has the customary meaning attached to it in covariant theory, and still more recently a fundamental system of modular covariants for two cogredient binary points and no forms has been found.‡

The present paper is limited to the binary case, although the writer has made considerable advance in the case of ternary forms and one cogredient point.

2. **Definitions.** Let $f_1, \cdots, f_l$ be any system of forms (homogeneous polynomials) in the arbitrary variables x_{01}, x_{02} with undetermined integral coefficients taken modulo p, where p is any prime. Let $c_1, c_2, \cdots$ denote the coefficients arranged in any order. Under the transformation

$$T_0: \qquad x_{0i} = \sum_{j=1}^{2} t_{ij}\, x'_{0j} \pmod{p} \qquad (i = 1, 2)$$

with integral coefficients, let f_i become the form f'_i and let $c'_1, c'_2, \cdots$ denote the coefficients of $f'_1, \cdots, f'_l$ corresponding in position to $c_1, c_2, \cdots$, respectively. Let (x_{k1}, x_{k2}), $k = 1, \cdots, n$, be transformed cogrediently with x_{01}, x_{02}, that is by the transformation

$$T_k: \qquad x_{ki} = \sum_{j=1}^{2} t_{ij}\, x'_{kj} \pmod{p} \quad (i = 1, 2;\ k = 1, \cdots, n).$$

A rational integral function

$$K(c_1, c_2, \cdots; x_{01}, x_{11}, \cdots, x_{n1}; x_{02}, x_{12}, \cdots, x_{n2})$$

* Presented to the Society (Providence), September 8, 1914.

† Dickson, these Transactions, vol. 14 (1913), pp. 299–305.

‡ Krathwohl, American Journal of Mathematics, October, 1914.

with integral coefficients taken modulo p is called a modular covariant of the points (x_{k1}, x_{k2}), $k = 1, \cdots, n$, and the forms $f_1, \cdots, f_l$ if, for every transformation T_k $(k = 0, 1, \cdots, n)$,

$$K(c_1', c_2', \cdots; x_{01}', x_{11}', \cdots, x_{n1}'; x_{02}', x_{12}', \cdots, x_{n2}')$$

$$\equiv |t_{ij}|^{\mu} K(c_1, c_2, \cdots; x_{01}, x_{11}, \cdots, x_{n1}; x_{02}, x_{12}, \cdots, x_{n2}) \pmod p$$

holds identically in $c_1, c_2, \cdots; x_{01}', \cdots, x_{n1}'; x_{02}', \cdots, x_{n2}'$ after $x_{01}, \cdots, x_{n1}$, $x_{02}, \cdots, x_{n2}$ are eliminated by means of the congruences T_k $(k = 0, \cdots, n)$ and $c_1', c_2', \cdots$ are replaced by their expressions in terms of $c_1, c_2, \cdots$. The exponent μ is called the index of K.

In this paper we establish the

THEOREM. *The set of all modular covariants of any system of forms in x_{01}, x_{02} and the cogredient points (x_{k1}, x_{k2}), $k = 1, \cdots, n$, is finite in the sense that they are all rational integral functions, with integral coefficients taken modulo p, of a finite number of covariants of the set.*

Professor Dickson has obtained the special case of this theorem where there is a system of forms but no cogredient points.* We extend his method of proof to secure the proof of the theorem as above stated.

3. **Method of proof.** In bare outline the proof of the theorem consists of two parts. First it is shown that for any modular covariant K_1 of the forms and the points there exists a modular covariant K_1' that has the same leader (defined in § 6) and the same index as K_1 has and that is a polynomial in covariants from a definite finite set S of modular covariants of the forms and the points. By taking the difference of K_1 and K_1' we obtain a new covariant K_2 for which we can build up K_2' by the same method as that used on K_1 and by the use of the same set S. The second part of the proof consists in showing that the repetition of this process a finite number of times on any given covariant K_1 leads to a covariant K_m where K_m is such that $K_m - K_m' = 0$. Thus it follows that $K_1 = \Sigma_{i=1}^{i=m} K_i'$ and therefore is expressible as a polynomial in covariants from the set S.

To carry out this method in detail we establish in § 4 a lemma which is an extension of the lemma† used by Professor Dickson in the case he considered. In § 5 we prove the theorem for n not greater than 1 and in § 6 we give the proof where n is any finite positive integer, including the value $n = 0$.

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* Dickson, loc. cit. Professor Dickson secures his theorem for forms in any finite number of variables.

† Dickson, American Journal of Mathematics, vol. 35 (1913), pp. 414–415.

Finiteness of the Modular Covariants

4. **Lemma.** *Any set S of functions of the type*

$$F = x_0^{e_0} x_1^{e_1} \cdots x_n^{e_n} \quad (e\text{'s integers} \geqq 0;\ e_0 \geqq e_1, \cdots, e_0 \geqq e_n) \tag{1}$$

contains a finite number of functions $F_1, \cdots, F_k$ such that each function F of the set S can be expressed as a product $F_i f$ where f is of the form (1), *with the same property of the exponents, but is not necessarily in the set S.*

For the moment designate the possible relations

$$e_i - e_{i+1} \geqq 0, \qquad e_i - e_{i+1} < 0 \qquad (i = 1, \cdots, n)$$

between the exponents of F as decreases and increases, respectively. Since the e's of F occur in a definite order, indicated in (1), the order in which the increases and decreases of a given F occur is unique. The functions F of S can be classified into sets S_j $(j = 1, \cdots, t;\ t \leqq n!)$ such that functions from no two sets will have their increases and decreases occurring in the same order, while the increases and decreases of functions in the same set occur in the same order.

Let S_k be one of the sets S_j, say the one in which there are no increases. It is evident that the functions of any set S_j can be written in the form of the functions of S_k by a substitution which amounts to a rearrangement of the x's. Write the functions $F^{(k)}$ of S_k in the form

$$\prod_{j=0}^{n} \left[\prod_{i=0}^{j} x_i \right]^{e_j - e_{j+1}} \qquad (e_{n+1} = 0).$$

Make the substitutions

$$\prod_{i=0}^{j} x_i = y_j, \qquad e_j - e_{j+1} = \epsilon_j \qquad (j = 1, \cdots, n) \tag{2}$$

on the variables and the exponents of $F^{(k)}$, respectively. We thus have a set S_k of functions

$$\phi^{(k)} = \prod_{j=0}^{n} y_j^{\epsilon_j} \qquad (\epsilon\text{'s integers} \geqq 0).$$

There exists* a finite number of functions $\phi_i^{(k)}$ $(i = 1, \cdots, h)$ belonging to the set S_k such that any function $\phi^{(k)}$ of the set S_k may be written

$$\phi^{(k)} = \phi_i^{(k)} f,$$

where

$$f = \prod_{j=0}^{n} y_j^{a_j} \qquad (a\text{'s integers} \geqq 0) \tag{3}$$

and i has some one of the values $1, \cdots, h$. Transforming (3) by the inverse

* See reference last cited; also Gordan, Journal de Mathématiques, ser. 5, vol. 6 (1900), pp. 141–149.

of (2_1), we have

$$f = \prod_{j=0}^{n} x_j^{\sum_{i=j}^{n} a_i}.$$

The lemma thus holds for the set S_k and similarly for each one of the t sets S_j; hence it holds for the set S.

5. **One cogredient point.** We make use of the known* rational integral modular covariants

$$L_i = x_{i1}^p x_{i2} - x_{i1} x_{i2}^p \qquad (i = 0, 1),$$

$$Q_i = \frac{x_{i1}^{p^2} x_{i2} - x_{i1} x_{i1}^{p^2}}{L_i} = x_{i1}^{p^2-p} + x_{i2}(\ ) \qquad (i = 0, 1),$$

$$M = x_{01} x_{12} - x_{02} x_{11}.$$

The covariants L_i and M have the index -1, while the Q_i are absolute.†

As any covariant is the sum of covariants that are homogeneous in each pair of variables separately, it is sufficient for us to consider covariants of the type $K(x_{01}, x_{02}, x_{11}, x_{12})$ of total degree ω_0 in x_{01}, x_{02} and of total degree ω_1 in x_{11}, x_{12}. If one of the ω's is zero, we have the case which Professor Dickson considered.

We shall call such a covariant K a regular covariant of the forms and the cogredient point if it has neither of the x_{i2} $(i = 1, 2)$ as a factor, while we shall call it irregular if it has either one of these variables as a factor or both as factors. Any irregular covariant is the product of $L_0^s L_1^t$ $(s, t = 0, 1, \cdots;$ s and t not both zero) by a regular covariant, since‡ any covariant that has the factor x_{i2} has the factor L_i. We need, therefore, to consider only regular covariants.

Let K be such a covariant and let it be of total degree ω_i in the two variables x_{i1}, x_{i2} $(i = 0, 1)$. We write K in the form

$$K = x_{i1}^{\omega_i} \sum_{j=0}^{\omega_k} S_j x_{k1}^{\omega_k - j} x_{k2}^j + x_{i2}(\ ) \qquad (\omega_i \geqq \omega_k;\ i, k = 0, 1;\ i \neq k), \tag{4}$$

where at least one S_j is not zero. We rewrite (4) in the notation

$$K = S x_{i1}^A x_{k1}^B x_{k2}^C + \cdots \qquad (S = S_0,\ A = \omega_i,\ B + C = \omega_k), \tag{5}$$

where B is the highest exponent which x_{k1} carries in a term containing x_{i1}^A. Denote the set of all regular covariants (5) with a fixed leader S and

$$A \equiv a, \qquad B \equiv b, \qquad C \equiv c \qquad (\text{mod } P) \qquad (P = p^2 - p),$$

* Dickson, *The Madison Colloquium Lectures on Mathematics*, pp. 35, 37. M is an algebraic covariant.

† Of index $\mu = 0$.

‡ Dickson, *The Madison Colloquium Lectures*, p. 35.

where each of the numbers a, b, c is at least zero and is less than P, by $[S, a, b, c]_i$. The number of such sets is finite since a, b, c are limited in value and the number of seminvariant leaders S is finite. These leaders are finite because each S is a polynomial in the coefficients c_i of the forms and thus all exponents can be reduced to $p - 1$ or less by use of Fermat's theorem. All covariants belonging to the same set S have the same index μ. By means of the Lemma (§ 4), we may select from the set $[S, a, b, c]$ a finite number t of covariants

$$K_h = Sx_{i1}^{A_h} x_{k1}^{B_h} x_{k2}^{C_h} + \cdots + x_{i2}(\,) \qquad (h = 1, \cdots, t), \tag{6}$$

such that for any covariant (5) of the set $[S, a, b, c]$ it is true that

$$A \geqq A_h, \qquad B \geqq B_h, \qquad C \geqq C_h, \qquad A - A_h \geqq C - C_h$$
$$(\text{for some value} \leqq t \text{ of } h).$$

But A and A_h (and likewise for B and B_h, C and C_h) are congruent to a modulo P; thus any covariant of the set $[S, a, b, c]$ has the form

$$K = Sx_{i1}^{A_h+uP} x_{k1}^{B_h+vP} x_{k2}^{C_h+wP} + \cdots \qquad (u \geqq w),$$

where u, v, w are positive integers or zero and h is some one of the integers $1, \cdots, t$.

From the absolute covariants Q_i, Q_k, M^P, and the covariant K_h, we construct a covariant of the same index as K and having the same initial term as K, and see that the difference

$$K' = K - K_h Q_i^{u-w} Q_k^v M^{wP} + \cdots$$

will be a covariant either irregular or with the leader

$$S' x_{i1}^{A} x_{k1}^{B'} x_{k2}^{C'} \qquad (A \geqq B' + C',\ B' < B,\ C' > C).$$

If K' is an irregular covariant, take out the factors L_i that occur and call the quotient K''. If K' is regular, call it K''. Hence

$$K = \psi_1(L_1, L_2, Q_1, Q_2, M, K_h, K''),$$

where ψ_1 is a polynomial in its arguments.

Arrange K'' in the form (6) and repeat for it the reduction process used on the regular covariant K. Whenever the exponent B in one of the regular covariants, say $K^{(e)}$, is zero, the next step in the process gives an irregular covariant with the factor x_{i2}, thus allowing a reduction of the degree in x_{i1} and x_{i2} by the removal of the factor L_i. If we let K_h $(h = 1, \cdots, \tau)$ serve for all sets $[S, a, b, c]$ as K_h $(h = 1, \cdots, t)$ served for the particular one, we see that every covariant K may be expressed as

$$K = \psi_m(K_1, \cdots, K_\tau, L_1, L_2, Q_1, Q_2, M),$$

where ψ_m is a polynomial in its arguments and is linear in the $K_1, \cdots, K_\tau$.

6. **Case of n cogredient points.** We use the L_i and the Q_i of § 5 with $i = 1, \cdots, n$. In addition to these we use the known modular covariants

$$M_{ik,n} = x_{i1} x_{k2}^{p^n} - x_{i2} x_{k1}^{p^n} \quad (i, k = 0, \cdots, n; i \neq k).$$

As in § 5 so here also we need consider only those covariants of the forms $f_1, \cdots, f_l$ and the cogredient points (x_{k1}, x_{k2}), $k = 1, \cdots, n$, which are homogeneous in each pair of variables separately.

We shall call such a covariant a regular covariant if it has none of the x_{i2} $(i = 0, \cdots, n)$ as factors, while we shall call it irregular if it has one or more such factors. Since any irregular covariant is the product of

$$\prod_{i=0}^{n} L_i^{s_i} \qquad (s_i = 0, 1, \cdots; \text{ at least one } s_i \neq 0)$$

and a regular covariant, we need to consider only regular covariants.

Let K be such a covariant and let ω_i be the order of K in x_{i1}, x_{i2}. Each x_{i1} will occur in at least one term of K with an exponent equal to the order ω_i. There exists some ω_i, say ω_0 for convenience in notation,* such that $\omega_0 \geqq \omega_j$ $(j = 1, \cdots, n)$. Write K with those terms leading that contain the factor $x_0^{\omega_0}$, the order of these terms being such that the sums of the exponents of the factors x_{i1} in them occur in the order of the decreasing natural numbers. The order among themselves of those terms for which such sums are equal is arbitrary. Denote the first term of K so written as the leader. By the help of zero exponents where necessary, write the leader of K so that every variable except x_{02} appears in it notationally as a factor. Arrange these factors in the ascending order of their subscripts so that we have

$$K = S \prod_{i=0}^{n} x_{i1}^{A_i} \prod_{j=1}^{n} x_{j2}^{B_j} + \cdots \qquad (A_0 = \omega_0,\ A_i + B_i = \omega_i,\ \omega_0 \geqq \omega_i). \tag{7}$$

Denote by

$$[S, a_0, a_1, \cdots, a_n; b_1, b_2, \cdots, b_n] \tag{8}$$

the set of all covariants (7) with a given seminvariant leader S and with

$$A_i \equiv a_i, \qquad B_i \equiv b_i \pmod{X}; \qquad X = p^{n+1}(p-1),$$

where each a_i and each b_i is at least zero and less than X. The number of sets (8) is finite. All covariants (7) belonging to the same set (8) have the same index μ. By the Lemma (§ 4), there exists a finite number of covariants

$$K_h = S \prod_{i=0}^{n} x_{i1}^{A_{ih}} \prod_{j=1}^{n} x_{j2}^{B_{jh}} + \cdots \qquad (h = 1, \cdots, t) \tag{9}$$

belonging to the set (8) such that the leader of any covariant (7) that belongs

* It is only slightly more cumbersome to carry the argument through for ω_0.

to set (8) can be expressed as the product of the leader of some covariant of (9) by f, where f is of the form (7), with the same property of the exponents, but has no factor S. Thus we may write

$$K = S \prod_{i=0}^{n} x_{i1}^{A_{ih}+q_i X} \prod_{j=1}^{n} x_{j2}^{B_{jh}+q'_j X} + \cdots. \tag{10}$$

Therefore

$$f = \left(\prod_{i=0}^{n} x_{i1}^{q_i} \prod_{j=1}^{n} x_{j2}^{q'_j} \right)^X$$
$$(q_0 \geqq q_1, \cdots, q_0 \geqq q_n;\ q_0 \geqq q'_1, \cdots, q_0 \geqq q'_n).$$

By the aid of zero exponents where necessary, let each variable except x_{02} appear notationally as a factor in f. Arrange the factors x_{i2} of f in the order of decreasing exponents. The order among themselves of those factors x_{i2} whose exponents are equal is arbitrary. Arrange the x_{i1} factors in the same order as their corresponding x_{i2}'s. Write f thus arranged in the notation

$$f = \left(x_{01}^{q_0} \prod_{i=1}^{n} x_{h_i 1}^{q_{h_i}} \prod_{j=1}^{n} x_{h_j 2}^{q'_{h_j}} \right)^X.$$

This may be written

$$f = \Big[x_{01}^{q_0 - q'_{h1}} \prod_{i=1}^{n} x_{h_i 1}^{q_{h_i}} (x_{01}\, x_{h_1 2})^{q'_{h1} - q'_{h2}} (x_{01}\, x_{h_1 2}\, x_{h_2 2})^{q'_{h2} - q'_{h3}} \cdots (x_{01} \cdots x_{h_{n-1} 2})^{q_{h_{n-1}} - q'_{h_n}} (x_{01} \cdots x_{h_n 2})^{q'_{h_n}} \Big]^X. \tag{11}$$

We now build up the absolute covariants

$$P_r = Q_0^{p^n - r} \prod_{k=h_1}^{h_r} M_{0k,n}^{p^2 - p} + \cdots \qquad (1 \leqq r \leqq n),$$

which we may write in the form

$$P_r = \left[x_{01} \prod_{k=h_1}^{h_r} x_{h_k 2} \right]^X + \cdots.$$

From the absolute covariants Q_0, Q_{h_i}, P_r, and the covariant K_h, which is one of the covariants (9), we construct a covariant of the same index as K and having the same initial term as K so that it follows that the difference

$$K' = K - K_h\, Q_0^{(q_0 - q'_1)p^n} \prod_{i=1}^{n} Q_{h_i}^{q_{h_i} - p^n} \prod_{r=1}^{n} P_r^{q'_{h_j} - q'_{h_{j+1}}} \qquad (q'_{h_{n+1}} = 0)$$

will be a covariant either irregular or with a leader that can be written in the form (10). If K' is irregular, take out the factors L_i that occur and call the quotient K''. If K' is regular, call it K''. Thus we have

$$K = \psi(L_1, \cdots, L_n, Q_1, \cdots, Q_n, K_h, P_1, \cdots, P_l)$$

where h is one of the values $1, \cdots, t$ and ψ is a polynomial in its arguments. Arrange K'' in the form (10) and subject it to the reduction process used on the regular covariant K, thus obtaining a regular covariant $K^{(\mathrm{IV})}$. After a finite number of repetitions of this process, we obtain either an irregular covariant or a regular one, say $K^{(q)}$, for which the sum of the exponents of the factors x_{i1} $(i = 1, \cdots, n)$ in its leader is less than the like sum for the leader of K. After repeating the process a finite number of times from this point, we obtain either an irregular covariant or one, say $K^{(l)}$, for which the sum of the exponents of the x_{i1} $(i = 1, \cdots, n)$ factors is zero. The next repetition of the reduction process reduces the exponent of x_{01} and gives an irregular covariant with the factor L_0 which, when removed, still further reduces the degree of x_{01}.

If we let K_h $(h = 1, \cdots, \tau)$ serve for all sets (8) as K_h $(h = 1, \cdots, t)$ served for one of them, we see that any covariant of the binary forms $f_1, \cdots, f_l$ and the n cogredient points is a polynomial in $K_1, \cdots, K_\tau; L_1, \cdots, L_n; Q_1, \cdots, Q_n; P_1, \cdots, P_n$, linear in $K_1, \cdots, K_\tau$.

The covariants P_r are polynomials in Q_0 and $M_{0k,\,n}$. It is of interest to note that by means of the recursion formula

$$M_{0k,\,n} = Q_k^{p^{n-2}} M_{0k,\,n-1} - L_k^{(p-1)p^{n-2}} M_{0k,\,n-2} \qquad (n > 1),$$

we are able to express any $M_{0k,\,n}$ as a rational integral function of $M_{0k,\,1}$, $M_{0k,\,0}$, Q_k, and L_k, linear in the $M_{0k,\,1}$, $M_{0k,\,0}$. It follows that any covariant K of the binary forms and the n points may be written as a polynomial in $K_1, \cdots, K_\tau; L_1, \cdots, L_n; Q_1, \cdots, Q_n; M_{01,\,1}, \cdots, M_{0n,\,1}; M_{01,\,0}, \cdots, M_{0n,\,0}$, linear in the $K_1, \cdots, K_\tau$.

ON THE DEGREE OF CONVERGENCE OF STURM-LIOUVILLE SERIES*

BY

DUNHAM JACKSON

After the degree of convergence of Fourier's series has been studied to a certain extent, as has been done by various authors, including the present writer, it is natural to inquire whether the results obtained are capable of extension to other series of characteristic solutions of homogeneous linear differential equations. A theorem in this connection has already been published by Tamarkine.† It is the purpose of the present paper to develop a greater variety of facts relating to a much less general differential equation than that which Tamarkine treats. In discussing expansion problems connected with an equation of the second order, Kneser,‡ following Liouville,§ establishes relations from which it is at once apparent that the theorem of Lebesgue,‖ which states that the remainder after n terms in the Fourier's series for a function satisfying a Lipschitz condition does not exceed a constant multiple of $(\log n)/n$, applies equally well to the development of such a function in Sturm-Liouville series of the form under consideration. It is to be shown how these relations can be refined so as to make it possible to carry over other theorems of the same nature from the Fourier's series to the more general one. The theorems susceptible of this generalization include that of Picard* based on the hypothesis that the function developed has a kth derivative of limited variation, and one proved by the author† for a function having

* Presented to the Society, December 30, 1913.

† Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 345–382; see especially p. 368. In connection with this paper, though not with reference to the particular point mentioned here, see also, in the same journal, Birkhoff, vol. 36 (1913), pp. 115–126, and Tamarkine, vol. 37 (1914), pp. 376–378.

‡ Mathematische Annalen, vol. 58 (1904), pp. 81–147; see p. 127.

§ Journal de mathématiques pures et appliquées, vol. 2 (1837), pp. 418–436.

‖ Bulletin de la société mathématique de France, vol. 38 (1910), pp. 184–210; p. 201.

* *Traité d' Analyse*, vol. 1, chapter IX, § 12. Picard is not directly concerned with the degree of convergence of the series, but nothing more is needed in this case than the formulas which he gives for the order of magnitude of the coefficients. The hypothesis concerning the function also is not exactly that stated here, but is essentially the same for the purposes of the proof. When the Lipschitz condition is used a knowledge of the order of magnitude of the coefficients is not enough; see Lebesgue, loc. cit., pp. 192–195.

† These Transactions, vol. 13 (1912), pp. 491–515, Theorem V. This paper will be referred to as A.

a $(k-1)$th derivative that satisfies a Lipschitz condition. The case of a function having a finite second derivative, which was treated by Liouville,* is included in that of functions having a first derivative of limited variation. The method employed is intimately related with that by which Horn† obtains extended asymptotic expressions for solutions of the differential equation.

For the proofs relating to the higher degrees of convergence, additional hypotheses, beyond that of mere continuity, are made concerning the functions which appear as coefficients in the differential equation, and it is assumed that the function developed vanishes with a sufficient number of its derivatives at the ends of the interval. The latter requirement may seem unduly restrictive, but an examination of the facts in the simplest case, that of the cosine-series,‡ shows that a limitation of this nature is required for the truth of the conclusions, and is not due merely to inadequacy of the method of treatment.

In the last part of the paper attention is given to the problem of representing a function that has a $(k-1)$th derivative satisfying a Lipschitz condition by means of a linear combination of a finite number of characteristic functions, with a higher degree of approximation than that afforded by the sum of the corresponding terms in the Sturm-Liouville series itself. Such a representation is obtained by summing the series by a method which the author had previously applied to Fourier's series.

1. Preliminary Statements

The differential equation with which we shall deal is the following:

$$\frac{d^2U}{dx^2} + [\rho^2 - \lambda(x)]\,U = 0. \tag{1}$$

Here $\lambda(x)$ is a function which is assumed at the outset to be continuous, and will be subjected to further restrictions as occasion demands, but only when such restrictions are explicitly mentioned. The parameter ρ^2 is not restricted to positive nor even to real values. The familiar transformation§ which reduces a more general equation to this form gives so directly the interpretation of our results with reference to the general equation that it is unnecessary to dwell upon the latter. The boundary conditions are

$$\begin{aligned} U'(0) - h'\,U(0) &= 0, \\ U'(\pi) + H'\,U(\pi) &= 0, \end{aligned} \tag{2}$$

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* Journal de Mathématiques pures et appliquées, vol. 2 (1837) pp. 16–35; see especially the sentence beginning at the foot of p. 18. See also Kneser, loc. cit., pp. 121–123.

† Mathematische Annalen, vol. 52 (1899), pp. 271–292.

‡ It is the cosine-series, rather than the complete Fourier's series, which is the true prototype of the expansions considered in this article.

§ See, e. g., Kneser, loc. cit., pp. 116–117.

the interval over which the variable x is to range being taken for convenience as that from 0 to π. The numbers h' and H' are real constants, not restricted as to sign; the notation is the classical one. The limiting cases obtained by letting one or both of these constants become infinite, that is, by making the solution U vanish at one or both ends of the interval, while requiring a discussion differing in details from the one that is to be given, offer nothing essentially new that is not provided for in the paper of Kneser already cited, and will not be treated separately here.

The following facts concerning the characteristic numbers and solutions of the system consisting of the equation (1) and the boundary conditions (2) will be assumed as well known.

There are infinitely many real* values of ρ^2 for which the system has a real solution not identically zero. Only a finite number of these values can be negative, and they have no cluster-point in the finite plane. If ρ_n represents the positive square root of the nth of them in algebraic order of magnitude, when n is large enough so that the corresponding value of ρ^2 is positive, then

$$\rho_n = n + \epsilon_n, \tag{3}$$

where†

$$\epsilon_n = O\left(\frac{1}{n}\right). \tag{4}$$

Two linearly independent solutions can not correspond to the same value of ρ^2. If the arbitrary constant factor in the nth characteristic solution $U_n(x)$ is suitably determined, the solution satisfies the integral equation

$$U_n(x) = \cos \rho_n x + \frac{h' \sin \rho_n x}{\rho_n} + \frac{1}{\rho_n}\int_0^x \lambda(t)\, U_n(t) \sin \rho_n (x - t)\, dt, \tag{5}$$

when the corresponding value of ρ^2 is positive. The maximum value of $U_n(x)$ in the interval $(0, \pi)$ remains finite as n becomes infinite,‡ so that the third as well as the second term on the right-hand side is very small when n is very large.

We shall be interested in the degree of convergence of the series

$$\sum_{n=0}^{\infty} \alpha_n U_n(x), \tag{6}$$

where

$$\alpha_n = \frac{\int_0^\pi \phi(x)\, U_n(x)\, dx}{\int_0^\pi [U_n(x)]^2\, dx}, \tag{7}$$

* The fact that there are no complex characteristic values will not be used explicitly.

† This notation means that the absolute value of ϵ_n does not exceed a constant multiple of $1/n$.

‡ This is readily deduced from (5) itself.

and $\phi(x)$ is a continuous function on which further restrictions will be imposed later. It will be assumed as known* that if this series converges uniformly, its value must necessarily be $\phi(x)$.

With this review of facts that are presupposed,† we go on to the detailed discussion of the problem in hand.

2. Degree of Convergence of the Series

The study of the degree of convergence of the series (6) involves an examination of the integral

$$\int_0^\pi \phi(x)\, U_n(x)\, dx, \tag{8}$$

which appears in the numerator of the expression (7) for the general coefficient. In evaluating this integral approximately, use will be made of the formula (5) for $U_n(x)$; but this identity will be written at greater length by substituting the whole right-hand side for the function U_n under the sign of integration, and repeating this process a number of times. It will be well to anticipate the discussion of the whole integral (8) by three lemmas relating to the individual terms that will be obtained.

Lemma I. *If $\phi(x)$ has a continuous‡ kth derivative of limited variation in the interval $0 \leqq x \leqq \pi$, and if*

$$\begin{aligned} \phi(0) = \phi'(0) = \cdots &= \phi^{(k-1)}(0) \\ = \phi(\pi) = \phi'(\pi) = \cdots &= \phi^{(k-1)}(\pi) = 0, \end{aligned} \tag{9}$$

then§

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = O\left(\frac{1}{n^{k+1}}\right), \tag{10}$$

and

$$\int_0^\pi \phi(x) \sin \rho_n x\, dx = O\left(\frac{1}{n^{k+1}}\right). \tag{11}$$

Consider first the integral containing $\cos \rho_n x$ in the integrand. Since ϕ vanishes at both ends of the interval, integration by parts gives at first

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = -\frac{1}{\rho_n} \int_0^\pi \phi'(x) \sin \rho_n x\, dx,$$

and after a sufficient number of repetitions,

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = \frac{1}{\rho_n^k} \int_0^\pi \phi^{(k)}(x) \cos\left(\rho_n x + \frac{k\pi}{2}\right) dx. \tag{12}$$

* See, e. g., Kneser, loc. cit., pp. 109–116, 123–124.

† For a concise exposition of the properties of functions of limited variation that will be used, see, e. g., E. B. Wilson, *Advanced Calculus*, pp. 309, 310.

‡ The assumption of continuity is not necessary, but is made for the sake of convenience.

§ Cf. Picard, loc. cit., where the corresponding proof is given for the case of the Fourier's series.

As the hypothesis is that $\phi^{(k)}(x)$ is of limited variation, let

$$\phi^{(k)}(x) = \phi_1(x) - \phi_2(x), \tag{13}$$

where ϕ_1 and ϕ_2 are positive or zero, continuous, and monotone increasing. By the second law of the mean,

$$\int_0^\pi \phi^{(k)}(x)\cos\left(\rho_n x + \frac{k\pi}{2}\right)dx = \phi_1(\pi)\int_\xi^\pi \cos\left(\rho_n x + \frac{k\pi}{2}\right)dx$$
$$- \phi_2(\pi)\int_\eta^\pi \cos\left(\rho_n x + \frac{k\pi}{2}\right)dx,$$

where ξ and η are numbers in the interval $(0, \pi)$. The absolute value of the expression just written down does not exceed

$$\frac{2}{\rho_n}[\phi_1(\pi) + \phi_2(\pi)].$$

Since, by (3) and (4), ρ_n is of the same order of magnitude as n, the relation (10) follows at once. The proof of (11) is precisely similar.

If $\phi^{(k)}(0) = 0$, it may be assumed that $\phi_1(0) = \phi_2(0) = 0$, and that the total variation v of $\phi^{(k)}(x)$ is equal to $\phi_1(\pi) + \phi_2(\pi)$. In this case, either of the integrals in the lemma is at most equal to $2v/\rho_n^{k+1}$. It will be convenient to use the letter c a number of times as a general notation for a positive constant, sometimes one and sometimes another, which is independent of n, x, and the function ϕ, though it may depend, for example, on the coefficient λ in the differential equation and on the coefficients h' and H' in the boundary conditions. With this convention, we may say that if $\phi^{(k)}(0) = 0$ each of the integrals remains inferior in absolute value to* cv/n^{k+1}.

Another interesting remark of a special nature is that when $k = 1$ the hypothesis (9), reducing here to

$$\phi(0) = \phi(\pi) = 0,$$

may be abandoned as far as the relation (10) is concerned. For

$$\int_0^\pi \phi(x)\cos\rho_n x\,dx = \frac{1}{\rho_n}[\phi(x)\sin\rho_n x]_0^\pi - \frac{1}{\rho_n}\int_0^\pi \phi'(x)\sin\rho_n x\,dx,$$

and $\sin\rho_n x$ vanishes when $x = 0$, while

$$\sin\rho_n\pi = \pm\sin\epsilon_n\pi = O\left(\frac{1}{n}\right),$$

by (4).

* This is more precise than the original statement of the lemma, because there it was not specified how the constant multiplier implied in the O-notation depends on the function ϕ; here we see that it can be taken proportional to the total variation of $\phi^{(k)}$.

LEMMA II. *If $\phi(x)$ satisfies the conditions of Lemma I, and if $\lambda(x)$ has a continuous $(k-1)$th derivative with limited variation* in $0 \leqq x \leqq \pi$, then*

$$\int_0^\pi \phi(x) \int_0^x \lambda(t_1) \cos \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1\, dx = O\left(\frac{1}{n^{k+1}}\right), \tag{14}$$

$$\int_0^\pi \phi(x) \int_0^x \lambda(t_1) \sin \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1\, dx = O\left(\frac{1}{n^{k+1}}\right). \tag{15}$$

By inversion of the order of integration the first of these two integrals may be given the form

$$\int_0^\pi \lambda(t_1) \cos \rho_n t_1 \int_0^\pi \phi(x) \sin \rho_n (x - t_1)\, dx\, dt_1.$$

Let the inner integral here be transformed by integration by parts. The function $\phi(x)$ vanishes with its first $k-1$ derivatives when $x = \pi$, but of course not, in general, when $x = t_1$. At the latter point, the sine of $\rho_n(x - t_1)$ vanishes, while the cosine is equal to 1. Hence

$$\begin{aligned}\int_{t_1}^\pi \phi(x) \sin \rho_n (x - t_1)\, dx &= \frac{1}{\rho_n}\phi(t_1) + \frac{1}{\rho_n}\int_{t_1}^\pi \phi'(x) \cos \rho_n (x - t_1)\, dx \\ &= \frac{1}{\rho_n}\phi(t_1) - \frac{1}{\rho_n^2}\int_{t_1}^\pi \phi''(x) \sin \rho_n (x - t_1)\, dx,\end{aligned}$$

and so on. Finally, if we let $k = 2\gamma$ or $2\gamma - 1$, according as k is even or odd,

$$\begin{aligned}\int_{t_1}^\pi \phi(x) \sin \rho_n (x - t_1)\, dx &= \frac{1}{\rho_n}\phi(t_1) - \frac{1}{\rho_n^3}\phi''(t_1) + \cdots \\ &+ \frac{(-1)^{\gamma-1}}{\rho_n^{2\gamma-1}}\phi^{(2\gamma-2)}(t_1) + \frac{1}{\rho_n^k}\int_{t_1}^\pi \phi^{(k)}(x) \sin\left[\rho_n(x - t_1) + \frac{k\pi}{2}\right] dx.\end{aligned} \tag{16}$$

By the use of (13) and the second law of the mean, it is seen that the absolute value of the last integral does not exceed

$$\frac{2}{\rho_n}[\phi_1(\pi) + \phi_2(\pi)],$$

whatever the value of t_1 may be.

We have now to multiply the several terms of (16) by $\lambda(t_1) \cos \rho_n t_1$ and integrate with respect to t_1 from 0 to π. The last term, in consequence of what we have just seen, will yield a quantity which is in order of magnitude $O(1/\rho_n^{k+1})$, and therefore $O(1/n^{k+1})$.

* The function $\lambda(x)$ is not subjected to any special restrictions at the ends of the interval.

Consider the first term of the integrated expression,

$$\frac{1}{\rho_n}\int_0^\pi \lambda(t_1)\phi(t_1)\cos\rho_n t_1\,dt_1. \tag{17}$$

The function $\lambda(t_1)\phi(t_1)$ has a $(k-1)$th derivative which is equal to

$$\lambda^{(k-1)}(t_1)\phi(t_1)+(k-1)\lambda^{(k-2)}(t_1)\phi'(t_1)+\cdots+\lambda(t_1)\phi^{(k-1)}(t_1). \tag{18}$$

Now $\phi^{(k-1)}(t_1)$, having a continuous derivative $\phi^{(k)}(t_1)$, is *a fortiori* of limited variation, and the same is true of the earlier derivatives of ϕ and of ϕ itself. Similarly, the derivatives of λ of orders from 0 to $k-2$ are of limited variation as well as the $(k-1)$th. Consequently the property of limited variation is possessed by the whole expression (18). From the corresponding expansions of the earlier derivatives of the product $\lambda\phi$ it is seen that each vanishes at both ends of the interval $(0,\pi)$. It is recognized thus that $\lambda\phi$ satisfies the conditions imposed on ϕ in Lemma I, except that k in the lemma is to be replaced by $k-1$. By that lemma, the integral in (17), without the factor $1/\rho_n$, is $O(1/n^k)$, and when divided by ρ_n becomes $O(1/n^{k+1})$.

Since $\lambda(t_1)\phi''(t_1)$ satisfies the conditions of Lemma I with k replaced by $k-2$, we have

$$\frac{1}{\rho_n^3}\int_0^\pi \lambda(t_1)\phi''(t_1)\cos\rho_n t_1\,dt_1 = O\left(\frac{1}{n^{k+2}}\right).$$

Each of the remaining terms can be disposed of in a similar manner; at each step, from this point on, the number of derivatives of limited variation known to be possessed by the integrand is diminished by 2, and this loss is compensated by the presence of a higher power of $1/\rho_n$ before the integral.

It appears then that (14) is true. The proof of (15) obviously follows the same lines.

Let us look again for a moment at the special case that $\phi^{(k)}(0)=0$. If v still represents the total variation of $\phi^{(k)}(x)$ in $(0,\pi)$,

$$|\phi^{(k)}(x)|\leqq v$$

throughout the interval. Hence the total variation of $\phi^{(k-1)}(x)$ in the interval can not exceed πv. The absolute value of $\phi^{(k-1)}(x)$ can not exceed the same quantity, since $\phi^{(k-1)}(0)=0$. In the same way the total variation of $\phi^{(k-2)}(x)$ and the maximum of its absolute value are less than or equal to $\pi^2 v$, and so on.

It is readily found that the total variation of $\lambda\phi$, for example, does not exceed v multiplied by a quantity independent of ϕ. We can write

$$\lambda\phi=(\lambda_1-\lambda_2)(\phi_1-\phi_2)=(\lambda_1\phi_1+\lambda_2\phi_2)-(\lambda_1\phi_2+\lambda_2\phi_1),$$

where λ_1, λ_2, ϕ_1, and ϕ_2 are monotone increasing and continuous, and the last two vanish at the point 0. It can be assumed also that λ_1 and λ_2 are positive or zero throughout the interval. Then if V is the total variation of $\lambda\phi$,

$$V \leqq \lambda_1(b)\phi_1(b) + \lambda_2(b)\phi_2(b) + \lambda_1(b)\phi_2(b) + \lambda_2(b)\phi_1(b)$$
$$\leqq 2[\lambda_1(b) + \lambda_2(b)]\pi^k v.$$

By similar reasoning, it is found that the $(k-1)$th derivative of $\lambda\phi$, expanded as in (18), and the $(k-2)$th derivative of $\lambda\phi''$, etc., have the same property, namely, that the total variation of each is inferior to v multiplied by a quantity independent of ϕ. Applying this fact in the preceding proof of Lemma II, we conclude that when $\phi^{(k)}(0) = 0$ the statement of the lemma can be made more precise by saying that the absolute value of each of the two integrals concerned remains inferior to cv/n^{k+1}.

LEMMA III. *If ϕ and λ satisfy the conditions of Lemma II, then**

$$\int_0^\pi \phi(x)\int_0^x \lambda(t_1)\sin\rho_n(x-t_1)\int_0^{t_1}\lambda(t_2)\sin\rho_n(t_1-t_2)\int_0^{t_2}\cdots$$
$$\int_0^{t_{s-1}}\lambda(t_s)\cos\rho_n t_s\sin\rho_n(t_{s-1}-t_s)\,dt_s\,dt_{s-1}\cdots dt_1\,dx = O\left(\frac{1}{n^{k+1}}\right), \tag{19}$$

and the same is true if $\cos\rho_n t_s$ *in the innermost integral is replaced by* $\sin\rho_n t_s$.

The proof is obtained by induction. The statement is already known to be true when $s = 1$, as it reduces then to Lemma II, the variable x taking the place of a variable t_0. We shall assume that the conclusion holds if s is replaced by $s-1$, and show then that it holds as stated. The steps in the passage from $s-1$ to s correspond exactly to those carried out in the proof of Lemma II. We begin by inverting the order of integration, confining our attention to the integral with $\cos\rho_n t_s$. The given expression is equal to

$$\int_0^\pi \lambda(t_s)\cos\rho_n t_s\int_{t_s}^\pi \lambda(t_{s-1})\sin\rho_n(t_{s-1}-t_s)\int_{t_{s-1}}^\pi\cdots$$
$$\int_{t_2}^\pi \lambda(t_1)\sin\rho_n(t_1-t_2)\int_{t_1}^\pi \phi(x)\sin\rho_n(x-t_1)\,dx\,dt_1\cdots dt_{s-1}\,dt_s.$$

The innermost integral here is precisely that which appears in (16). By the aid of that formula, we express the $(s+1)$-fold integral as the sum of $\gamma+1$ terms, of which the last is immediately seen to be $O(1/n^{k+1})$, while the others

* In this multiple integral, the notation for which is suggestive rather than complete, $s-3$ more factors of the form $\lambda(t_i)\sin\rho_n(t_{i-1}-t_i)$ are to be understood. It will be seen a few lines below how an integral of this form is obtained as a result of successive substitutions in (5).

are shown to be $O(1/n^{k+1})$, or even $O(1/n^{k+2})$, by means of the assumed lemma for s-fold integrals, with ϕ replaced successively by $\lambda\phi$, $\lambda\phi''$, $\cdots$, $\lambda\phi^{(2\gamma-2)}$. Thus the relation (19) is established, and clearly remains true if $\sin \rho_n t_s$ is written instead of $\cos \rho_n t_s$.

In the special case that $\phi^{(k)}(0) = 0$, it is readily shown by the use of the corresponding refinement of Lemma II that each of the two integrals in the present lemma remains inferior in absolute value to cv/n^{k+1}.

We are now in a position to deal with the integral (8). If, on the right-hand side of (5), the function $U_n(t)$ under the integral sign is expressed at length by means of (5) itself, the following more extended formula results:

$$
\begin{aligned}
U_n(x) = \cos \rho_n x &+ \frac{h' \sin \rho_n x}{\rho_n} + \frac{1}{\rho_n}\int_0^x \lambda(t_1)\cos \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1 \\
&+ \frac{h'}{\rho_n^2}\int_0^x \lambda(t_1)\sin \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1 \\
&+ \frac{1}{\rho_n^2}\int_0^x \lambda(t_1)\sin \rho_n (x - t_1)\int_0^{t_1} \lambda(t_2)\, U_n(t_2)\sin \rho_n (t_1 - t_2)\, dt_2\, dt_1.
\end{aligned}
\tag{20}
$$

There is still an integral on the right-hand side containing U_n under the integral sign, and the step just taken can be repeated. This process is to be continued until an expression is obtained in which the integral involving U_n is preceded by a factor $1/\rho_n^{k+1}$, where k is an appropriate integer. Let this expression be multiplied by $\phi(x)$ and integrated from 0 to π with regard to x. The last of the resulting terms will be $O(1/n^{k+1})$, inasmuch as the integral, without the factor $1/\rho_n^{k+1}$, remains finite for all values of n. The integrals which appear in the earlier terms are explicit expressions of the types considered in Lemmas I, II, and III. If ϕ and λ are restricted as in those lemmas, each integral, even without the power of $1/\rho_n$ which stands before it, will be $O(1/n^{k+1})$. If λ is supposed provided with only $k-2$ continuous derivatives of limited variation, instead of $k-1$, so that k is to be replaced by $k-1$ in applying Lemmas II and III, each term will still be $O(1/n^{k+1})$ after the factors $1/\rho_n$ have been multiplied in. The conclusion may be stated thus:

LEMMA IV. *If ϕ satisfies the conditions of Lemma I, and if λ has a continuous $(k-2)$th derivative of limited variation in $0 \leqq x \leqq \pi$, then*

$$
\int_0^\pi \phi(x)\, U_n(x)\, dx = O\left(\frac{1}{n^{k+1}}\right). \tag{21}
$$

If, in addition, λ has a continuous $(k-1)$th derivative of limited variation in the interval,

$$
\int_0^\pi \phi(x)\, U_n(x)\, dx = \int_0^\pi \phi(x)\cos \rho_n x\, dx + O\left(\frac{1}{n^{k+2}}\right). \tag{22}
$$

If $\phi^{(k)}(0) = 0$, the constant multiplier implied in the O-notation can be taken as v, the total variation of $\phi^{(k)}(x)$, multiplied by a quantity independent* of ϕ.

Let the case $k = 1$ be brought up again for special consideration at this point. The hypothesis for the first part of Lemma IV takes the form that ϕ has a continuous first derivative of limited variation and that

$$\phi(0) = \phi(\pi) = 0,$$

while nothing more than continuity is required of λ. The point to be made is that in this case the special restriction on ϕ at the ends of the interval is unnecessary. Let $U_n(x)$ be expanded to five terms as in (20), multiplied by $\phi(x)$, and integrated. It has already been pointed out, in connection with the proof of Lemma I, that

$$\int_0^\pi \phi(x) \cos \rho_n x \, dx = O\left(\frac{1}{n^2}\right),$$

whether ϕ vanishes at the ends of the interval or not. As for the other terms, the application of Lemma II with $k = 0$ gives all that is required.†

The discussion so far has related to the numerator of the expression (7) for α_n. For the denominator we need only the simple relation given by Liouville:‡

$$\int_0^\pi [U_n(x)]^2 \, dx = \frac{\pi}{2} + O\left(\frac{1}{n}\right). \tag{23}$$

The proof is simply this: In consequence of (5),

$$\int_0^\pi [U_n(x)]^2 \, dx = \int_0^\pi \cos^2 \rho_n x \, dx + O\left(\frac{1}{n}\right),$$

and

$$\int_0^\pi \cos^2 \rho_n x \, dx = \frac{\pi}{2} + \frac{\sin 2\rho_n \pi}{4\rho_n} = \frac{\pi}{2} + O\left(\frac{1}{n}\right).$$

We have the materials now for one theorem on the degree of convergence

* It is to be noticed that the formulas on which all our demonstrations depend hold only for values of n *from a certain point on*, but it is readily seen that the conclusions are correct for all positive values of n. This remark is not trivial in the case of such a proposition as the one to which this note is appended. It is asserted, for example, that

$$\int_0^\pi \phi(x) U_1(x) \, dx$$

does not exceed a constant multiple of v; this is true with the hypotheses that are stated, since the maximum of $|\phi|$ does not exceed a constant multiple of v, but would not generally be true otherwise.

† If this lemma and (20) had not already been written out at length, it would have been still simpler to operate with the three-term expression (5) directly.

‡ Loc. cit., p. 430.

of the series (6). If ϕ and λ satisfy the hypotheses for the first part of Lemma IV, it follows from (21) and (23) that

$$\alpha_n = O\left(\frac{1}{n^{k+1}}\right),$$

and as the maximum of $|U_n(x)|$ remains finite when n becomes infinite,

$$|\alpha_n U_n(x)| \leqq \frac{\beta}{n^{k+1}},$$

where β is independent of n and x. It follows that the series $\sum \alpha_n U_n(x)$ converges uniformly,* in which case, as has already been stated, it must converge to the value $\phi(x)$. If we set

$$\sigma_n(x) = \sum_{\nu=0}^{n} \alpha_\nu U_\nu(x),$$

then

$$|\phi(x) - \sigma_n(x)| = \left|\sum_{\nu=n+1}^{\infty} \alpha_\nu U_\nu(x)\right| \leqq \sum_{\nu=n+1}^{\infty} \frac{\beta}{\nu^{k+1}} < \beta \int_n^\infty \frac{dt}{t^{k+1}} = \frac{\beta}{kn^k}.$$

That is:

THEOREM I. *If $\phi(x)$ has a continuous kth derivative of limited variation in the interval $0 \leqq x \leqq \pi$, while ϕ itself and its first $k-1$ derivatives vanish for $x = 0$ and for $x = \pi$, and if, furthermore, the function $\lambda(x)$ which appears as a coefficient in the differential equation has a continuous $(k-2)$th derivative of limited variation in $0 \leqq x \leqq \pi$, then*

$$\phi(x) = \sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly throughout the interval.

Whenever the O-notation is used in the present paper in a relation involving a function of x, it will be understood, without being repeated on every occasion, that the relation holds uniformly, that is, that the implicit constant multiplier is independent of x as well as of n. In the relation just written down it depends on k, λ, h', H', and ϕ. In the case that $\phi^{(k)}(0) = 0$ its dependence on ϕ is completely characterized by saying that it is proportional to the total variation of $\phi^{(k)}$.

When $k = 1$, the theorem is true without the restriction that

$$\phi(0) = \phi(\pi) = 0,$$

in consequence of the facts previously established for this special case. That the restrictions at the ends of the interval are not superfluous in general is shown by the simplest examples. If

$$\lambda(x) = 0, \qquad h' = H' = 0, \qquad \phi(x) = \sin x,$$

* Of course this is assured at the outset by well-known general theorems, but it appears incidentally here.

for instance, so that the series is the cosine-series for $\sin x$, the hypotheses of Theorem I, except those relating to the end-points, are satisfied for any value of k, but the remainder in the series

$$\sin x = \frac{4}{\pi}\left[1 - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \cdots - \frac{\cos 2nx}{4n^2 - 1} - \cdots\right]$$

at the point $x = \frac{1}{2}\pi$ does not approach zero faster than $1/n^2$.

A further theorem can be obtained by operating with (22). Let the equation (12) be recalled, and compared with the following, which is obtained in the same way:

$$\int_0^\pi \phi(x)\cos nx\,dx = \frac{1}{n^k}\int_0^\pi \phi^{(k)}(x)\cos\left(nx + \frac{k\pi}{2}\right)dx.$$

By (3) and (4) we may write

$$\frac{1}{\rho_n^k} = \frac{1}{n^k}\left(\frac{n}{\rho_n}\right)^k = \frac{1}{n^k}\left(1 - \frac{\epsilon_n}{\rho_n}\right)^k = \frac{1}{n^k} + O\left(\frac{1}{n^{k+2}}\right).$$

On the other hand,

$$\begin{aligned}\cos\left(\rho_n x + \frac{k\pi}{2}\right) &= \cos\left(nx + \frac{k\pi}{2}\right) - (1 - \cos\epsilon_n x)\cos\left(nx + \frac{k\pi}{2}\right)\\ &\qquad - \sin\epsilon_n x \sin\left(nx + \frac{k\pi}{2}\right)\\ &= \cos\left(nx + \frac{k\pi}{2}\right) - \sin\epsilon_n x\sin\left(nx + \frac{k\pi}{2}\right) + O\left(\frac{1}{n^2}\right).\end{aligned}$$

Now $\sin\epsilon_n x$ is a positive monotone increasing function in $(0, \pi)$, as soon as n is sufficiently large, and so, if we write

$$\phi^{(k)}(x) = \phi_1(x) - \phi_2(x),$$

as in (13), we shall have

$$\begin{aligned}&\int_0^\pi \phi^{(k)}(x)\sin\epsilon_n x\sin\left(nx + \frac{k\pi}{2}\right)dx\\ &\quad = \phi_1(\pi)\sin\epsilon_n\pi\int_\xi^\pi \sin\left(nx + \frac{k\pi}{2}\right)dx - \phi_2(\pi)\sin\epsilon_n\pi\int_\eta^\pi \sin\left(nx + \frac{k\pi}{2}\right)dx\\ &\quad = O\left(\frac{1}{n^2}\right),\end{aligned}$$

since $\sin\epsilon_n\pi$ and both the integrals are separately $O(1/n)$. By combination of these relations,

$$\frac{1}{\rho_n^k}\int_0^\pi \phi^{(k)}(x)\cos\left(\rho_n x+\frac{k\pi}{2}\right)dx$$

$$=\left[\frac{1}{n^k}+O\left(\frac{1}{n^{k+2}}\right)\right]\left[\int_0^\pi \phi^{(k)}(x)\cos\left(nx+\frac{k\pi}{2}\right)dx+O\left(\frac{1}{n^2}\right)\right]$$

$$=\frac{1}{n^k}\int_0^\pi \phi^{(k)}(x)\cos\left(nx+\frac{k\pi}{2}\right)dx+O\left(\frac{1}{n^{k+2}}\right).$$

That is,

$$\int_0^\pi \phi(x)\cos\rho_n x\,dx=\int_0^\pi \phi(x)\cos nx\,dx+O\left(\frac{1}{n^{k+2}}\right);$$

the relation (22) remains true if $\cos\rho_n x$ is replaced by $\cos nx$. The integral on the right is itself $O(1/n^{k+1})$. From (23),

$$\frac{1}{\int_0^\pi [U_n(x)]^2\,dx}=\frac{2}{\pi}+O\left(\frac{1}{n}\right).$$

Hence, by multiplication,

$$\alpha_n=\frac{2}{\pi}\int_0^\pi \phi(x)\cos nx\,dx+O\left(\frac{1}{n^{k+2}}\right).$$

To go one step further,

$$U_n(x)=\cos\rho_n x+O\left(\frac{1}{n}\right)=\cos nx+O\left(\frac{1}{n}\right),$$

and so

$$\alpha_n U_n(x)=\frac{2}{\pi}\cos nx\int_0^\pi \phi(t)\cos nt\,dt+O\left(\frac{1}{n^{k+2}}\right). \tag{24}$$

Up to this point the original hypotheses about the function ϕ have been retained. It is for a somewhat different class of functions that the last relations are to be used. Suppose that $\phi(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition,

$$|\phi^{(k-1)}(x_2)-\phi^{(k-1)}(x_1)|\leqq\mu|x_2-x_1|, \tag{25}$$

where x_1 and x_2 are any two values in the closed interval $(0,\pi)$ and μ is a constant. This derivative is *a fortiori* continuous and of limited variation, and if we suppose that ϕ and its first $k-2$ derivatives vanish at 0 and π, and that λ has a continuous derivative of order $k-2$ with limited variation, the hypotheses for (24) will be fulfilled with k replaced by $k-1$.

The function $\phi(x)$ has been defined up to the present only in the interval from 0 to π. Let this definition be extended by setting $\phi(x)=\phi(-x)$ for $-\pi\leqq x\leqq 0$, and then making $\phi(x+2\pi)=\phi(x)$ for all real values of x. The periodic function so defined will have a $(k-1)$th derivative

satisfying a Lipschitz condition everywhere* if k is odd, and this will be true for even values of k as well if the hypotheses previously made are supplemented by requiring that $\phi^{(k-1)}(0) = \phi^{(k-1)}(\pi) = 0$.

The Fourier's series for $\phi(x)$ will involve only cosine-terms, since ϕ is even, and will have for its general term precisely the expression on the right-hand side of (24), with the O-term omitted. Let this general term of the Fourier's series be denoted by $a_n \cos nx$, the sum of the first $n+1$ terms by $s_n(x)$. It is known that†

$$\phi(x) - s_n(x) = \sum_{\nu=n+1}^{\infty} a_\nu \cos \nu x = O\left(\frac{\log n}{n^k}\right). \tag{26}$$

Remembering that k in (24) is to be replaced by $k-1$, we have

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{\beta}{n^{k+1}}, \tag{27}$$

where β is independent of n and x. It is understood that‡ $k \geqq 1$; it appears from the relations just written down that the Sturm-Liouville series $\sum \alpha_n U_n(x)$ converges uniformly, and that

$$\begin{aligned}
|\phi(x) - \sigma_n(x)| &\leqq |\phi(x) - s_n(x)| + |[\phi(x) - \sigma_n(x)] - [\phi(x) - s_n(x)]| \\
&= |\phi(x) - s_n(x)| + \left|\sum_{\nu=n+1}^{\infty} [\alpha_\nu U_\nu(x) - a_\nu \cos \nu x]\right| \\
&\leqq |\phi(x) - s_n(x)| + \sum_{\nu=n+1}^{\infty} \frac{\beta}{\nu^{k+1}} \\
&= O\left(\frac{\log n}{n^k}\right) + O\left(\frac{1}{n^k}\right) = O\left(\frac{\log n}{n^k}\right).
\end{aligned}$$

This may be formulated as follows:

THEOREM II. *If $\phi(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition throughout the interval $0 \leqq x \leqq \pi$, while ϕ itself and its first $k-2$ derivatives, and, in case k is even, the $(k-1)$th derivative also, vanish for*

* It is seen at once that ϕ itself is continuous, and readily shown that the successive derivatives through the $(k-2)$th exist and are continuous, even at the points $0, \pi, 2\pi \cdots$. The $(k-2)$th derivative will be an even or an odd function according as k is even or odd. In the latter case its right-hand and left-hand derivatives at the point 0 will be equal; in the former they will be the negatives of each other, and so equal only if they vanish. Similar reasoning applies to the point π. The others need not be considered separately, because of the periodicity of ϕ. The proof that the Lipschitz condition is satisfied offers no difficulty.

† For the case $k = 1$, see Lebesgue, loc. cit., p. 201; for the general case, D. Jackson, loc. cit., also D. Jackson, these Transactions, vol. 14 (1913), pp. 343–364, Theorem X. The latter paper will be referred to as B.

‡ The meaning of the hypothesis in the case $k = 1$ would be, of course, that ϕ itself satisfies a Lipschitz condition.

$x = 0$ and for $x = \pi$, and if $\lambda(x)$ has a continuous $(k-2)$th derivative of limited variation for $0 \leqq x \leqq \pi$, then

$$\phi(x) = \sigma_n(x) + O\left(\frac{\log n}{n^k}\right)$$

uniformly throughout the interval.

If k is even, so that it is assumed that $\phi^{(k-1)}(0) = 0$, or if k is odd and this assumption is added to those already made, it will be seen, on following through the work with this in view, that the remainder does not exceed $c\mu(\log n)/n^k$, where c is independent of ϕ, and μ is the coefficient in the Lipschitz condition which $\phi^{(k-1)}(x)$ satisfies,* and it is assumed that $n \geqq 2$. It is essential for this purpose to note from the papers referred to for the Fourier's series that the constant factor implicit in the O-symbol in (26) may be taken as μ multiplied by a quantity independent of ϕ.

The suggested refinement of Theorem II may be dismissed with these few lines, as far as general values of k are concerned. In the simplest case, $k = 1$, the theorem is needed for an application in its more precise form, and it will be well to state this simple result separately and to give the details of the proof from the beginning.†

THEOREM II*a*. *If $\phi(x)$ satisfies the Lipschitz condition*

$$|\phi(x_2) - \phi(x_1)| \leqq \mu|x_2 - x_1| \tag{28}$$

throughout the interval $0 \leqq x \leqq \pi$, and if $\phi(0) = 0$, then, in the whole interval,

$$|\phi(x) - \sigma_n(x)| \leqq \frac{c\mu \log n}{n}, \qquad n \geqq 2,$$

where c is independent of x, n, and ϕ.

The restriction on $\lambda(x)$ is merely that of continuity.

In accordance with the hypothesis, $|\phi(x)| \leqq \pi\mu$, and $\phi(x)$ can be written in the form

$$\phi(x) = \phi_1(x) - \phi_2(x), \tag{29}$$

where ϕ_1 and ϕ_2 are continuous and monotone increasing, and

$$\phi_1(0) = \phi_2(0) = 0, \qquad \phi_1(\pi) \leqq \pi\mu, \qquad \phi_2(\pi) \leqq \pi\mu.$$

The expression (5) for $U_n(x)$ will be sufficient for our purpose. Let the second term be multiplied by $\phi(x)$ and integrated; we find

* It is obvious that the total variation of $\phi^{(k-1)}(x)$ in $(0, \pi)$ is at most $\pi\mu$.

† This will involve some repetition of what has gone before, which is perhaps compensated by the gain in clearness.

$$\left|\int_0^\pi \phi(x)\sin\rho_n x\,dx\right| = \left|\phi_1(\pi)\int_\xi^\pi \sin\rho_n x\,dx - \phi_2(\pi)\int_\eta^\pi \sin\rho_n x\,dx\right|$$
$$\leqq \frac{4\pi\mu}{\rho_n} \leqq \frac{c\mu}{n},$$

and hence

$$\left|\frac{h'}{\rho_n}\int_0^\pi \phi(x)\sin\rho_n x\,dx\right| \leqq \frac{c\mu}{n^2};$$

as has already been explained, the letter c will be used to represent a number independent of n and ϕ, and, in a relation where x occurs, independent also of x, but will stand for different numbers of this sort in different lines, and, on occasion, even in the same line. Consider the third term:

$$\int_0^\pi \phi(x)\int_0^x \lambda(t)\,U_n(t)\sin\rho_n(x-t)\,dt\,dx$$
$$= \int_0^\pi \lambda(t)\,U_n(t)\cos\rho_n t\int_t^\pi \phi(x)\sin\rho_n x\,dx\,dt$$
$$- \int_0^\pi \lambda(t)\,U_n(t)\sin\rho_n t\int_t^\pi \phi(x)\cos\rho_n x\,dx\,dt,$$

and as each of the integrals extended from t to π is seen by the use of (29) and the second law of the mean to be in absolute value not greater than $c\mu/n$, it follows that the whole expression satisfies an inequality of the same form. Consequently

$$\left|\frac{1}{\rho_n}\int_0^\pi \phi(x)\int_0^x \lambda(t)\,U_n(t)\sin\rho_n(x-t)\,dt\,dx\right| \leqq \frac{c\mu}{n^2}.$$

Return now to the first term of (5); here it is to be pointed out that

$$\left|\int_0^\pi \phi(x)\cos\rho_n x\,dx - \int_0^\pi \phi(x)\cos nx\,dx\right|$$
$$\leqq \left|\int_0^\pi (\cos\epsilon_n x - 1)\,\phi(x)\cos nx\,dx\right| + \left|\int_0^\pi \phi(x)\sin\epsilon_n x\sin nx\,dx\right|$$
$$\leqq \frac{c\mu}{n^2} + \left|\phi_1(\pi)\sin\epsilon_n\pi\int_\xi^\pi \sin nx\,dx - \phi_2(\pi)\sin\epsilon_n\pi\int_\eta^\pi \sin nx\,dx\right|$$
$$\leqq \frac{c\mu}{n^2},$$

the last inequality but one being obtained as soon as n is so large that* $\epsilon_n \leqq \frac{1}{2}$ and $\sin\epsilon_n x$ is monotone increasing in $(0, \pi)$. To sum up the inequalities obtained thus far,

$$\int_0^\pi \phi(x)\,U_n(x)\,dx = \int_0^\pi \phi(x)\cos nx\,dx + r_1(n, x), \tag{30}$$

* The point where this begins to be true is of course independent of ϕ.

where the remainder r_1 is such that

$$|r_1(n, x)| \leqq \frac{c\mu}{n^2}.$$

In consequence of (23), which was proved in four lines, we may write

$$\frac{1}{\int_0^\pi [U_n(x)]^2\, dx} = \frac{2}{\pi} + r_2(n), \tag{31}$$

where

$$|r_2(n)| \leqq \frac{c}{n}.$$

Furthermore,

$$U_n(x) = \cos \rho_n x + r_3(n, x) = \cos nx + r_4(n, x), \tag{32}$$

where

$$|r_3(n, x)| \leqq \frac{c}{n}, \qquad |r_4(n, x)| \leqq \frac{c}{n}.$$

As the second law of the mean shows that

$$\left| \int_0^\pi \phi(x) \cos nx\, dx \right| \leqq \frac{c\mu}{n},$$

multiplication of (30), (31), and (32) gives

$$\alpha_n U_n(x) = \frac{2}{\pi} \cos nx \int_0^\pi \phi(t) \cos nt\, dt + r_5(n, x), \tag{33}$$

where

$$|r_5(n, x)| \leqq \frac{c\mu}{n^2}. \tag{34}$$

Let $\phi(x)$ be defined outside of $(0, \pi)$ so as to be an even function of period 2π for all real values of x. This function will satisfy (28) everywhere. If $a_n \cos nx$ is the general term of its Fourier's series, which contains no sine-terms, and

$$s_n(x) = \sum_{\nu=0}^{n} a_n \cos nx,$$

then*

$$|\phi(x) - s_n(x)| \leqq \frac{c\mu \log n}{n}, \qquad n \geqq 2.$$

The relation (33), with (34), states that

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{c\mu}{n^2}.$$

* Lebesgue, loc. cit., p. 201; D. Jackson, in the paper A, Theorem V.

This has been established only for values of n from a certain point on, this point being unspecified, but independent of ϕ. On taking account of the latter circumstance, it is readily seen that the relation holds also for the positive values of n previously neglected. For the inequalities

$$|\alpha_n U_n(x)| \leqq c\mu, \qquad |a_n \cos nx| \leqq c\mu,$$

are seen at once to hold for all these values of n, since $|\phi(x)| \leqq \pi\mu$, and $U_n(x)$, whether represented by (5) or not, is continuous and not identically zero; it follows that

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq c\mu,$$

and it is immaterial whether we write $c\mu$ or $c\mu/n^2$, since only a finite number of values of n are concerned. Consequently, for $n \geqq 2$,

$$\begin{aligned}
|\phi(x) - \sigma_n(x)| &\leqq |\phi(x) - s_n(x)| + |[\phi(x) - \sigma_n(x)] - [\phi(x) - s_n(x)]| \\
&= |\phi(x) - s_n(x)| + \left| \sum_{\nu=n+1}^{\infty} [\alpha_\nu U_\nu(x) - a_\nu \cos \nu x] \right| \\
&\leqq |\phi(x) - s_n(x)| + \sum_{\nu=n+1}^{\infty} \frac{c\mu}{n^2} \\
&\leqq \frac{c\mu \log n}{n} + \frac{c\mu}{n} \leqq \frac{c\mu \log n}{n}.
\end{aligned}$$

Now let $\phi(x)$ be any function whatever that is continuous in $0 \leqq x \leqq \pi$ and vanishes for $x = 0$. Represent by δ any positive quantity, and let $\omega(\delta)$ denote the maximum of $|\phi(x_2) - \phi(x_1)|$ for values of x_1 and x_2 in the interval subject to the restriction that $|x_2 - x_1| \leqq \delta$. It is obvious at once that $\omega(\delta)$ is a monotone increasing function of δ, and that

$$\lim_{\delta=0} \omega(\delta) = 0.$$

It may be assumed further that $\omega(\delta)/\delta$ does not approach zero as δ approaches zero, otherwise ϕ would have a vanishing derivative throughout the interval and so would vanish identically itself. The relation just obtained is to be used to establish the following proposition:

THEOREM III. *If $\phi(x)$ is continuous in $0 \leqq x \leqq \pi$ and*

$$|\phi(x_2) - \phi(x_1)| \leqq \omega(\delta) \tag{35}$$

whenever

$$|x_2 - x_1| \leqq \delta,$$

and if $\phi(0) = 0$, then

$$|\phi(x) - \sigma_n(x)| \leqq c\omega\left(\frac{\pi}{n}\right) \log n, \qquad n \geqq 2.$$

The proof, after Theorem IIa has been established, is that given by Lebesgue* for the corresponding property of Fourier's series. We begin by deriving the following lemma:

LEMMA V. *If*

$$|\phi(x)| \leqq \epsilon$$

throughout the interval $0 \leqq x \leqq \pi$, *then*

$$|\phi(x) - \sigma_n(x)| \leqq c\epsilon \log n, \qquad n \geqq 2.$$

It is obviously sufficient to show that $|\sigma_n(x)|$ itself can not exceed $c\epsilon \log n$. Now it follows at once from (5), with (3) and (4), that

$$\left| \int_0^\pi \phi(x) U_n(x)\, dx - \int_0^\pi \phi(x) \cos \rho_n x\, dx \right| \leqq \frac{c\epsilon}{n},$$

or, since $|\cos \rho_n x - \cos nx| \leqq c/n$, that

$$\left| \int_0^\pi \phi(x) U_n(x)\, dx - \int_0^\pi \phi(x) \cos nx\, dx \right| \leqq \frac{c\epsilon}{n}.$$

From this and (31) and (32),

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{c\epsilon}{n}.$$

Therefore

$$|\sigma_n(x) - s_n(x)| \leqq \sum_{\nu=1}^{n} \frac{c\epsilon}{\nu} \leqq c\epsilon \log n,$$

and as it is well known† that

$$|s_n(x)| \leqq c\epsilon \log n,$$

the lemma follows at once.

Now let a function $\psi(x)$ be defined as equal to $\phi(x)$ at the points $x = i\pi/n$, $i = 0, 1, \cdots, n$, and linear between these points. Because of (35),

$$|\phi(x) - \psi(x)| \leqq 2\omega\left(\frac{\pi}{n}\right) \tag{36}$$

throughout the interval. Furthermore, $\psi(x)$ satisfies (28) with

$$\mu = \frac{\omega(\pi/n)}{\pi/n}. \tag{37}$$

Let us denote by $\sigma_n(x)$ the sum of the first $n + 1$ terms of the Sturm-Liouville series for $\phi(x)$, and by $\sigma_{1n}(x)$ and $\sigma_{2n}(x)$ the corresponding sums formed for $\psi(x)$ and $\phi(x) - \psi(x)$ respectively. Of course

$$\sigma_n(x) = \sigma_{1n}(x) + \sigma_{2n}(x).$$

* Loc. cit., pp. 201–202.

† See, e. g., Lebesgue, loc. cit., pp. 196–197.

By (37) and Theorem IIa,

$$|\psi(x) - \sigma_{1n}(x)| \leqq \frac{c \log n}{n} \cdot \frac{\omega(\pi/n)}{\pi/n} = c\omega\left(\frac{\pi}{n}\right)\log n.$$

By (36) and Lemma V,

$$|\sigma_{2n}(x)| \leqq c\omega\left(\frac{\pi}{n}\right)\log n.$$

As it follows from (36) *a fortiori* that

$$|\phi(x) - \psi(x)| \leqq c\omega\left(\frac{\pi}{n}\right)\log n,$$

the identity

$$\phi(x) - \sigma_n(x) = \phi(x) - \psi(x) + \psi(x) - \sigma_{1n}(x) - \sigma_{2n}(x)$$

gives the demonstration of Theorem III at once.

If $\phi(x)$ is continuous, but $\phi(0) \neq 0$, let

$$\phi(x) = \phi(0) + \chi(x).$$

Theorem I is applicable to the Sturm-Liouville series for the constant $\phi(0)$, with $k = 1$; for of course a constant has a first derivative of limited variation, and it was pointed out in connection with that theorem that for $k = 1$ the function developed need not vanish at the ends of the interval. Hence the remainder after n terms of the series is $O(1/n)$. On the other hand, Theorem III may be applied to the function $\chi(x)$. If $\omega(\delta)$, formed for this function, is such that the ratio of δ to $\omega(\delta)$ remains finite as δ approaches zero,* then

$$\frac{1}{n} = O\left[\omega\left(\frac{\pi}{n}\right)\right] = O\left[\omega\left(\frac{\pi}{n}\right)\log n\right],$$

and $\phi(x)$ still has the property that

$$\phi(x) = \sigma_n(x) + O\left[\omega\left(\frac{\pi}{n}\right)\log n\right],$$

but it is no longer true that the constant multiplier in the O-symbol is independent of ϕ.

3. Summation of the Series

The theorem expressed in (26), concerning the degree of convergence of the Fourier's series for a function $\phi(x)$ of period 2π having a $(k-1)$th derivative that satisfies a Lipschitz condition, was proved by an indirect method. It was obtained as a consequence of the two following propositions:

(a) If $\phi(x)$ is a function of the character described, it is possible to define for each positive integral value of n a finite trigonometric sum of the nth order

* This assumption goes somewhat beyond the earlier one, that $\omega(\delta)/\delta$ shall not approach zero, that is, that the ratio of δ to $\omega(\delta)$ shall not become infinite.

at most which represents $\phi(x)$ approximately with a maximum error not exceeding a constant multiple of* $1/n^k$.

(*b*) If $\phi(x)$ is any function† of period 2π which can be approximately represented by a finite trigonometric sum of the nth or lower order with an error nowhere exceeding ϵ, and if $s_n(x)$ is the corresponding partial sum of the Fourier's series for $\phi(x)$, then, for $n \geqq 2$,

$$|\phi(x) - s_n(x)| \leqq K\epsilon \log n,$$

where K is an absolute constant.‡

It may not be unprofitable to observe that each of these propositions, which are interesting for their own sake, has its analogue in the theory of the Sturm-Liouville series. For (*b*) this analogue reads as follows:

If $\phi(x)$ can be approximately represented throughout the closed interval $(0, \pi)$ by a linear combination $\Sigma_n(x)$ of the characteristic functions $U_0(x)$, $U_1(x), \cdots, U_n(x)$, with constant coefficients, with an error nowhere exceeding ϵ, and if $\sigma_n(x)$ denotes the corresponding partial sum of the Sturm-Liouville series for $\phi(x)$, then, for $n \geqq 2$,

$$|\phi(x) - \sigma_n(x)| \leqq c\epsilon \log n$$

throughout the interval.

Here, as everywhere in this paper, c denotes a constant independent of ϕ, but depending conceivably on $\lambda(x)$ and on the coefficients in the boundary conditions.

The truth of the assertion is an immediate consequence of Lemma V. The partial sum of the Sturm-Liouville series for $\phi(x)$ is obtained by adding those for $\Sigma_n(x)$ and for $\phi(x) - \Sigma_n(x)$; the partial sum for $\Sigma_n(x)$ is $\Sigma_n(x)$ itself, and an error can arise only in the partial sum for $\phi(x) - \Sigma_n(x)$, to which Lemma V applies.

The generalization of (*a*) is somewhat less trivial. Let us recall the method by which that theorem itself was proved in the paper A.§

If k is the integer which appears in the statement of the theorem, and n is any positive integer, let κ be the smallest integer for which $2\kappa - k > 1$, and m the largest integer for which $\kappa(m-1)$ does not exceed n, and let

$$S_n(x) = h_m \int_{-\pi/2}^{\pi/2} [(-1)^{k+1}\phi(x+2ku) + (-1)^k k\phi(x+2(k-1)u) + \cdots + k\phi(x+2u)] \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du, \tag{38}$$

* D. Jackson, Dissertation, Göttingen 1911, Theorem VII; A, Theorem III; B, Theorem III.

† Integrable in the sense of Lebesgue.

‡ Lebesgue, loc. cit., p. 201; also, earlier, Lebesgue, Annales de la Faculté des Sciences de l'Université de Toulouse pour les sciences mathématiques et les sciences physiques, series 3, vol. 1 (1909), pp. 25–117; pp. 116–117.

§ The neater proof given in B seems less convenient for the present purpose.

where the numerical coefficients in the integrand are the binomial coefficients corresponding to the exponent k, the last being omitted, and h_m is a constant defined by the equation

$$\frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} \left[\frac{\sin mu}{m \sin u} \right]^{2\kappa} du.$$

It is shown in A* that $S_n(x)$ is a trigonometric sum in x of order not higher than $\kappa(m-1)$, and so not higher than n, and that $S_n(x)$ represents $\phi(x)$ with the required degree of approximation.

This information as to the form of $S_n(x)$ will not be enough for us here. We shall show that†

$$S_n(x) = \int_{-\pi}^{\pi} \phi(t)\, T_n(t-x)\, dt, \tag{39}$$

where

$$T_n(t-x) = \sum_{\nu=0}^{n} A_{n\nu} \cos \nu(t-x), \tag{40}$$

and the coefficients $A_{n\nu}$ are constants independent of ϕ. It will be sufficient to show that each of the k terms of which $S_n(x)$ is composed in (38) has this form. Consider any one of these terms, say that containing $\phi(x+2ru)$. The binomial coefficient and the factor $\pm h_m$ may be left out of account. The change of variable $t = x + 2ru$ gives

$$\int_{-\pi/2}^{\pi/2} \phi(x+2ru) \left[\frac{\sin mu}{m \sin u} \right]^{2\kappa} du = \frac{1}{2r} \int_{x-r\pi}^{x+r\pi} \phi(t) \left[\frac{\sin m \dfrac{t-x}{2r}}{m \sin \dfrac{t-x}{2r}} \right]^{2\kappa} dt.$$

Because of the periodicity of the functions involved, this is, except for the irrelevant constant factor $1/(2r)$, the same as

$$\int_{-r\pi}^{r\pi} \phi(t) \left[\frac{\sin m \dfrac{t-x}{2r}}{m \sin \dfrac{t-x}{2r}} \right]^{2\kappa} dt.$$

Let the interval of integration be broken up into r intervals, each of length 2π, and let these be reduced to a common interval by change of variable. In this way the last integral can be brought into the form

$$\int_{-r\pi}^{-(r-2)\pi} \phi(t) \sum_{j=0}^{r-1} \left[\frac{\sin m \left(\dfrac{t-x}{2r} + \dfrac{2j\pi}{2r} \right)}{m \sin \left(\dfrac{t-x}{2r} + \dfrac{2j\pi}{2r} \right)} \right]^{2\kappa} dt. \tag{41}$$

* The notation has been changed somewhat.

† Cf. B, pp. 347–348, where only the case $k = 1$ is taken up.

Now it is a familiar fact* that $[(\sin \frac{1}{2}mv)/(m \sin \frac{1}{2}v)]^2$ is a finite trigonometric sum in v, of order $m-1$, involving only cosines, and so, if this expression is raised to the κth power, there will be obtained a sum of the same character, of order $\kappa(m-1)$. That is, each term under the sign of summation in the integral just written down has the form

$$\sum_{i=0}^{\kappa(m-1)} B_i \cos i\left(\frac{t-x}{r}+\frac{2j\pi}{r}\right),$$

where the coefficients B_i are constants. On performing the summation with regard to j, a double sum is obtained which can be written thus:

$$(42) \qquad \sum_{i=0}^{\kappa(m-1)} \left\{ B_i \left[\cos i\frac{t-x}{r}\sum_{j=0}^{r-1}\cos\frac{2ij\pi}{r} - \sin i\frac{t-x}{r}\sum_{j=0}^{r-1}\sin\frac{2ij\pi}{r} \right]\right\}.$$

Since

$$\sum_{j=0}^{r-1}\cos\frac{2ij\pi}{r}$$

is zero unless i is divisible by r, and the corresponding sum of sines is always zero,† the expression (42) really involves no sines of multiples of $(t-x)/r$ at all, and the cosines of only such multiples of $(t-x)/r$ as are at the same time integral multiples of $t-x$. It has the same form as the right-hand side of (40), with the coefficients for which the second subscript is greater than n/r all equal to zero. As the integrand in (41), regarded as a function of t, is now seen to have the period 2π, the interval of integration may equally well be taken as that from $-\pi$ to π. To justify (39) and (40) it remains only to combine the terms corresponding to the several values of r.

The approximating property of $S_n(x)$, stated precisely, is this: If $\phi(x)$, of period 2π, has a $(k-1)$th derivative satisfying the condition (25), then, for all positive integral values of n and all values of x,

$$|\phi(x) - S_n(x)| \leqq \frac{K_k \mu}{n^k},$$

where K_k is a constant depending only on k.

Of course the number k enters into the definition of $S_n(x)$. If ϕ had a derivative of order $l-1$ satisfying a Lipschitz condition, $l > k$, it would not in general be true of this $S_n(x)$ that $|\phi(x) - S_n(x)|$ remains inferior to a constant multiple of $1/n^l$; to attain this degree of approximation, it would be necessary to define a new S_n by means of a new T_n. But it is readily seen, on following through the demonstration in A, that the function $T_n(t-x)$ formed for any particular value of k applies equally well for any *smaller*

* See Fejér, Mathematische Annalen, vol. 58 (1904), pp. 51–69; p. 53.

† See, e. g., Bôcher, *Introduction to the theory of Fourier's series*, Annals of Mathematics, ser. 2, vol. 7 (1906), pp. 81–152; p. 135.

value of k, and this is important for what follows. It will be well to give a summary of the facts that will be needed about the functions S_n and T_n in the form of a lemma.

LEMMA VI. *The positive integer k' being regarded as fixed, there exists for every positive integral value of n a cosine-sum $T_n(t-x)$, of the form* (40), *such that if $\phi(x)$ is any function of period 2π whose $(l-1)$th derivative, $l \leqq k'$, everywhere satisfies the condition*

$$|\phi^{(l-1)}(x_2) - \phi^{(l-1)}(x_1)| \leqq \mu |x_2 - x_1|,$$

then the corresponding $S_n(x)$, defined by (39), *satisfies for all values of x the relation*

$$|\phi(x) - S_n(x)| \leqq \frac{g\mu}{n^l},$$

where g depends only on k'.

For the purpose of obtaining a theorem on the approximate representation of a function which has a $(k-1)$th derivative satisfying a Lipschitz condition, by means of linear combinations of the functions $U_n(x)$, the number k' in the lemma is to be set equal to $k+1$, while l is to be given the values $k+1$ and k successively.

The former value is used in studying the behavior of the coefficients $A_{n\nu}$. Let $\cos \nu x$ be substituted for $\phi(x)$ in the lemma. It satisfies the relation

$$\left| \frac{d^k}{dx^k} \cos \nu x_2 - \frac{d^k}{dx^k} \cos \nu x_1 \right| \leqq \nu^{k+1} |x_2 - x_1|,$$

as is seen at once by applying the mean-value theorem to the kth derivative. Therefore

$$\left| \cos \nu x - \int_{-\pi}^{\pi} T_n(t-x) \cos \nu t \, dt \right| \leqq \frac{g\nu^{k+1}}{n^{k+1}}.$$

On the other hand, it follows from (40) that for $\nu \geqq 1$

$$\int_{-\pi}^{\pi} T_n(t-x) \cos \nu t \, dt = \pi A_{n\nu} \cos \nu x,$$

so that

$$|\cos \nu x - \pi A_{n\nu} \cos \nu x| \leqq \frac{g\nu^{k+1}}{n^{k+1}}$$

for all values of x, or, setting $x = 0$ in particular,

$$|1 - \pi A_{n\nu}| \leqq \frac{g\nu^{k+1}}{n^{k+1}}. \tag{43}$$

The same reasoning applied when $\nu = 0$ shows that

$$A_{n0} = \frac{1}{2\pi}$$

precisely.

It appears on substituting (40) in (39) that for any function $\phi(x)$

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t)\,dt$$

$$+ \sum_{\nu=1}^{n} \left\{ A_{n\nu} \left[\cos \nu x \int_{-\pi}^{\pi} \phi(t) \cos \nu t\,dt + \sin \nu x \int_{-\pi}^{\pi} \phi(t) \sin \nu t\,dt \right] \right\}$$

$$= a_0 + \sum_{\nu=1}^{n} d_{n\nu} (a_\nu \cos \nu x + b_\nu \sin \nu x),$$

where a_ν, b_ν are the Fourier coefficients of ϕ, and the numbers $d_{n\nu} = \pi A_{n\nu}$ are independent of ϕ. That is, the functions $S_n(x)$ are formed by applying a particular method of summation to the Fourier's series for ϕ. Rewritten in the new notation, (43) states that

$$|1 - d_{n\nu}| \leqq \frac{g\nu^{k+1}}{n^{k+1}}. \tag{44}$$

For the sake of uniformity we will introduce a coefficient $d_{n0} = 1$.

We are now in a position to define the approximating function desired. Let it be assumed that $\phi(x)$ and $\lambda(x)$ satisfy the hypotheses of Theorem II, and let α_n denote as before the general coefficient in the Sturm-Liouville series for $\phi(x)$. The function to be used is the following:

$$\Sigma_n(x) = \sum_{\nu=0}^{n} d_{n\nu}\, \alpha_\nu\, U_\nu(x).$$

It will be shown that as n becomes infinite $\Sigma_n(x)$ converges uniformly to a function $\psi(x)$, which is then necessarily continuous, so rapidly that

$$|\psi(x) - \Sigma_n(x)|$$

does not exceed a constant multiple of $1/n^k$. When this has been done, it will remain to be proved that $\psi(x)$ and $\phi(x)$ are identical.

Suppose $\phi(x)$ defined for values of x outside of the interval $(0, \pi)$ so as to be an even function of period 2π. This function will have everywhere a $(k-1)$th derivative satisfying a Lipschitz condition of the form (25), so that Lemma VI is applicable with $l = k$. Since ϕ is even, $S_n(x)$ now has the form

$$S_n(x) = \sum_{\nu=0}^{n} d_{n\nu}\, a_\nu \cos \nu x,$$

where

$$a_\nu = \frac{2}{\pi}\int_0^{\pi} \phi(t) \cos \nu t\,dt, \qquad \nu \geqq 1,$$

with a corresponding formula for a_0.

Consider the difference

$$\Sigma_{n+p}(x) - \Sigma_n(x),$$

where p is any positive integer. It is equal to

$$(45)\qquad \begin{aligned} S_{n+p}(x) - S_n(x) + \sum_{\nu=1}^{n} (d_{n+p,\,\nu} - d_{n\nu})[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x] \\ + \sum_{\nu=n+1}^{n+p} d_{n+p,\,\nu}[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x]. \end{aligned}$$

By Lemma VI,

$$|S_{n+p}(x) - S_n(x)| = |[\phi(x) - S_n(x)] - [\phi(x) - S_{n+p}(x)]| \leqq \frac{2g\mu}{n^k}.$$

From (44),

$$|d_{n+p,\,\nu} - d_{n\nu}| \leqq \frac{2g\nu^{k+1}}{n^{k+1}},$$

and from (27), the correctness of which was established under the hypotheses that we are using now,

$$(46)\qquad |\alpha_\nu U_\nu(x) - a_\nu \cos \nu x| \leqq \frac{\beta}{\nu^{k+1}},$$

where β is a constant. Hence the absolute value of each term under the first sign of summation in (45) is less than or equal to $2g\beta/n^{k+1}$, and the absolute value of the sum of n terms does not exceed $2g\beta/n^k$. As for the other sum, it follows from (44) that

$$|d_{n+p,\,\nu}| \leqq 1 + g,$$

for all values of n, p, and ν, while (46) is still satisfied, and so

$$\left|\sum_{\nu=n+1}^{n+p} d_{n+p,\,\nu}[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x]\right|$$

$$\leqq \beta(1+g) \sum_{\nu=n+1}^{n+p} \frac{1}{\nu^{k+1}} < \beta(1+g) \int_n^\infty \frac{dt}{t^{k+1}} = \frac{\beta(1+g)}{kn^k}.$$

Combining these facts, we see that the absolute value of $\Sigma_{n+p}(x) - \Sigma_n(x)$ does not exceed $1/n^k$ multiplied by a quantity independent of x, n, and p. Consequently $\Sigma_n(x)$ does uniformly approach a continuous limiting function $\psi(x)$ as n becomes infinite, and

$$\psi(x) = \Sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly for all values of x in the interval from 0 to π.

There would be no difficulty in showing that if $\phi^{(k-1)}(0) = 0$, the absolute value of $\psi(x) - \Sigma_n(x)$ does not exceed $c\mu/n^k$, where c is independent of ϕ; but as in other cases of the same sort we will not go into detail on this point.

It is still to be proved that the limiting function $\psi(x)$ is identical with $\phi(x)$. It might be interesting to know whether it is true that a convergent series will always be summable by means of the factors $d_{n\nu}$, and that the value so obtained will always be equal to the sum of the series. We shall leave this general question aside, and confine our attention to the special problem in hand, making use of the fact that $s_n(x)$, $S_n(x)$, and $\sigma_n(x)$ all converge to the value $\phi(x)$. The proof is as follows:

Let x be any number in the interval $0 \leqq x \leqq \pi$.

Let ϵ be any positive quantity.

Let $S_n^q(x)$ denote the sum of the first $q+1$ terms of $S_n(x)$, and $\Sigma_n^q(x)$ the sum of the first $q+1$ terms of $\Sigma_n(x)$, when $n \geqq q$.

Two integers n and q are to be chosen subject to seven conditions; the possibility of fulfilling these conditions will be made clear after the conditions themselves have been written down.

Let q be a number such that

$$|\phi(x) - s_q(x)| < \epsilon/7, \tag{47}$$

$$|\sigma_q(x) - \phi(x)| < \epsilon/7, \tag{48}$$

$$|[\Sigma_n(x) - \Sigma_n^q(x)] - [S_n(x) - S_n^q(x)]| < \epsilon/7 \tag{49}$$

for every value of $n > q$. Let q, once chosen, be held fast, and let a number $n > q$ be chosen so that

$$|s_q(x) - S_n^q(x)| < \epsilon/7, \tag{50}$$

$$|\Sigma_n^q(x) - \sigma_q(x)| < \epsilon/7, \tag{51}$$

$$|S_n(x) - \phi(x)| < \epsilon/7, \tag{52}$$

$$|\psi(x) - \Sigma_n(x)| < \epsilon/7. \tag{53}$$

That (47), (48), (52), and (53) can be satisfied, is an immediate consequence of the convergence of the respective sums involved. In (50),

$$s_q(x) - S_n^q(x) = \sum_{\nu=1}^{q} (1 - d_{n\nu})\, a_\nu \cos \nu x,$$

and for a fixed value of q the right-hand member involves n only in the differences $1 - d_{n\nu}$, and approaches zero as n becomes infinite. Similar reasoning applies to (51). In the remaining condition (49),

$$\Sigma_n(x) - \Sigma_n^q(x) = \sum_{\nu=q+1}^{n} d_{n\nu}\, \alpha_\nu\, U_\nu(x),$$

$$S_n(x) - S_n^q(x) = \sum_{\nu=q+1}^{n} d_{n\nu}\, a_\nu \cos \nu x,$$

and the difference of these expressions is, except as to notation, the second sum in (45), and is inferior in absolute value to a quantity which is independent of n and approaches zero as q becomes infinite. Hence (49) also can be fulfilled. Of course it will be essential that all seven conditions be satisfied simultaneously, but as each by itself is satisfied for *all* values of q or of n from some point on, there will be no difficulty in finding a single value of q for the first three, and then a single value of n for the last four.

With the justification of the seven inequalities, the proof is practically complete. It is found upon adding that the sum of the quantities inclosed in bars on the left-hand sides is simply $\psi(x) - \phi(x)$, and as the absolute value of this sum can not exceed the sum of the absolute values, it follows that

$$|\phi(x) - \psi(x)| < \epsilon$$

for suitable values of q and n. But ϕ and ψ are independent of q and n, and as ϵ is arbitrarily small, it must be that $\psi(x) = \phi(x)$ exactly. As this has been proved for a value of x which is any value in $(0, \pi)$, the equation is an identity. The conclusion is as follows:

THEOREM IV. *If $\phi(x)$ and $\lambda(x)$ satisfy the hypotheses of Theorem II, there exists for every positive integral value of n a linear combination $\Sigma_n(x)$ of the functions $U_\nu(x)$, $\nu = 0, 1, \cdots, n$, with constant coefficients, such that*

$$\phi(x) = \Sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly throughout the interval $0 \leqq x \leqq \pi$.

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SINGULAR INTEGRAL EQUATIONS OF THE VOLTERRA TYPE*

BY

CLYDE E. LOVE

1. Introduction. It has been shown by Evans† that the equation

$$\phi(x) = f(x) + \int_x^\infty K(x, t)\phi(t)\,dt \tag{1}$$

has a unique finite and integrable solution provided that, for all values of $x \geqq a$ and of $t \geqq x$, (a) the functions $f(x)$ and $K(x, t)$ are bounded and continuous; (b) the integral

$$\int_x^\infty |K(x, t)|\,dt$$

exists; (c) a constant b can be found such that

$$\int_x^\infty |K(x, t)|\,dt \leqq N < 1 \qquad (x \geqq b).$$

Equations of the form (1) sometimes arise,‡ however, for which the conditions of Evans's theorem are not satisfied. Various cases in which this is true are considered in the present paper. In each instance an attempt is made not merely to prove the existence of a continuous solution, but also to determine its behavior for large values of x.

The independent variables x and t will be restricted to real values. Functions of the single variable x will be considered for values of x lying either on the range

I $\qquad x \geqq a$

or on the range

I_b $\qquad a \leqq x \leqq b,$

* Presented to the Society, September 8, 1914.

† Atti della Reale Accademia dei Lincei, ser. 5, vol. 20 (1911), pp. 656–662. Cf. also: Ibid., vol. 20 (1911), pp. 7–11.

‡ For instance, in the theory of linear differential equations. Cf. Dini, Annali di Matematica, ser. 3, vol. 2 (1899), pp. 297–324; ibid., vol. 3 (1900), pp. 125–183; also papers by the present writer: Annals of Mathematics, ser. 2, vol. 15 (1914), pp. 145–156, and American Journal of Mathematics, vol. 36 (1914), pp. 151–166.

where b is arbitrary; functions of the two variables x, t will be considered for values of x and t in the domain

$$T \qquad x \geqq a, \qquad t \geqq x.$$

For brevity we shall speak of these three regions as the interval I, the interval I_b, and the region T respectively.

The functions $f(x)$ and $K(x, t)$ are assumed throughout to be continuous in I and in T respectively.

2. Theorem 1. *If the integral*

$$\int_x^\infty |K(x, t)f(t)|dt$$

converges in I, converges uniformly in I_b, and has in I the property that

$$\int_x^\infty |K(x, t)f(t)|dt < \theta|f(x)| \qquad (0 < \theta < 1), \tag{2}$$

where θ is a constant independent of x, then equation (1) *has in I a continuous solution $\phi(x)$ expressible in the form*

$$\phi(x) = f(x)\rho(x), \tag{3}$$

where $\rho(x)$ is a continuous function such that

$$|\rho(x)| < 1/(1 - \theta).$$

The method of iteration,* when applied to the equation (1), leads to consideration of the series

$$\phi(x) = \sum_{n=0}^\infty u_n(x), \tag{4}$$

where

$$u_0(x) = f(x),$$

$$\begin{aligned} u_n(x) &= \int_x^\infty \int_t^\infty \cdots \int_{t_{n-3}}^\infty \int_{t_{n-2}}^\infty K(x, t)K(t, t_1) \\ &\qquad \cdots K(t_{n-3}, t_{n-2})K(t_{n-2}, t_{n-1})f(t_{n-1})\,dt_{n-1}\,dt_{n-2}\cdots dt_1\,dt \\ &= \int_x^\infty K(x, t)u_{n-1}(t)\,dt \qquad (n = 1, 2, \cdots). \end{aligned}$$

We shall first show by mathematical induction that, in I, $u_n(x)$ is continuous and satisfies the inequality

$$|u_n(x)| < \theta^n|f(x)| \qquad (n = 1, 2, \cdots). \tag{5}$$

* Cf., for example, Bôcher, *An Introduction to the Study of Integral Equations* (1909), p. 14.

Assume that $u_{n-1}(x)$ is continuous in I, and that

$$|u_{n-1}(x)| \leqq \theta^{n-1}|f(x)|.$$

Then $u_n(x)$ is continuous in I and satisfies (5). For, since the integral

$$\int_x^\infty |K(x,t)f(t)|dt$$

converges uniformly in I_b, we can find for every value of $\epsilon > 0$ a value of μ independent of x, such that

$$\int_\mu^{\mu'} |K(x,t)f(t)|dt < \epsilon, \tag{6}$$

where μ' has any value greater than μ, and x lies in I_b. Now

$$\left|\int_\mu^{\mu'} K(x,t)u_{n-1}(t)\,dt\right| \leqq \int_\mu^{\mu'} |K(x,t)u_{n-1}(t)|dt$$
$$\leqq \theta^{n-1}\int_\mu^{\mu'} |K(x,t)f(t)|dt < \epsilon,$$

by (5) and (6). But this last relation establishes the uniform convergence of $u_n(x)$ in I_b, and hence its continuity at all points in I, since an interval I_b can always be found so as to include any preassigned point in I. Further,

$$|u_n(x)| \leqq \int_x^\infty |K(x,t)u_{n-1}(t)|dt \leqq \theta^{n-1}\int_x^\infty |K(x,t)f(t)|dt < \theta^n|f(x)|,$$

by (2).

To complete the proof by induction we need only note that our assumption regarding $u_{n-1}(x)$ is justified, by hypothesis, for $n = 1$.

As a consequence of (5), we have

$$\sum_{n=0}^{\infty} |u_n(x)| < \frac{1}{1-\theta}|f(x)|, \tag{7}$$

so that the series $\sum_{n=0}^{n=\infty} u_n(x)$ is absolutely convergent in I, and $\phi(x)$ as given by (4) is defined in that interval. Further, the same series is uniformly convergent in I_b, by Weierstrass's test, since

$$|u_n(x)| < N\theta^n,$$

where N is the maximum value of $|f(x)|$ in I_b. The function $\phi(x)$ is therefore the sum of a uniformly convergent series of continuous functions, and is itself continuous.

Let us now multiply the series for $\phi(t)$ by $K(x,t)$ and integrate term by term from x to ∞. That this is allowable appears from the following theorem:*

* Bromwich, *Infinite Series* (1908), p. 453.

If the series $\sum_{n=0}^{n=\infty} f_n(x)$ *converges uniformly in any fixed interval* $a \leqq x \leqq b$, *where* b *is arbitrary, and if* $F(x)$ *is continuous for all finite values of* x, *then*

$$\int_a^\infty F(x)\left[\sum_{n=0}^{\infty} f_n(x)\right] dx = \sum_{n=0}^{\infty} \int_a^\infty F(x) f_n(x)\, dx,$$

provided that the integral

$$\Phi(x) = \int_a^\infty |F(x)| \sum_{n=0}^{\infty} |f_n(x)|\, dx,$$

converges.

For, if we take $a = x$, $x = t$, $F(t) = K(x, t)$, $f_n(t) = u_n(t)$, it is only necessary to show that the integral $\Phi(x)$ converges. But this follows at once from (7) and (2).

Upon performing the term-by-term integration mentioned above, we obtain the equation

$$\int_x^\infty K(x, t)\phi(t)\, dt = \sum_{n=1}^{\infty} u_n(x) = \phi(x) - f(x),$$

which shows that $\phi(x)$ as given by (4) is a solution of (1).

From (7) it appears at once that we may write $\phi(x)$ in the form (3). The function $f(x)$ cannot vanish in I, by (2), so that $\rho(x)$ is continuous. This completes the proof of the theorem.

That the solution thus obtained is not necessarily the only continuous solution* appears from the following example. The hypotheses of the theorem are evidently satisfied by the equation

$$\phi(x) = e^x + \tfrac{1}{2}e^{2x} \int_x^\infty e^{-2t}\phi(t)\, dt,$$

since

$$\int_x^\infty |K(x, t) f(t)|\, dt = \tfrac{1}{2}e^{2x} \int_x^\infty e^{-t}\, dt = \tfrac{1}{2}e^x.$$

The equation evidently has the infinity of solutions

$$\phi(x) = 2e^x + ce^{3x/2},$$

where c is an arbitrary constant, the solution given by the theorem being obtained by taking $c = 0$.

Similar examples can be constructed for each of the cases still to be treated.

In case the hypotheses of the theorem are not satisfied for all values of

* It may be remarked that in the case treated by Evans (loc. cit.) there may likewise be more than one continuous solution. The equation

$$\phi(x) = 1 + \tfrac{1}{2}e^x \int_x^\infty e^{-t}\phi(t)\, dt,$$

which evidently satisfies the conditions of Evans's theorem, has the family of continuous solutions

$$\phi(x) = 2 + ce^{x/2},$$

where c is an arbitrary constant.

x in I, but only for sufficiently large values, say $x \geqq a' > a$, we may proceed by "prolongement"* to obtain a continuous solution valid throughout I. Write the equation (1) in the form

$$\phi(x) = f_1(x) + \int_x^{a'} K(x, t)\phi(t)\,dt, \tag{8}$$

where

$$f_1(x) = f(x) + \int_{a'}^{\infty} K(x, t)\phi(t)\,dt.$$

Our theorem may be used to determine $\phi(x)$ for $x \geqq a'$, after which $f_1(x)$ becomes a known function, and (8) reduces to a *regular* Volterra equation whose solution in the interval $a \leqq x \leqq a'$ may be found by the usual methods.

A similar remark will apply to each of the later theorems.

3. Theorem 2. *If there exists a continuous positive real function $h(x)$ such that the integrals*

$$\int_x^{\infty} |K(x, t)f(t)|\,dt, \qquad \int_x^{\infty} |K(x, t)|h(t)\,dt$$

converge in I, converge uniformly in I_b, and satisfy in I the inequalities

$$\int_x^{\infty} |K(x, t)f(t)|\,dt < h(x), \tag{9}$$

$$\int_x^{\infty} |K(x, t)|h(t)\,dt < \theta h(x) \qquad (0 < \theta < 1), \tag{10}$$

where θ is a constant independent of x, then equation (1) *has in I a continuous solution $\phi(x)$ expressible in the form*

$$\phi(x) = f(x) + h(x)\rho(x), \tag{11}$$

where $\rho(x)$ is a continuous function such that

$$|\rho(x)| < 1/(1 - \theta).$$

COROLLARY. *If the integrals*

$$\int_x^{\infty} |K(x, t)f(t)|\,dt, \qquad \int_x^{\infty} |K(x, t)|\,dt$$

converge in I, converge uniformly in I_b, and satisfy in I the inequalities

$$\int_x^{\infty} |K(x, t)f(t)|\,dt < N, \qquad \int_x^{\infty} |K(x, t)|\,dt < \theta,$$

where N is a constant, then equation (1) *has in I a continuous solution $\phi(x)$ expressible in the form*

$$\phi(x) = f(x) + \rho(x),$$

where

$$|\rho(x)| < N/(1 - \theta).$$

* Cf. Evans, loc. cit., p. 662.

To prove the theorem, consider again the series (4). We note first that $u_n(x)$ is continuous in I and satisfies the inequality

$$|u_n(x)| < \theta^{n-1} h(x) \qquad (n = 1, 2, \cdots). \tag{12}$$

For, if we assume that $u_{n-1}(x)$ has these properties, the same will be true for $u_n(x)$, by argument similar to that of § 2. But the assumption holds for $u_1(x)$, by hypothesis.

It follows from (12) that the series (4) converges absolutely in I and uniformly in I_b. By making obvious modifications in the argument of § 2, we see that $\phi(x)$ as given by (4) is continuous in I, and satisfies equation (1) in the same interval.

By (12), we have

$$\sum_{n=1}^{\infty} |u_n(x)| < \frac{1}{1-\theta} h(x),$$

so that

$$|\phi(x) - f(x)| < \frac{1}{1-\theta} h(x).$$

The form (11) for $\phi(x)$ results at once.

The corollary is merely that case of the theorem in which we may take

$$h(x) = N.$$

4. Theorem 3. *If there exists a continuous positive real function $\eta(x)$ for which the integral*

$$\int_x^{\infty} \eta(t)\, dt$$

exists, and such that, in T,

$$|K(x, t) f(t)| < |f(x)| \eta(t), \tag{13}$$

then equation (1) *has in I a continuous solution $\phi(x)$ expressible in the form*

$$\phi(x) = f(x)[1 + \epsilon(x)],$$

where $\epsilon(x)$ is continuous and has the property that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

We may evidently assume without loss of generality that

$$\int_x^{\infty} \eta(t)\, dt < \theta \qquad (0 < \theta < 1). \tag{14}$$

For, if this is not true throughout I, it will be true in the interval $x \geqq a'$, where a' is sufficiently large. After establishing the existence of a continuous solution of (1) in this latter interval, we may by prolongement* extend it to all values of x in I.

* Cf. § 2.

It results from (13) and (14) that

$$\int_x^\infty |K(x,t)f(t)|dt < |f(x)| \int_x^\infty \eta(t)\,dt < \theta|f(x)| \qquad (0 < \theta < 1).$$

Further, the integral

$$\int_x^\infty |K(x,t)f(t)|dt$$

converges uniformly in I_b, by the following (Weierstrass's) test.†

Suppose that for all values of x in the interval $a \leqq x \leqq b$ the function $F(x,t)$ satisfies the condition

$$|F(x,t)| < M(t),$$

where $M(t)$ is a positive function, independent of x. Then if the integral

$$\int_x^\infty M(t)\,dt$$

converges, the integral

$$\int_x^\infty F(x,t)\,dt$$

is absolutely and uniformly convergent for all values of x in the interval $a \leqq x \leqq b$.

Let us write

$$\int_x^\infty |K(x,t)f(t)|dt = \int_x^b |K(x,t)f(t)|dt + \int_b^\infty |K(x,t)f(t)|dt.$$

The test applies at once to the integral last written if we take

$$m = b, \qquad F(x,t) = |K(x,t)f(t)|, \qquad M(t) = N\eta(t),$$

where N is the maximum value of $|f(x)|$ in I_b.

Thus the conditions of Theorem 1 are satisfied in the present case, and by that theorem there exists a continuous solution of (1) of the form

$$\phi(x) = f(x)\rho(x).$$

If we put

$$\rho(x) = 1 + \epsilon(x),$$

it appears by (4) that

$$f(x)\epsilon(x) = \sum_{n=1}^\infty u_n(x),$$

and by (7) we have

$$|f(x)\epsilon(x)| \leqq \sum_{n=1}^\infty |u_n(x)| < \frac{\theta}{1-\theta}|f(x)|,$$

or

$$|\epsilon(x)| < \theta/(1-\theta).$$

† Cf., for instance, Bromwich, *Infinite Series*, p. 434.

Now θ may be taken arbitrarily small if x be taken sufficiently large, so that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

This completes the proof.

5. Theorem 4. *If there exist functions $h(x)$ and $\eta(x)$* such that the inequalities*

$$|K(x,t)f(t)| < h(x)\eta(t), \tag{15}$$

$$|K(x,t)|h(t) < h(x)\eta(t) \tag{16}$$

are satisfied in T, then equation (1) *has in I a continuous solution $\phi(x)$ expressible in the form*

$$\phi(x) = f(x) + h(x)\epsilon(x),$$

where $\epsilon(x)$ is continuous and has the property that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

COROLLARY. *If there exists a function $\eta(x)$ such that the inequalities*

$$|K(x,t)f(t)| < \eta(t), \qquad |K(x,t)| < \eta(t)$$

are satisfied in T, then equation (1) *has in I a continuous solution $\phi(x)$ expressible in the form*

$$\phi(x) = f(x) + \epsilon(x),$$

where

$$\lim_{x=\infty} \epsilon(x) = 0.$$

From (15) and (16) it appears that (9) and (10) are satisfied. The integrals

$$\int_x^\infty |K(x,t)f(t)|dt \quad \text{and} \quad \int_x^\infty |K(x,t)|h(t)\,dt$$

are readily seen to satisfy the other conditions of Theorem 2. Hence equation (1) has a continuous solution

$$\phi(x) = f(x) + h(x)\epsilon(x),$$

where

$$\epsilon(x) = \rho(x) = \frac{u_1(x)}{h(x)} + \frac{1}{h(x)}\sum_{n=2}^{\infty} u_n(x).$$

Now by (15)

$$\frac{|u_1(x)|}{h(x)} < \theta,$$

and by (12)

$$\frac{1}{h(x)}\sum_{n=2}^{\infty} |u_n(x)| < \frac{\theta}{1-\theta},$$

* The functions $h(x)$ and $\eta(x)$ have the same properties as in Theorem 2 and Theorem 3 respectively.

so that

$$|\epsilon(x)| < \theta + \theta/(1-\theta).$$

The desired result follows as in § 4.

To prove the corollary, take $h(x) = 1$.

6. Application. The foregoing results may be used in a variety of ways to discuss particular equations of the form (1). The following may be mentioned as a typical application:

Let $p(x)$, $q(x)$, and $r(x)$ be polynomials in x such that for large values of x

$$R[p(x)] < R[q(x)] < R[r(x)],$$

where $R[x]$ means the real part of x. If for sufficiently large values of x, and for values of $t \geqq x$, we may write, in Poincaré's sense,*

$$f(x) \sim e^{q(x)} x^{\alpha}\left(a_0 + \frac{a_1}{x} + \cdots\right),$$

$$K(x,t) \sim e^{p(x)} x^{\beta} e^{-r(t)} t^{\gamma}\left(b_{0,0} + \frac{b_{1,0}}{x} + \frac{b_{0,1}}{t} + \frac{b_{2,0}}{x^2} + \cdots\right),$$

where α, β, γ, $a_0, \cdots, b_{0,0}, \cdots$ are determinate constants, then for the same values of x the equation (1) has a continuous solution $\phi(x)$ such that we may write

$$\phi(x) \sim f(x).$$

To prove this statement, we note first that x may be taken so large that

$$|K(x,t)| < N_1 |e^{p(x)} x^{\beta}|\,|e^{-r(t)} t^{\gamma}|, \qquad |f(x)| < N_2 |e^{q(x)} x^{\alpha}|,$$

where N_1 and N_2 are certain constants. Thus

$$|K(x,t) f(t)| < N_1 N_2 |e^{p(x)} x^{\beta} e^{q(t)-r(t)} t^{\alpha+\gamma}|. \tag{17}$$

Let us now apply Theorem 4, taking

$$h(x) = |e^{p(x)} x^{\beta}|, \qquad \eta(x) = N|e^{q(x)-r(x)}|x^k,$$

where N is the larger of the quantities N_1 and $N_1 N_2$, and k is the larger of the quantities

$$R[\alpha+\beta], \qquad R[\alpha+\gamma].$$

Since

$$R[q(x)] < R[r(x)],$$

the integral

$$\int_x^{\infty} \eta(t)\,dt$$

*Acta Mathematica, vol. 8 (1886), p. 296.

exists. We have directly, by (17), the inequality

$$|K(x,t)f(t)| < h(x)\eta(t).$$

Also

$$|K(x,t)|h(t) < N_1|e^{p(x)}\, x^{\beta}\, e^{p(t)-r(t)}\, t^{\beta+\gamma}| < h(x)\eta(t),$$

since

$$R[p(x)] < R[q(x)]. \tag{18}$$

Thus the hypotheses of Theorem 4 are satisfied, and equation (1) has a continuous solution of the form

$$\begin{aligned}\phi(x) &= f(x) + h(x)\epsilon(x)\\ &= e^{q(x)}\, x^{\alpha}\left[a_0 + \frac{a_1}{x} + \cdots + \frac{a_n + \omega_n(x)}{x^n}\right] + |e^{p(x)}\, x^{\beta}|\epsilon(x),\end{aligned}$$

where n is an arbitrary positive integer and

$$\lim_{x=\infty} \omega_n(x) = 0, \qquad \lim_{x=\infty} \epsilon(x) = 0.$$

Put

$$\bar{\epsilon}(x) = \frac{\epsilon(x)|e^{p(x)}\, x^{\beta}|}{e^{p(x)}\, x^{\beta}},$$

$$\bar{\omega}_n(x) = \omega_n(x) + e^{p(x)-q(x)}\, x^{\beta-\alpha+n}\, \bar{\epsilon}(x).$$

Then we have

$$\phi(x) = e^{q(x)}\, x^{\alpha}\left[a_0 + \frac{a_1}{x} + \cdots + \frac{a_n + \bar{\omega}_n(x)}{x^n}\right].$$

It appears by (18) that

$$\lim_{x=\infty} \bar{\omega}_n(x) = 0,$$

whence we may write

$$\phi(x) \sim e^{q(x)}\, x^{\alpha}\left(a_0 + \frac{a_1}{x} + \cdots\right).$$

Thus $\phi(x)$ has an asymptotic expansion coinciding with that of $f(x)$, which was to be proved.

It is believed that the present theorems may prove useful in the study of certain problems in the theory of linear differential equations. (The writer hopes to consider applications of this type in a later paper.)

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ON THE REDUCTION OF INTEGRO-DIFFERENTIAL EQUATIONS*

BY

GRIFFITH C. EVANS

Certain integro-differential equat ons can be completely treated by first resolving an integral equation, and second, integrating a differential equation. In so far, such equations do not constitute a new problem in analysis, because they are reducible to equations of simpler types. From this point of view Volterra discusses the equation

$$\nabla^2 V(x, y, z|t) + \int_{t_0}^{t} f(t, \tau) \nabla^2 V(x, y, z|\tau) d\tau = F(x, y, z, t)$$

which is satisfied by the potential V of an electric field in an isotropic medium, in the so-called "static case" of hysteresis.† By solving this equation, regarded as an integral equation, for $\nabla^2 V$, we are led to the differential equation of Poisson to determine V.

We may however have the case that an integro-differential equation whose solution is subject to certain boundary conditions is reducible to one or more problems in equations of more elementary character only when account is taken of these conditions. This is the point of view to be developed in the present paper. It is a point of view which can be extended to more general types of functional equations.

We shall treat, in the following pages, the reduction of boundary value problems in the case of the integro-differential equation of parabolic type, and quite briefly that of hyperbolic type, to problems in linear differential and integral equations separately. It is worth while perhaps to specially mention § 2, in which the generalization of the integro-differential equation of the second order to an equation of the first order involving integration

* Presented to the Society, December, 1912, and December, 1913. Since the first writing, the paper has been entirely revised, partly to include new material, and partly to take advantage of the material of Prof. W. A. Hurwitz's thesis, *Randwertaufgaben bei Systemen von linearen partiellen Differentialgleichungen erster Ordnung*, Diss. Göttingen, 1910, of which the author was ignorant at the time of the first writing.

† V. Volterra, *Lectures delivered at the celebration of the 20th anniversary of the foundation of Clark University* (Sept., 1911), p. 75; *Sulle equazioni integro differenziali*, Rendiconti della R. Accademia dei Lincei, vol. 18 (1909), p. 174; *Sulle equazioni integro-differenziali della teoria dell'elasticità*, ibid., vol. 18 (1909), p. 297.

around an arbitrary path demands a corresponding restatement and treatment of the non-homogeneous linear partial differential equation of parabolic type.*

1. The Integro-Differential Equation of Parabolic Type

1. The region and its corresponding equation. For the parabolic *differential* equation the earliest boundary value problems referred to an open rectangle as the boundary. Afterwards the results were extended to curves of a more general nature.

In connection with the integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = \int_{t_0}^{t} A(t,\tau)\frac{\partial u(x,\tau)}{\partial \tau}d\tau$$

there is a boundary value problem based upon the open rectangle: if under proper conditions a chain of values is given along the three sides of this rectangle, the given equation then uniquely determines them along the fourth side. In fact, for all the partial integro-differential equations heretofore considered, the boundaries have been of this sort. And if we wish to extend the results to a more general type of region, it is also opportune for us, as it turns out, to consider a slightly more general type of integro-differential equation.

We shall take as the open boundary the curves $t = t_0$ on the left, and the curves $x = \xi_2(t)$ above, and $x = \xi_1(t)$ below. We shall assume that $\xi_1(t)$ and $\xi_2(t)$ and their first and second derivatives remain single valued, finite and continuous; moreover that $\xi_2(t) > \xi_1(t)$ for any value of $t \geqq t_0$. The derivatives $\xi_1'(t)$ and $\xi_2'(t)$ are to vanish only a finite number of times in the interval $t_0 \leqq t \leqq t_1$. For simplicity we shall impose also the unnecessary restriction that $x = \xi_1(t)$ shall have no maximum and $x = \xi_2(t)$ no minimum, in the interval $t_0 < t < t_1$.†

We shall adopt Professor Hurwitz's convenient notation ${}_a\Omega_b$ for the space bounded by these curves and by the straight lines $t = a$ on the left and $t = b$ on the right.

Along the open boundary is to be given the continuous chain of values

$$(\alpha)\quad F(x, T_x) = \begin{cases} f(x) & \text{when } t = t_0 \qquad [\xi_1(t_0) \leqq x \leqq \xi_2(t_0)] \\ \left.\begin{array}{ll} \phi_1(t) & \text{when } x = \xi_1(t) \\ \phi_2(t) & \text{when } x = \xi_2(t) \end{array}\right\} & (t_0 \leqq t \leqq t_1) \end{cases}$$

* For an extensive treatment of the differential equation of parabolic type see Hurwitz, loc. cit., on which our § 1 is based, and also for a treatment that demands different conditions E. E. Levi, *Sul l'equazione del calore*, Annali di Matematica, vol. 14 (1908), p. 187.

† This restriction may be removed by an extension of the present method of proof, or by a continuous prolongation of the equation and its solution.

where the functions $f(x)$, $\phi_1(t)$, and $\phi_2(t)$ are supposed to have continuous first derivatives in the range under consideration, and T_x is used to denote the t-coördinate of the left-hand point on the boundary whose other coördinate is x. When it is necessary to use the t-coördinate of the right-hand boundary point whose other coördinate is x, we shall denote it by t_x, t_{1x}, etc., according as the right-hand limit of the region is formed by $t = t$, $t = t_1$, etc.

In connection with this region and these boundary values, we shall investigate the equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = \int_{T_x}^{t} A(x,t,\tau)\frac{\partial u(x,\tau)}{\partial \tau}\,d\tau. \tag{1}$$

We shall say that a function $u(x,t)$ is regular with respect to this equation if within the region and on its boundary u and $\partial u/\partial x$ remain finite and continuous, and within the region $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ exist except on a regularly distributed set of curves,* and are integrable linearly in the Riemann sense.

2. The theorem of reducibility for equation (1). In equation (1) let $A(x,t,\tau)$ and its first partial derivatives with respect to the three arguments be finite and continuous functions within and on the boundary of the region, and inside that boundary let $\partial A(x,t,\tau)/\partial\tau = A_1(x,t,\tau)$ have a derivative in regard to x that remains finite and continuous, except on a regularly distributed set of curves.

Under these conditions, there is one and only one regular solution $u(x,t)$ of the equation (1), *which joins on continuously to the boundary values* (α). *It can be reduced to the form*

$$u(x_1,t_1) = u'(x_1,t_1) - \iint_{t_0\Omega t_1} k(x_1,t_1|x,t)\,u'(x,t)\,dx\,dt, \tag{2}$$

in which $u'(x,t)$ is the regular solution of the differential equation

$$\frac{\partial u'(x,t)}{\partial t} - \frac{\partial^2 u'(x,t)}{\partial x^2} = -A(x,t,T_x)F(x,T_x) \tag{3}$$

which takes on the boundary values (α), *and $k(x_1,t_1|x,t)$ is the function associated to*

$$\begin{aligned} K(x_1,t_1|x,t) = \frac{1}{2\sqrt{\pi}}\Big\{ & g(x,t|x_1,t_1)A(x,t,t) \\ & - \int_t^{t_{1x}} A_1(x,\tau,t)\,g(x,\tau|x_1,t_1)\,d\tau \Big\} \end{aligned} \tag{4}$$

by means of Volterra's relation

* See M. Bôcher, *Introduction to Integral Equations* (1909), p. 3.

$$
\begin{aligned}
k(x_1, t_1 | x, t) + K(x_1, t_1 | x, t) &= \iint_{{}_t\Omega_{t_1}} K(x_1, t_1 | \xi, \tau)\, k(\xi, \tau | x, t)\, d\xi\, d\tau \\
&= \iint_{{}_t\Omega_{t_1}} k(x_1, t_1 | \xi, \tau)\, K(\xi, \tau | x, t)\, d\xi\, d\tau .
\end{aligned} \tag{5}
$$

By $g(x, t | x_1, t_1)$ is denoted the Green's function for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.^*$$

It shou'd be noticed that $u'(x, t)$ depends on the boundary conditions, but that $K(x_1, t_1 | x, t)$ is independent of them, and is determined merely by the coefficients of the equation and the nature of the region. We shall now give a proof of this theorem.

3. The equivalence of the integral and the integro-differential equations. By means of an integration by parts, the equation (1) may be rewritten in the form

$$
\begin{aligned}
\frac{\partial u(x, t)}{\partial t} - \frac{\partial^2 u(x, t)}{\partial x^2} &= A(x, t, t)\, u(x, t) - A(x, t, T_x)\, F(x, T_x) \\
&\quad - \int_{T_x}^{t} A_1(x, t, \tau)\, u(x, \tau)\, d\tau .
\end{aligned} \tag{6}
$$

We have the theorem:† *If $\phi(x, y)$ is continuous in the given region, and if in the neighborhood of a point (x_1, y_1) within that region $\partial\phi/\partial x$ exists and is continuous, then the function*

$$\bar{u}(x_1, t_1) = \frac{1}{2\sqrt{\pi}} \iint_{{}_{t_0}\Omega_{t_1}} g(x, t | x_1, t_1)\, \phi(x, t)\, dx\, d$$

is regular in the neighborhood of the point (x_1, y_1) and in that neighborhood satisfies the differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \phi(x, t).$$

Moreover it vanishes on the open boundary of the region Ω.

If we examine the function $A(x, t, T_x)\, F(x, T_x)$ we notice that its

---

* We may write

$$g(x, t | x_1, t_1) = h_{01/2} + g',$$

where g' is regular and

$$h_{\alpha\beta}(x, t | x_1, t_1) = \frac{(x_1 - x)^\alpha}{(t_1 - t)^\beta}\, e^{-\frac{(x_1 - x)^2}{4(t_1 - t)}}$$

(see for instance loc. cit. Levi, § 7).

† This is a slight but obvious generalization of Hurwitz's theorem VIII *b* (Hurwitz, loc. cit., p. 85), to which the same proof applies.

derivative in regard to x exists and is continuous except on a finite number of lines parallel to the t-axis, namely, those lines which pass through points where $\xi_1'(t)$ or $\xi_2'(t)$ vanishes, and the two lines $x = \xi_1(t_0)$ and $x = \xi_2(t_0)$. Hence we know that the regular solution $u'(x, t)$ of (3), which satisfies the given conditions (α), exists and is uniquely determined. If we write then $u(x, t) = u'(x, t) + v(x, t)$ we have the following integro-differential equation to determine $v(x, t)$;

$$\begin{aligned}\frac{\partial v(x,t)}{\partial t} - \frac{\partial^2 v(x,t)}{\partial x^2} &= A(x,t,t)v(x,t) - \int_{T_x}^{t} A_1(x,t,\tau)v(x,\tau)\,d\tau \\ &\quad + A(x,t,t)u'(x,t) - \int_{T_x}^{t} A_1(x,t,\tau)u'(x,\tau)\,d\tau,\end{aligned} \tag{7}$$

with the condition that it be regular, and join on continuously to zero boundary values.

By means of the customary analysis with Green's theorem, we find as a necessary condition on $v(x, t)$ that it satisfy the equation

$$\begin{aligned}v(x_1, t_1) = \frac{1}{2\sqrt{\pi}} \iint_{t_0 \Omega t_1} g(x, t|x_1, t_1) \Big\{ & A(x,t,t)v(x,t) \\ & - \int_{T_x}^{t} A_1(x,t,\tau)v(x,\tau)\,d\tau \Big\}\,dx\,dt \\ + \frac{1}{2\sqrt{\pi}} \iint_{t_0 \Omega t_1} g(x, t|x_1, t_1) \Big\{ & A(x,t,t)u'(x,t) \\ & - \int_{T_x}^{t} A_1(x,t,\tau)u'(x,\tau)\,d\tau \Big\}\,dx\,dt,\end{aligned} \tag{8}$$

whence, if we add $u'(x_1, t_1)$ to both members, it follows that we must have

$$\begin{aligned}u(x_1, t_1) = u'(x_1, t_1) + \frac{1}{2\sqrt{\pi}} \iint_{t_0 \Omega t_1} g(x, t|x_1, t_1) \Big\{ & A(x,t,t)u(x,t) \\ & - \int_{T_x}^{t} A_1(x,t,\tau)u(x,\tau)\,d\tau \Big\}\,dx\,dt.\end{aligned} \tag{9}$$

This, by a permissible change in the order of integration, since $g(x, t|x_1, t_1)$ is a continuous function divided by $\sqrt{t_1 - t}$, may be written in the form

$$u(x_1, t_1) = u'(x_1, t_1) + \iint_{t_0 \Omega t_1} K(x_1, t_1|x, t)\,u(x, t)\,dx\,dt, \tag{10}$$

where $K(x_1, t_1|x, t)$ is given by (4). It is necessary then that $u(x_1, t_1)$ be a solution of (10) within the region.

On the other hand if $u(x_1, t_1)$ is continuous and a solution of (10), it will be regular and a solution of (1), and will join on continuously to the boundary values (α). In fact

$$\frac{\partial u}{\partial x_1} = \frac{\partial u'}{\partial x_1} + \frac{1}{2\sqrt{\pi}} \iint_{t_0 \Omega t_1} \frac{\partial g(x, t | x_1, t_1)}{\partial x_1} \left\{ A(x, t, t) u(x, t) - \int_{T_x}^{t} A_1(x, t, \tau) u(x, \tau) d\tau \right\} dx\, dt,$$

and is therefore finite and continuous. Hence if we put

$$\phi(x, t) = A(x, t, t) u(x, t) - \int_{T_x}^{t} A_1(x, t, \tau) u(x, \tau) d\tau \tag{11}$$

we see that

$$\frac{\partial \phi(x, t)}{\partial x} = \frac{\partial}{\partial x}[A(x, t, t) u(x, t)] - \int_{T_x}^{t} \frac{\partial}{\partial x}[A_1(x, t, \tau) u(x, \tau)] d\tau + A_1(x, t, T_x) u(x, T_x) \frac{dT_x}{dx}$$

represents a function of x, t which is continuous except for discontinuities regularly distributed. On some of these lines of discontinuity $\partial\phi/\partial x$ will become infinite. But referring to the theorem of Professor Hurwitz already quoted, we see that the function $\bar{u}(x, t)$ determined by giving $\phi(x, t)$ the value from (11) will be regular and a solution of equation (7), which takes on zero boundary values. Moreover, as we see from equation (9) this $\bar{u}(x, t)$ is precisely $u(x, t) - u'(x, t)$ where $u(x, t)$ is the supposed continuous solution of (10), and $u'(x, t)$ is the $u'(x, t)$ already specified. Hence $u(x, t)$ will satisfy (6), and therefore (1), will be regular, and will take on the given boundary values (α). Our assertion is therefore proved.

It may be noticed here that the discontinuities in the derivatives of highest order in the integro-differential equation (1) appear as projected through the region from points on the boundary, on account of the dependence of the equation on the chosen region, a fact which has no analog in the theory of the parabolic differential equation.

4. The integral equation (10) and its solution in closed form. It merely remains to show that (10) has a unique continuous solution, which can be written in the form (2).

We can show directly that equation (10) has one and only one continuous solution. In fact, the solution may be written in the form of an absolutely and uniformly convergent series of continuous functions*

* Evans, *Sull'equazione integro-differenziale di tipo parabolico,* Rendiconti della R. Accademia dei Lincei, vol. 21 (1912), p. 28. The proof applies with obvious modifications to this case.

$$u(x_1, t_1) = \sum_{n=1}^{\infty} u^{(n)}(x_1, t_1), \tag{12}$$

where

$$u^{(n)}(x_1, t_1) = \iint_{t_0\Omega_{t_1}} K(x_1, t_1|x, t)\, u^{(n-1)}(x, t)\, dx\, dt, \qquad n > 1,$$

$$u^{(1)}(x_1, t_1) = u'(x_1, t_1).$$

In order to reduce the solution to the closed form (3), we go through a process which incidentally gives us another proof of its existence. We must determine the function $k(x_1, t_1|x, t)$ which satisfies Volterra's relation (equation (5)).

To this purpose let us consider the series formed in the usual way

$$-\sum_{1}^{\infty} K_i(x_1, t_1|x, t) \tag{13}$$

in which

$$K_1(x_1, t_1|x, t) = K(x_1, t_1|x, t)$$

$$K_i(x_1, t_1|x, t) = \iint_{t\Omega_{t_1}} K(x_1, t_1|\xi, \tau)\, K_{i-1}(\xi, \tau|x, t)\, d\xi\, d\tau.$$

Of this series every term but the first remains finite. In fact for the second term, since from the explicit expression for $K(x_1, t_1|x, t)$

$$|K(x_1, t_1|\xi, \tau)| \leqq \sqrt{\frac{M}{t_1 - \tau}},$$

in which M is some constant, we have

$$\iint_{t\Omega_{t_1}} |K(x_1, t_1|\xi, \tau)\, K(\xi, \tau|x, t)|\, dt \leqq M^2 A \int_t^{t_1} \frac{d\tau}{\sqrt{t_1 - \tau}\sqrt{\tau - t}} \leqq \pi M^2 A, \tag{14}$$

where A is some constant. Hence that term remains finite. The $(n+2)$th term is in absolute value not greater than the expression

$$\frac{C\{D(t_1 - t)\}^n}{n!}, \qquad n \geqq 0,$$

where C and D are certain constants,—provided of course that we are dealing with integrable functions. If we neglect for the moment this proviso, we see that the series, starting with the second term, is absolutely and uniformly convergent.

The first term is integrable over the given x, t region for any given values

of x_1, t_1, and is continuous in the neighborhood of any point $(x_1, t_1|x, t)$ provided that we do not have both $x_1 = x$, and $t_1 = t$. It is understood of course that $t_1 \geqq t$. Let us now consider the remaining terms in regard to continuity, and incidentally therefore in regard to integrability.

The second term, as we have shown, remains finite, and is continuous except when we happen to have $x_1 = x$, $t_1 = t$. Let us then examine the value of this term in the four dimensional neighborhood of a point

$$(X, T|X, T),$$

where (X, T) is within Ω, and show that by taking our neighborhood small enough, we can make the value of this term as small as we please $(\leqq \eta)$ provided that we keep $t_1 \geqq t$; in other words, that the second term is continuous at any point $(X, T|X, T)$, if it is given there the value zero.

By referring to the explicit expression for $K(x_1, t_1|x, t)$, we see that the only term in $K_2(x_1, t_1|x, t)$ that requires examination is

$$\frac{1}{4\pi}\int\int_{t\Omega_{t_1}} \frac{A(x, \tau, \tau)\, A(x, t, t)}{\sqrt{t - \tau}\sqrt{\tau - t}}\, e^{-(x_1-\xi)^2/4(t_1-\tau)-(\xi-x)^2/4(\tau-t)}\, d\xi\, d\tau.$$

Let us divide the region of integration for ξ into three parts, c_1 to $X - \Delta/2$, $X - \Delta/2$ to $X + \Delta/2$, and $X + \Delta/2$ to c_2, where c_1 and c_2 are respectively the minimum and maximum values of x in Ω. If we write $|A(x, t, t)| < N$, and make use of the inequality (14), we see that the above expression is in absolute value less than the expression

$$\frac{N^2\Delta}{4} + \frac{N^2}{4\pi}\int_{c_1}^{X-\Delta/2} d\xi \int_t^{t_1} \frac{e^{-(x_1-\xi)^2/4(t_1-\tau)-(\xi-x)^2/4(\tau-t)}}{\sqrt{t_1 - \tau}\sqrt{\tau - t}}\, d\tau$$
$$+ \frac{N^2}{4\pi}\int_{X+\Delta/2}^{c_2} d\xi \int_t^{t_1} \frac{e^{-(x_1-\xi)^2/4(t_1-\tau)-(\xi-x)^2/4(\tau-t)}}{\sqrt{t_1 - \tau}\sqrt{\tau - t}}\, d\tau.$$

Let us now choose $|x_1 - X|$ and $|x - X| \leqq \Delta/4$, and Δ small enough so that the first term of the expression is less than $\eta/3$. In the two integrals, the values of $|x_1 - \xi|$ and $|\xi - x|$ will always be at least as great as $\Delta/4$, and the integrands therefore less in absolute value than a certain constant, independent of the choice of x and x_1 within the given interval. Hence if we take $|t_1 - T|$ and $|t - T| < \epsilon$, keeping $t_1 \geqq t$, we may choose ϵ small enough to make each integral separately in absolute value less than $\eta/3$, and thus verify the desired inequality.

Accordingly the second term in (13) is continuous within the region. Hence, as is immediately verifiable, all later terms are continuous in the same region.

If now we define $k(x_1, t_1|x, t)$ as the series (13), it will obviously satisfy

the symmetrical relations (5). The continuous function $u(x, t)$ defined by (2) is now seen to be the same as the $u(x, t)$ determined by (10); for from the symmetry of the relations (5) and of the equations (2) and (10), it follows, in the usual way, that the $u(x, t)$ defined by (2) is the unique continuous solution of (10). Thus the closed form of the solution is established.

2. The Generalized Differential and Integro-Differential Equations of Parabolic Type

5. Introduction. The complication of the methods of §§ 1–4, which are the classical methods, is due to the fact that we must insure the existence of $\partial^2/\partial x^2$ and $\partial/\partial t$ for functions which we wish to consider as possible solutions of any of the various parabolic equations. The possibility of avoiding these complications is suggested by trying to consider the complex of derivatives of highest order as a single differential operator

$$\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t},$$

rather than as the combination of two.

Such a definition may be reached in several ways: we may for instance in forming this expression require that the increments of x and t continue to satisfy a certain relation as they approach zero.* It is more convenient for us, however, because closer to a possible physical interpretation and less laborious in its analysis, to define the operator by substituting for the parabolic differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t) \tag{1}$$

the generalized equation

$$\int_s \left[\frac{\partial u}{\partial x} dt + u dx \right] = \int\int_\sigma f(x, t)\, dx\, dt \tag{1'}$$

in which s is an arbitrary closed curve of a kind later to be described, and σ is the enclosed region.† Similarly for the integro-differential equation

* We can generalize Green's theorem so that it applies to this operator. The corresponding operator has been defined by H. Petrini in the elliptic case for Poisson's equation, *Les dérivées premiers et secondes du potentiel*, Acta Matematica, vol. 31 (1908), p. 127–332, see introduction and page 181.

† Compare Professor Bôcher's treatment of the equation $\Delta^2 u = 0$ by means of the relation

$$\int_s \partial u / \partial n = 0.$$

M. Bôcher, *On harmonic functions in two dimensions*, Proceedings of the American Academy of Science, vol. 41 (1905–06).

We may regard the passage from equation (1) to equation (2) as equivalent to defining

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) = \lim_{\sigma = 0} \frac{1}{\sigma} \int_s \left[\frac{\partial}{\partial x} dt + (\ )\, dx \right].$$

The corresponding definition is used by Ignatowsky, *Die Vektoranalysis*, Leipzig (1909–10), in his treatment of Δ^2, but his treatment is not exact at all points.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_{T_x}^{t} A(x, t, \tau) u(x, \tau) d\tau$$

we can substitute the generalized equation

$$\int_s \left[\frac{\partial u}{\partial x} dt + u dx \right] = \int\int_\sigma d\sigma \int_{T_x}^{t} A(x, t, \tau) u(x, \tau) d\tau .$$

By establishing the corresponding generalization of Green's theorem we can give a treatment of the boundary value problems for these equations. It is not necessary that any derivative of $f(x, t)$ should exist, or that $f(x, t)$ should satisfy any restriction of the type of Levi's condition,*—a result that applies also to the corresponding generalization of the non-homogeneous elliptic equation. Likewise in the treatment of the integro-differential equation, it is unnecessary that any derivative of $A(x, t, \tau)$ should exist, or that $A(x, t, \tau)$ should satisfy Levi's condition.

6. Regular functions, standard curves, and Green's theorem. We say that a function is regular with reference to a generalized parabolic equation and a given region if with its first derivative with respect to x, it is continuous within and on the boundary of the region. The generalized parabolic equations are defined with reference to the integration of regular functions along certain classes of curves within the given region. These curves may be called standard curves and defined as follows.

A standard curve is a closed curve which does not cut itself at any point, composed of a finite number of branches. For each branch, the coördinates of a point are given by two functions $\phi(q)$ and $\psi(q)$, throughout a finite interval for q; $\phi(q)$ and $\psi(q)$ are assumed to be continuous throughout the closed interval, and therefore finite, with their derivatives of the first order. It is assumed that $\phi'(q)$ and $\psi'(q)$ vanish only at a finite number of points, and do not both vanish at the same point. Hence no line parallel to the x-axis or t-axis can cut a standard curve in more than a finite number of points, unless it includes itself a branch of the curve.

We are now in a position to state Green's theorem:

If u and v are regular, and f and g are continuous, throughout a given simply connected region, and if for any standard curve s contained wholly within the region

$$\int_s \left[\frac{\partial u}{\partial x} dt + u dx \right] = \int\int_\sigma f d\sigma$$

(2)

$$\int_s \left[-\frac{\partial v}{\partial x} dt + v dx \right] = \int\int_\sigma g d\sigma$$

then

(3) $$\int_s \left[\left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) dt + uv dx \right] = \int\int_\sigma (vf + ug) dx\, dt .$$

* Levi, loc. cit., § 8, Art. 23.

7. Proof of Green's theorem. Let us form the functions

$$U = \int_0^{t'} dt' \int_0^{x'} u(x+x', t+t')\,dx', \qquad V = \int_0^{t''} dt'' \int_0^{x''} v(x+x'', t+t'')\,dx''$$

$$F = \int_0^{t'} dt' \int_0^{x'} f(x+x', t+t')\,dx', \qquad G = \int_0^{t''} dt'' \int_0^{x''} g(x+x'', t+t'')\,dx''$$

keeping $|t'| < a$, $|x'| < a$, $|t''| < a$, $|x''| < a$, where a is a positive constant less than the smallest distance of any point of s to the boundary of the given region.

If we displace the standard curve s parallel to itself by an amount whose components are x', t', it will still remain within the region, and equation (1) will give us, by the corresponding change of variable,

$$\int_s \left[\frac{\partial u(x+x', t+t')}{\partial x}\,dt + u(x+x', t+t')\,dx \right] = \int\int_\sigma f(x+x', t+t')\,dx\,dt.$$

If we integrate this equation from 0 to x' in regard to x' and from 0 to t' in regard to t' we shall have

$$(4) \qquad \int_s \left[\frac{\partial U(x, x'; t, t')}{\partial x}\,dt + U(x, x'; t, t')\,dx \right] = \int\int_\sigma F(x, x'; t', t')\,dx\,dt,$$

since all the functions involved are continuous. But by a change of variable $\xi = x + x'$, $\eta = t + t'$, the expression for U becomes

$$U = \int^{t+t'} d\eta \int_x^{x+x'} u(\xi, \eta)\,d\xi,$$

so that we have the formulæ

$$\frac{\partial U}{\partial x} = \int_t^{t+t'} \{u(x+x', \eta) - u(x, \eta)\}\,d\eta,$$

$$\frac{\partial U}{\partial t} = \int_x^{x+x'} \{u(\xi, t+t') - u(\xi, t)\}\,d\xi,$$

$$\frac{\partial^2 U}{\partial x^2} = \int_t^{t+t'} \left\{ \frac{\partial u(x+x', \eta)}{\partial x} - \frac{\partial u(x, \eta)}{\partial x} \right\} d\eta,$$

all functions which are finite and continuous in σ and on s. Hence

$$\int\int_\sigma \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} \right) d\sigma = \int_s \left[\frac{\partial U}{\partial x}\,dt + U\,dx \right].$$

But F is continuous in σ, and therefore from (4) we have

$$\frac{\partial U(x, x'; t, t')}{\partial t} - \frac{\partial^2 U(x, x'; t, t')}{\partial x^2} = F(x, x'; t, t').$$

Similarly, we find the equation

$$\frac{\partial V(x, x''; t, t'')}{\partial t} + \frac{\partial^2 V(x, x''; t, t'')}{\partial x^2} = G(x, x''; t, t'').$$

From these two equations it follows by means of Green's theorem, as ordinarily stated, that

$$\begin{aligned}\int_{s'} \Bigg[\Bigg\{ & V(x, x''; t, t'') \frac{\partial U(x, x'; t, t')}{\partial x} \\ & - U(x, x'; t, t') \frac{\partial V(x, x''; t, t'')}{\partial x} \Bigg\} dt + U(x, x'; t, t') \\ & \times V(x, x''; t, t'') dx \Bigg] = \iint_\sigma \Big\{ V(x, x''; t, t'') F(x, x'; t, t') \\ & + U(x, x'; t, t') G(x, x''; t, t'') \Big\} d\sigma. \end{aligned} \tag{5}$$

Referring to the definitions of U and V, F and G, we see that

$$\frac{\partial^2 U}{\partial x' \, \partial t'} = u(x + x', t + t'), \qquad \frac{\partial^2 V}{\partial x'' \, \partial t''} = v(x + x'', t + t''),$$

$$\frac{\partial^2 F}{\partial x' \, \partial t'} = f(x + x', t + t'), \qquad \frac{\partial^2 G}{\partial x'' \, \partial t''} = g(x + x'', t + t'').$$

Hence if we differentiate (5) in regard to x', t', x'', t'', under the integral sign, as we have a right to do, since the resulting functions are all finite and continuous, we get the formula

$$\begin{aligned}\int_s \Bigg[\Bigg\{ & v(x + x'', t + t'') \frac{\partial u(x + x', t + t')}{\partial x} \\ & - u(x + x', t + t') \frac{\partial v(x + x'', t + t'')}{\partial x} \Bigg\} dt + u(x + x', t + t') \\ & \times v(x + x'', t + t'') dx \Bigg] = \iint_\sigma \Big\{ v(x + x'', t + t'') \\ & \times f(x + x', t + t') + u(x + x', t + t') g(x + x'', t + t'') \Big\} d\sigma. \end{aligned}$$

If in this we put $x' = x'' = t' = t'' = 0$ we get equation (3), which we wished to prove.

If the boundary of the region is a standard curve the theorem will still hold if we

take s as the boundary itself. This can be shown by extending the function u across s in the following manner.

Let us isolate each vertex of the curve, and each point of the curve where there is a vertical tangent, unless it is an interior point of a vertical branch of the curve, by a small circle. Consider the portion of the curve between two successive circles. If this portion is vertical, we shall define u and $\partial u/\partial x$ throughout a small neighborhood on the outside of the curve by means of the values which they possess on the vertical line itself for the same value of x.

If the portion is not vertical it will not have a vertical tangent at any interior point. Let us denote by $r(t)$ the values of $\partial u/\partial x$ along this portion of the curve, and define u throughout a small neighborhood on the outside of the curve as that solution of the equation

$$\frac{\partial u}{\partial x} = r(t)$$

which takes on along the curve itself the already known values of u.

If now we surround each circle by a slightly larger circle, and take as a contour s' the curve composed, first, of the portions of s between the successive circles, last constructed, and second of the portions of these circles interior to the region, we shall have for s', if we take the circles small enough, a standard curve, and it will lie wholly within a region for which u and $\partial u/\partial x$ are continuous. Green's theorem will then hold for s'. We get Green's theorem for s if we let the circles approach zero as a limit.

8. The generalized parabolic differential equation. As the region to be considered, we shall understand that enclosed by the lines $t = t_0$ on the left, and $t = t_1$ on the right, and by the curves $x = \xi_2(t)$ above and $x = \xi_1(t)$ below. We shall assume that $\xi_1(t)$, $\xi_2(t)$ and their first derivatives remain single-valued, finite, and continuous, and that $\xi_2(t) > \xi_1(t)$, $t \geqq t_0$. The functions $\xi_1(t)$ and $\xi_2(t)$ are to have only a finite number of maxima and minima in the interval $t_0 \leqq t \leqq t_1$.

Along the open boundary is to be given the continuous chain of values (α) of § 1, under the conditions there stated. The function $f(x, t)$ is to be finite, and continuous within the given region.

Under these conditions there is one and only one regular solution $u(x, t)$ of the generalized parabolic differential equation (1′), *which joins on continuously to the boundary values* (α). *It has the form*

$$u(x_1, t_1) = u'(x_1, t_1) + \frac{1}{2\sqrt{\pi}} \int_{t_0}^{t_1}\int_{\Omega} g(x, t|x_1, t_1) f(x, t)\, dx\, dt \tag{6}$$

where $u'(x_1, t_1)$ is the regular solution of the equation

$$\frac{\partial u'(x, t)}{\partial t} = \frac{\partial^2 u'(x, t)}{\partial x^2}. \tag{7}$$

which joins on continuously to the boundary values (α),* *and* $g(x, t|x_1, t,)$ *is the Green's function for this same equation.*

From Green's theorem it follows that under the given conditions the solution of (1′) must necessarily have the form (6). On the other hand we can show directly that (6) does give us a solution which takes on the boundary values (α). In fact we shall show this if we show that *the function*

$$z(x_1, t_1) = \frac{1}{2\sqrt{\pi}} \iint\limits_{t_0 \Omega t_1} g(x, t|x_1, t_1) f(x, t)\, dx\, dt \tag{8}$$

is regular with respect to (1′), *takes on continuously the value zero on the open boundary, and satisfies* (1′).

Let us write

$$z(x, \tau) = Z(x, t) + w(x, t)$$

and

$$Z(x_1, t_1) = \frac{1}{2\sqrt{\pi}} \iint\limits_{t_0 \Omega t_1} \frac{1}{\sqrt{t_1 - t}} e^{-(x_1 - x)^2/4(t - t_1)} f(x, t)\, dx\, dt.$$

We know that $Z(x_1, t_1)$ is regular with regard to (1′) since with its derivative in regard to x it is continuous within and on the boundary of Ω.† It is also a solution of (1′), as we can show by performing the indicated integration.

9. Formulæ of integration. Let l be a straight line lying wholly within Ω, of which the end points are (a_1, b_1) and (a_2, b_2), with $b_2 > b_1$, and let x' be the x coördinate of the point on that line whose other coördinate is t, i. e.

$$x' = \frac{a_2 - a_1}{b_2 - b_1}(t - b_1) + a_1.$$

If we enclose this line in a small parallelogram, which we allow to vanish, we can easily deduce the following formulæ.

If l is perpendicular to the t-axis, we have

$$\begin{aligned} &\int_{a_1}^{a_2} dx_1 \iint\limits_{t_0 \Omega t_1} h_{0\,1/2}(x, t|x_1, t_1) f(x, t)\, dx\, dt \\ &\qquad = \lim_{\delta = 0} \iint\limits_{t_0 \Omega t_1 - \delta} dx\, dt\, f(x, t) \int_{a_1}^{a_2} h_{0\,1/2}(x, t|x_1, t_1)\, dx_1. \end{aligned} \tag{I}$$

If l is not perpendicular to the t-axis, we have

* For the existence and uniqueness of $u'(x, t)$ under these conditions see G. E. Levi, loc. cit., §§ 2, 5.

† See E. E. Levi, loc. cit., p. 234. See also § 7.

$$(\mathrm{II}_a)\quad \int_{a_1{}^l a_2} dx_1 \iint_{t_0{}^\Omega t_1} h_{0\,1/2}(x, t|x_1, t_1) f(x, t)\, dx\, dt = \iint_{t_0{}^\Omega b_2} dx\, dt f(x, t) \int_{x'{}^l a_2} h_{0\,1/2}(x, t|x_1, t_1)\, dx_1,$$

$$(\mathrm{II}_b)\quad \int_{b_1{}^l b_2} dt_1 \iint_{t_0{}^\Omega t_1} h_{1\,3/2}(x, t|x_1, t_1) f(x, t)\, dx\, dt = \iint_{t_0{}^\Omega b_2} dx\, dt f(x, t) \int_{t'{}^l b_2} h_{1\,3/2}(x, t|x_1, t_1)\, dt_1.$$

In these formulæ, as before, we write

$$h_{\alpha\beta}(x, t|x_1, t_1) = \frac{(x_1 - x)^\alpha}{(t_1 - t)^\beta} e^{(x_1-x)^2/4(t_1-t)},$$

from which it follows that

$$\frac{\partial}{\partial x_1} h_{0\,1/2}(x, t|x_1, t_1) = -\tfrac{1}{2} h_{1\,3/2}(x, t|x_1, t_1).$$

10. Application to equation (1′). With reference to these formulæ let us form the expression

$$\int_C \left[\frac{\partial Z}{\partial x_1} dt_1 + Z dx_1\right],$$

where C denotes a closed polygon of a finite number of sides, the boundary of which does not anywhere cut itself. If we denote by ${}_tC$ the portion of the contour composed of points of which the t-coördinates are *greater* than the value t, and by b the maximum t-coördinate for any point of the contour, we shall have the equation

$$\int_C \left[\frac{\partial Z(x_1, t_1)}{\partial x_1} dt_1 + Z(x_1, t_1)\, dx_1\right] = \frac{1}{2\sqrt{\pi}} \iint_{t_0{}^\Omega b} dx\, dt f(x, t) \int_{{}_tC} \left[\frac{\partial h_{0\frac{1}{2}}(x, t|x_1, t_1)}{\partial x_1} dt_1 + h_{0\,1/2}(x, t|x_1, t_1)\, dx_1\right].$$

The curvilinear integral of the second member of the above equation is however, as is well known, equal to

$$2\sqrt{\pi} \quad \text{if the point } (x, t) \text{ is inside } C,$$

$$0 \quad \text{if the point } (x, t) \text{ is outside } C,$$

since $h_{0\frac{1}{2}}$ is itself a solution in x_1, t_1 of the homogeneous partial differential equation (7). The boundary points form an aggregate of zero external

measure, and since the curvilinear integral remains finite, contribute nothing to the value of the second member. If then we divide ${}_{t_0}\Omega_b$ into two parts, one consisting of the portion of Ω inside C, and the other the portion of Ω outside C, we observe that the value of the double integral over the latter portion is zero, on account of the zero factor in the integrand. Hence we have

$$\int_C \left[\frac{\partial Z(x_1, t_1)}{\partial x_1} dt_1 + Z(x_1, t_1)\, dx_1 \right] = \int\int_C f(x, t)\, dx\, dt,$$

the double integral being extended over the interior of C.

Let us now finally consider any standard curve entirely within Ω. In the first place it is rectifiable, and its length is by definition the limit of the perimeter of an inscribed polygon as the vertices of the polygon become infinite in number, in such a way as to make the polygon approach the contour. In order to construct the approximate polygons in a perfectly definite manner, we may imagine the ith branch of the standard curve s to have the coördinates of its points expressed in terms of the parameter q, as we have already described, and take as the vertices of the polygon the n_i *interior* points of division corresponding to equally spaced values of q in the interval (the vertices of the curve therefore are not vertices of the polygon). By taking every n_i large enough, we get in this way a polygon of the kind already described, namely one of a finite number of sides which does not cut itself. If we denote the total approximate polygon by C_n, we shall have

$$\int_{C_n} \left[\frac{\partial Z}{\partial x} dt + Z dx \right] = \int\int_{C_n} f(x, t)\, dx\, dt.$$

As each n_i becomes infinite the right-hand member approaches the double integral over σ. The left-hand member approaches the integral over s, since the functions Z and $\partial Z/\partial x$ are finite and continuous, and the curve s is rectifiable. In fact the integral around s may be regarded as the limit of a sum, the values of the functions being taken at the vertices of the polygons; and from the continuity of the functions u and $\partial u/\partial x$ in the two-dimensional region it follows that the integral around C_n can also be made to differ by as little as we please from that sum, and from its limit, by taking all the n_i large enough.

The function Z is therefore a solution of equation (1′).

Let $w(x, t)$ be that solution of (7) which has on the open boundary of Ω the same values as $Z(x, t)$, reversed in sign. Since $Z(x, t)$ is finite and continuous within and on the boundary of Ω, $w(x, t)$ will be regular with respect to (7), as the existence theorem for (7) tells us. The function

$$z(x, t) = Z(x, t) + w(x, t)$$

will therefore be regular with respect to (1′), will take on zero values on the boundary of Ω, and will satisfy (1′), for any standard curve lying wholly within the boundary. It follows moreover from Green's theorem that it must be expressible in the form (8). This is what was to be proved.

11. A generalized integro-differential equation. Let us consider now an integro-differential equation of parabolic type corresponding to the region Ω. If we denote by $\int_{t_0 t_1}$ integration along that portion of a line parallel to the axis of t which is inside the region Ω and between the lines $t = t_0$ and $t = t_1$, the integral equation that we desire to consider may be written

$$\int_s \left[\frac{\partial u}{\partial x} dt + u dx \right] = \int\int_\sigma dx\, dt \int_{t_0 t} A(x, t, \tau) u(x, \tau) d\tau, \tag{9}$$

where by s is meant any closed contour of the kind that we have already specified.

If $u(x, t)$ is continuous, the quantity

$$\int_{t_0 t} A(x, t, \tau) u(x, \tau) d\tau$$

will be a continuous function of x and t.* Hence by the theorem of § 8, in order that $u(x, t)$ be the regular solution of (9), which takes on the boundary conditions (α) it is necessary and sufficient that it be a continuous solution of the equation

$$\begin{aligned} u(x_1, t_1) &= u'(x_1, t_1) \\ &\quad + \frac{1}{2\sqrt{\pi}} \int\int_{t_0 \Omega t_1} dx\, dt\, g(x, t | x_1, t_1) \int_{t_0 t} A(x, t, \tau) u(x, \tau) d\tau. \end{aligned} \tag{10}$$

This equation, however, is merely an integral equation similar to that already discussed in part 1. For by rewriting this equation in the form

$$\begin{aligned} u(x_1, t_1) &= u'(x_1, t_1) \\ &\quad + \frac{1}{2\sqrt{\pi}} \int_{c_1}^{c_2} dx \int_{t_0 t_1} dt\, g(x, t | x_1, t_1) \int_{t_0 t} A(x, t, \tau) u(x, \tau) d\tau, \end{aligned}$$

* According to our conditions the boundaries $x = \xi_1(t)$ and $x = \xi_2(t)$ cannot have portions consisting of horizontal straight lines. If it is desired to admit these, we must also generalize the theorem of § 4 to apply to functions $f(x, t)$ which are finite and discontinuous in a regular manner, — a generalization which is immediate.

and interchanging the order of integration with respect to t and τ we have the equation

$$u(x_1, t_1) = u'(x_1, t_1) + \frac{1}{2\sqrt{\pi}} \int_{c_1}^{c_2} dx \int_{t_0 t_1} d\tau\, u(x, \tau) \int_{\tau t_1} g(x, t | x_1, t_1) A(x, t, \tau)\, dt,$$

which is again the same as

$$u(x_1, t_1) = u'(x_1, t_1) + \frac{1}{2\sqrt{\pi}} \iint_{t_0 \Omega t_1} dx\, dt\, u(x, t) \int_{t t_1} g(x, \tau | x_1, t_1) A(x, \tau, t)\, d\tau. \tag{11}$$

But this equation, according to the method of part 1, has a unique continuous solution of the kind there specified. We have then the following

THEOREM. *There is one and only one regular solution $u(x, t)$ of the equation* (9) *which joins on continuously to the boundary values* (α). *It may be reduced to the form*

$$u(x_1, t_1) = u'(x_1, t_1) - \iint_{t_0 \Omega t_1} k(x_1, t_1 | x, t)\, u'(x, t)\, dx\, dt,$$

where $k(x_1, t_1 | x, t)$ is the function associated to

$$K(x_1, t_1 | x, t) = \frac{1}{2\sqrt{\pi}} \int_{t t_1} g(x, \tau | x_1, t_1) A(x, \tau, t)\, d\tau,$$

by means of Volterra's relation.

3. The Integro-Differential Equation of Hyperbolic Type

12. In this last section we consider very briefly the hyperbolic integro-differential equation of the following simple type,*

$$\frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\partial^2 u(x, t)}{\partial t^2} = \int_0^t A(x, t, \tau) \frac{\partial^2 u(x, \tau)}{\partial \tau^2}\, d\tau, \tag{1}$$

in which $A(x, t, \tau)$ is a continuous function, with continuous first derivatives, in a region

$$0 \leqq \tau \leqq t \leqq r$$

$$a - r + t \leqq x \leqq a + r - t.$$

* This equation has been treated by the method of partial solutions by V. Volterra, *Vibrazioni elastiche nel caso della eredità*, Rendiconti della R. Accademia dei Lincei, vol. 21 (1912), p. 1. See also V. Volterra, *Lectures delivered at the celebration of the 20th anniversary of the foundation of Clark University* (1911), p. 80.

The region of variation for the point (x, t) is thus a triangle whose vertex is at the point (a, r) and whose base is on the x-axis bounded by the points $(a - r, 0)$ and $(a + r, 0)$. The solution of this equation is to be regular,* and is to satisfy the boundary conditions

$$u(x, 0) = f(x),$$

$$(\beta) \qquad \left[\frac{\partial u(x, t)}{\partial t}\right]_{t=0} = f_1(x),$$

in which $f(x)$, $f_1(x)$ and their first derivatives are continuous functions of x in the range of values $a - r \leqq x \leqq a + r$.

As long as we base our treatment on the characteristic function of Riemann, which is itself regular, there is no need of generalizing the equation in order to get rid of the second derivatives. We therefore limit ourselves in this section to a treatment that corresponds to § 1, and content ourselves with a bare mention of the resulting theorem.

THEOREM. *Under the given conditions for $A(x, t, \tau)$ there is one and only one regular solution of* (1), $u(x, t)$, *which joins on continuously to the boundary values* (β). *It may be reduced to the form*

$$(2) \qquad u(x, t) = u'(x, t) - \int_{x-t}^{x+t} d\xi \int_0^{\tau_x} l(x, t|\xi, \tau) u'(\xi, \tau) d\tau,$$

where

$$\tau_x = t - x + \xi \quad \text{for} \quad \xi \leqq x$$

$$\tau_x = t + x - \xi \quad \text{for} \quad \xi \geqq x,$$

where $u'(x, t)$ is the regular solution of the equation

$$(3) \qquad \begin{aligned} &\frac{\partial^2 u'(x, \tau)}{\partial x^2} - \frac{\partial^2 u'(x, t)}{\partial t^2} - A(x, t, \tau)\frac{\partial u'(x, t)}{\partial t} \\ &\qquad + \left[\frac{\partial A(x, t, \tau)}{\partial \tau}\right]_{\tau=t} u'(x, \tau) \\ &\qquad = -A(x, t, 0) f_1(x) + \left[\frac{\partial A(x, t, \tau)}{\partial \tau}\right]_{t=0} f(x) \end{aligned}$$

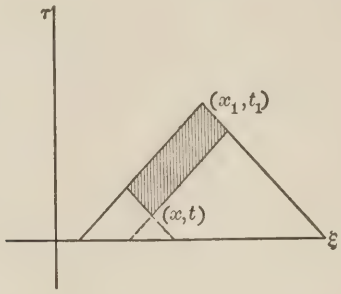
that takes on the boundary conditions (β), *and where $l(x, t|\xi, \tau)$ is the function associated to*

$$(4) \qquad L(x_1, t_1|x, t) = \frac{1}{2}\int_t^{t_x} G(x_1, t_1|x, \tau) \frac{\partial^2 A(x, t, \tau)}{\partial t^2} d\tau$$

* That is, is to be continuous, with continuous first derivatives, and integrable second derivatives ($\partial^2/\partial x^2$ and $\partial^2/\partial t^2$) in the given region $0 \leqq t \leqq r$, $a - r + t \leqq x \leqq a + r - t$.

by means of Volterra's relation. By $G(x_1, t_1|x, t)$ *is denoted Riemann's characteristic function for equation* (3).*

In the case of the region that we are considering Volterra's relation may be written as



$$(5) \qquad l(x_1, t_1|x, t) + L(x_1, t_1|x, t) = \iint L(x_1, t_1|\xi, \tau)\, l(\xi, \tau|x, t)\, d\xi\, d\tau,$$

the double integration being extended over the shaded area in the accompanying diagram.

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* That is, the solution of the equation adjoint to the first member of (3), which takes on the value 1 for $x = x_1$, $y = y_1$, and satisfies the conditions

$$\sqrt{2}\frac{dG}{ds_1} = \frac{A(x, t, t)}{2} G \text{ along the line } x - t = x_1 - t_1$$

and

$$\sqrt{2}\frac{dG}{ds_2} = \frac{A(x, t, t)}{2} G \text{ along the line } x + t = x_1 + t_1.$$

INVARIANTS IN THE THEORY OF NUMBERS*

BY

L. E. DICKSON

1. Polynomials in the coefficients of a form $f(x_1, \cdots, x_n)$ which have the invariantive property with respect to all linear homogeneous transformations on $x_1, \cdots, x_n$ with integral coefficients taken modulo p, where p is a prime, are called formal invariants modulo p of f if the coefficients of f are independent variables, but are called modular invariants if the coefficients of f are integers taken modulo p. The concept of formal invariants modulo p was introduced by Hurwitz;† but the only known results concerning them relate to the binary quadratic and cubic forms.‡ On the contrary, a simple and effective theory of modular invariants has been given by the writer. A new method of deriving modular invariants from seminvariants is given in § 8. But the main purpose of this paper is to present a simple general method of constructing formal invariants. The method is applicable also to formal seminvariants and, more generally, to the invariants of any linear congruence group (§ 7). Moreover, the new point of view forms an adequate basis for a general theory of formal invariants.

Construction of Formal Invariants

2. The method of construction will first be illustrated by the simple example of the binary quadratic form

$$Q = ax^2 + bxy + cy^2$$

for the case of modulus 2. The only real points (i. e., with integral coördinates) modulo 2 are

$$P_1 = (1, 0), \qquad P_2 = (0, 1), \qquad P_3 = (1, 1).$$

The corresponding values of Q are

$$a, \qquad c, \qquad s = a + b + c.$$

By the interchange of x and y in Q, a and c are interchanged, also P_1 and P_2.

* Presented to the Society at Providence, September 7, 1914.

† Archiv der Mathematik und Physik, ser. 3, vol. 5 (1903), p. 25.

‡ Dickson, *On Invariants and the Theory of Numbers, Madison Colloquium Lectures*, American Mathematical Society, pp. 40–54.

By the transformation $x' = x + y$, $y' = y$, Q is replaced modulo 2 by a form with the same a and b, but with c replaced by s; then c and s are interchanged, as well as P_2 and P_3. Since any binary linear transformation with integral coefficients modulo 2 is generated by the preceding two, it follows that any such transformation gives rise to a permutation of a, c, s amongst themselves, and the same permutation of P_1, P_2, P_3. Hence *any symmetric function of a, c, s is a formal invariant modulo 2 of Q.*

The elementary symmetric functions are, modulo 2,

$$b, \qquad q = a^2 + ac + c^2 + (a + c)b, \qquad k = ac(a + b + c).$$

They form* a fundamental system of rational integral formal invariants modulo 2 of Q.

3. Consider the system of forms Q and

$$l = \eta x + \xi y.$$

The values of l at the points P_1, P_2, P_3 are $\eta, \xi, \eta + \xi$, respectively. The latter undergo the same permutation as the P's when l is transformed linearly (§ 2). Hence, if ϕ is any polynomial in two arguments,

$$\phi(a, \eta), \qquad \phi(c, \xi), \qquad \phi(s, \eta + \xi)$$

are permuted amongst themselves when Q and l are transformed linearly modulo 2. Hence any symmetric function of these three ϕ's is a formal invariant modulo 2 of Q and l.

Taking $\phi(a, \eta)$ to be η, $a\eta$, and $a\eta^2$, in turn, and employing the elementary symmetric functions in each case, we get the following formal invariants of Q and l:

$$\lambda = \xi^2 + \xi\eta + \eta^2, \qquad \pi = \xi\eta(\xi + \eta), \qquad j = (a + b)\xi + (b + c)\eta,$$

$$t = ac\xi\eta + s(a\eta + c\xi)(\xi + \eta), \qquad u = (a + b)\xi^2 + (b + c)\eta^2,$$

$$v = ac\xi^2\eta^2 + s(a\eta^2 + c\xi^2)(\xi^2 + \eta^2),$$

and $k\pi$, $k\pi^2$. From $\phi = a\eta^4$, we get

$$U = (a + b)\xi^4 + (b + c)\eta^4 = \lambda u + j\pi.$$

Also t and v can be expressed in terms of simpler invariants:

$$t = q\lambda + j^2 + bu, \qquad v = q\lambda^2 + u^2 + bU.$$

Whether or not the reduced set $b, q, k, \lambda, \pi, j, u$ form a fundamental system of formal invariants modulo 2 of Q and l has not been investigated.

But if we pass to modular invariants by regarding a, b, c, ξ, η to be integers

* *Madison Colloquium Lectures*, p. 42.

taken modulo 2, we have $k \equiv bq$, $\pi \equiv 0$, $u \equiv j$, and the reduced set b, q, λ, j form* a fundamental system of modular invariants of Q and l.

4. Passing to the general case, let

$$f_1(x_1, \cdots, x_n), \quad \cdots, \quad f_t(x_1, \cdots, x_n)$$

be forms of total degrees $d_1, \cdots, d_t$ in the independent variables $x_1, \cdots, x_n$. Let the modulus be p and let g_i be the greatest common divisor of $p-1$ and d_i. Set $q_i = (p-1)/g_i$. Then for any integer ρ not divisible by p,

$$\rho^{d_i q_i} = (\rho^{p-1})^{d_i/g_i} \equiv 1 \qquad (\text{mod } p),$$

by Fermat's theorem. Hence, since f_i is of degree d_i,

$$[f_i(\rho x_1, \cdots, \rho x_n)]^{q_i} \equiv [f_i(x_1, \cdots, x_n)]^{q_i} \qquad (\text{mod } p).$$

Using homogeneous coördinates, let $(x_1, \cdots, x_n)$ be the same point as $(\rho x_1, \cdots, \rho x_n)$, when ρ is any integer not divisible by p. The preceding formula shows that $f_i^{q_i}$ has a definite value at each real point (i. e., one with integral coördinates taken modulo p). The values at the various real points are merely permuted amongst themselves by any linear homogeneous transformation on $x_1, \cdots, x_n$ with integral coefficients taken modulo p. In fact, the real points are permuted by such a transformation. We thus have the

THEOREM. *We obtain a formal invariant modulo p of the system of forms $f_i(x_1, \cdots, x_n)$, $i = 1, \cdots, t$, if we take any symmetric function with integral coefficients of the quantities*

$$\phi(f_1^{q_1}, \cdots, f_t^{q_t})$$

given by substituting in turn for the x's the coördinates of the

$$P = 1 + p + p^2 + \cdots + p^{n-1}$$

real points. Here ϕ is any polynomial in its t arguments with integral coefficients, and q_i is the quotient of $p-1$ by the greatest common divisor of $p-1$ and the degree of f_i.

Each q_i is unity if $p = 2$ (cf. §§ 2, 3); also if $p = 3$ and each f_i is of even degree. For example, if $t = 1$ and

$$f = f_1 = ax^2 + 2bxy + cy^2,$$

and $p = 3$, the values of f at the $P = 4$ real points $(1, 0)$, $(0, 1)$, $(1, 1)$, $(-1, 1)$ are respectively

$$a, \qquad c, \qquad a - b + c, \qquad a + b + c.$$

* *Madison Colloquium Lectures*, p. 57.

The elementary symmetric functions reduce modulo 2 to zero and

$$ac - b^2, \qquad (a + c)(a + b - c)(a - b - c),$$
$$ac(a + b + c)(a - b + c),$$

respectively. The first is the discriminant D of f. The last two are $\gamma_2 = \Gamma$ and $a\gamma_0 = J$ in the notations of the *Madison Colloquium Lectures*, pages 43–45.

A further evident invariant is the product of all non-proportional linear functions of a, b, c with integral coefficients taken modulo 3. Hence its quotient

$$B = b(b + a)(b - a)(c - a)(c - b)(c + b)$$

by ΓJ is a formal invariant. In the place just cited it was proved that D, Γ, J, B form a fundamental system of formal invariants modulo 3 of f.

If $p = 5$, the sum of the products by twos of the squares of the values of f at the six real points is

$$3(ac - b^2)^2 \qquad (\text{mod } 5).$$

Hence as before we are led to the algebraic discriminant, as well as to invariants peculiar to the number theory case.

5. The present method is particularly useful for forms in three or more variables, since the construction of formal or modular invariants by earlier methods is excessively laborious and dependent largely upon special devices.

As an illustration, consider

$$F(x) = a_1 x_2 x_3 + a_2 x_1 x_3 + a_3 x_1 x_2 + b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2$$

for the modulus 2. Its values v_j for the seven real points are

$$b_1, \quad b_2, \quad b_3, \quad a_1 + b_2 + b_3, \quad a_2 + b_1 + b_3, \quad a_3 + b_1 + b_2,$$
$$\sum_{i=1}^{3} (a_i + b_i).$$

Since their sum is zero modulo 2, we readily get

$$\sum v_1 v_2 = \sum a_1 b_1 + \sum a_1^2 + a_1 a_2 + a_1 a_3 + a_2 a_3 = \alpha,$$
$$\sum v_1^2 v_2 = \sum v_1 v_2 v_3 = \sum a_1 b_1^2 + \sum a_1^2 b_1 + \sum a_1 a_2^2,$$
$$\sum v_1 v_2 v_3 v_4 = \alpha \sum b_1^2 + \sum a_1 b_1 (a_1 + b_1)(b_2 + b_3) + \sum a_1 a_2 b_1 b_2$$
$$+ (a_1 a_2 a_3 + a_1 a_2 b_3 + a_1 a_3 b_2 + a_2 a_3 b_1) \sum a_1$$
$$+ \sum b_1^4 + \sum b_1^2 b_2^2 + b_1 b_2 b_3 \sum b_1.$$

The second is congruent to the sum of the formal invariants

$$\Delta = a_1 a_2 a_3 + \sum a_1^2 b_1, \qquad \Delta_1 = a_1 a_2 a_3 + \sum a_1 b_1^2 + \sum a_1 a_2^2,$$

which, with a modular invariant J, are given on pages 69 and 74 of the *Madison Colloquium Lectures*. Note that Δ is the discriminant of F.

If we take each a_i and b_i to be an integer modulo 2, we find that our third invariant reduces to the modular invariant $J + \alpha + 1$. Set $A = \alpha + \Delta + 1$. It is proved on page 76 of the same book that Δ, A, and J form a fundamental system of modular invariants of F.

As shown indirectly in § 6, we cannot construct a fundamental system of modular invariants of F by use of the real points only, but succeed if we use also points whose coördinates are Galois imaginaries.

Criteria for the Equivalence of Modular Forms

6. An important application of modular invariants is that to the question of the equivalence of two modular forms under linear transformation with integral coefficients taken modulo p. We can treat this question by means of the idea underlying the foregoing construction of invariants, without actually constructing them. For example, consider the ternary quadratic form modulo 2. As well known, such a form is equivalent to one and but one of the forms

$$x_1 x_2 + x_3^2\,[4], \qquad x_1 x_2 + x_1^2 + x_2^2\,[6], \qquad x_1 x_2\,[2], \qquad x_1^2\,[4], \qquad 0\,[0].$$

After each form we have given the number of real points for which it is congruent to unity modulo 2. Hence no two of these forms are equivalent, with a possible exception in the case of the first and fourth. To show by the same method that the latter are not equivalent, we employ the points each of whose coördinates are $0, 1, j$ or $j+1$, where j is the Galois imaginary for which

$$j^2 + j + 1 \equiv 0 \pmod 2. \tag{1}$$

Now x_1^2 vanishes for just five such points, viz., $(0, 0, 1)$ and $(0, 1, a)$, where $a = 0, 1, j$ or $j + 1$. But $x_1 x_2 + x_3^2$ vanishes for just nine such points, viz., $(0, b, 0)$, $(b, 0, 0)$, $(1, b, b^2)$, where $b = 1, j$, or $j + 1$. Hence the two forms are not equivalent.

The second of our five forms vanishes for just nine such points, viz., $(0, 0, 1)$, $(1, j, a)$, $(1, j + 1, a)$. Finally, $x_1 x_2$ vanishes only for nine such points. Hence the five forms vanish for respectively 6, 8, 4, 2, 14 imaginary points whose coördinates depend upon j. Thus *the classes of conics modulo* 2 *are completely characterized by the number of their imaginary points whose coördinates are functions of a root of* (1).

We readily find the characteristic invariant I_k for the kth class, i. e., an invariant with the value unity for any form in the kth class and the value zero for the remaining forms. For the class represented by our second form,

I_2 is the sum of the products six at a time of the values of F for the seven real points. For the class represented by $x_1 x_2$, I_3 is the sum of the products by twos of those values. We get $I_1 + I_4$ by using the products by fours; to get I_1 itself, we employ imaginary points as above. Finally,

$$1 + I_1 + I_2 + I_3 + I_4 = I_5.$$

Construction of Formal Seminvariants

7. The preceding method is readily extended to the general case of the formal invariants of any system of forms under any group of linear transformations modulo p.

By way of illustration, consider the form Q of § 2 and the group composed of the identity and

$$T: \qquad x' \equiv x + y, \qquad y' \equiv y \qquad (\text{mod } 2).$$

It is therefore the question of the formal seminvariants of Q. Since T leaves unaltered the point P_1 of § 2 and interchanges the points P_2 and P_3, a and any symmetric function of c and s are formal seminvariants. The elementary symmetric functions are $a + b$ and cs. These with a form in fact a fundamental system of formal seminvariants modulo 2 of Q (*Madison Colloquium*, page 42).

Similarly, the system of forms Q and l (§ 3) have the formal seminvariants modulo 2

$$a, \qquad \eta, \qquad \phi(c, \xi) + \phi(s, \eta + \xi), \qquad \phi(c, \xi) \cdot \phi(s, \eta + \xi),$$

where ϕ is any polynomial in its two arguments.

Derivation of Modular Invariants from Seminvariants

8. The method will be illustrated for the binary quadratic form Q (§ 2) and the modulus 2. The coefficients are integers taken modulo 2. To make the treatment self-contained, we shall first derive a fundamental system of modular seminvariants. The transformation

$$(x + ty, y): \qquad x' = x + ty, \qquad y' = y$$

replaces Q by $ax^2 + bxy + c'y^2$, where $c' = c + (a + b)t$. Hence a and b are seminvariants of Q. If $a + b \equiv 1$, we take $t = c$ and have $c' \equiv 0$. If $a + b \equiv 0$, then $c' \equiv c$ and the value of c is given by

$$J = (1 + a + b)c.$$

Since J has the same value for equivalent forms Q, it is a seminvariant. The first column of the following table gives a representative of each class $c_1, \cdots, c_6$ of forms Q.

Representative of the Class	Determined by Seminvariants	Characteristic Seminvariant
$Q_1 = x^2 + xy + y^2$	$a = b = 1, \quad J = 1$	$\pi = abJ \equiv abc$
$Q_2 = x^2 + xy$	$a = b = 1, \quad J = 0$	$ab(J+1) = \pi + ab$
$Q_3 = y^2$	$a = b = 0, \quad J = 1$	$(a+1)(b+1)J = \pi + J$
$Q_4 \equiv 0$	$a = b = 0, \quad J = 0$	$(a+1)(b+1)(J+1) = I$
$Q_5 = x^2$	$a = 1, \quad b = 0$	$a(b+1)$
$Q_6 = xy$	$a = 0, \quad b = 1$	$(a+1)b$

By definition, the characteristic seminvariant of a class c_i is one having the value unity for each form of the class c_i and the value zero for each form in the remaining classes. In the present case, the characteristic seminvariant of class c_i was found by inspection from the values of the seminvariants in the second column which specify that class,—it is a product in which any factor is one of those seminvariants or the sum of it and unity, according as the seminvariant is 1 or 0 for the class.

To derive the invariants of Q from its characteristic seminvariants, we separate the forms Q into classes C_i of equivalent forms under the group G of all binary linear transformations modulo 2. Now G is generated by $(x + y, y)$ and (y, x). Since classes $c_1, \cdots, c_4$ are each composed of a single form, and since the forms Q_1 and Q_4 are unaltered by (y, x), while Q_2 is changed into a form of c_6, and Q_3 into Q_5, we see that the classes under G are

$$C_1 = c_1, \qquad C_2 = c_2 + c_6, \qquad C_3 = c_3 + c_5, \qquad C_4 = c_4.$$

Hence, for $i = 1$ or 4, the characteristic invariant I_i for class C_i is the characteristic seminvariant for c_i. But I_2 is the sum of the characteristic seminvariants for c_2 and c_6, and I_3 the sum of those for c_3 and c_5. Thus the characteristic invariants are

$$I_1 = \pi, \qquad I_2 = \pi + b, \qquad I_3 = \pi + J + ab + a = I + b + 1, \qquad I_4 = I.$$

In § 2 it was shown that b, q, k form a fundamental system of formal invariants modulo 2 of Q. Taking the coefficients to be integers modulo 2, we obtain the modular invariants b, $q \equiv \delta$, $k \equiv b\delta$, where

$$b = I_1 + I_2, \qquad \delta = ac + (b+1)(a+c) = I_1 + I_3.$$

Conversely, from b and δ we get

$$\pi = b\delta, \qquad I = (b+1)(\delta+1).$$

This new method of deriving all modular invariants from the seminvariants is more direct and simpler than the method employed in the *Madison Colloquium*, pages 28–32.

ADDENDA AND ERRATA, VOLUMES, 11, 14

Volume 11

Page 25. Ernest B. Lytle.* *Proper multiple integrals over iterable fields.*

1. Consider the fundamental relation

$$(A) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}\, \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f$$

where f is any limited function defined over the limited field $\mathfrak{A}$, and where the integrals are Pierpont integrals and hence are applicable to fields which are not metric (defined in these Transactions, vol. 11 (1910), p. 26). The problem under consideration here is to find the most general conditions upon the field $\mathfrak{A}$ under which relation (A) is true.

Pierpont has shown† that relation (A) is true when the field $\mathfrak{A}$ is metric. In a former paper‡ the author found a more general class of fields called iterable for which relation (A) holds true. This class of iterable fields includes all metric fields and some non-metric fields.

In this note I show that iterability of the field $\mathfrak{A}$, that is,

$$\int \overline{\mathfrak{C}} = \overline{\mathfrak{A}},$$

is also a necessary condition that the class of limited functions over $\mathfrak{A}$ simultaneously satisfy relation (A).

2. Theorem 1. *If all limited functions defined over $\mathfrak{A}$ satisfy relation (A), then $\mathfrak{A}$ is iterable.*

Consider the contraposite, if $\mathfrak{A}$ is not iterable then not all limited functions over $\mathfrak{A}$ satisfy (A); and we see that to prove Theorem 1 it is only necessary to show there exists a limited function over a non-iterable field which does not satisfy (A). Such a function is as follows:

Example 1. In $\mathfrak{A} = \{x, y\}$, let $0 \leqq x \leqq 1$; for rational x let $0 \leqq y \leqq 1$, and for irrational x let $0 \leqq y \leqq \frac{1}{2}$. Let $f(x, y) = 1$ over this $\mathfrak{A}$.

This $\mathfrak{A}$ is not iterable, for

$$\underline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} = \tfrac{1}{2}, \qquad \overline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} = 1,$$

* Presented to the Society (Madison), September 8, 1913.

† These Transactions, vol. 7 (1906), p. 167.

‡ These Transactions, vol. 11 (1910), p. 25.

and therefore $\int_{\mathfrak{B}} \mathfrak{C}$ does not even exist. Further, $f(x, y)$ here does not satisfy relation (A), for

$$\underline{\int}_{\mathfrak{A}} f = 1, \qquad \text{and} \qquad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = \tfrac{1}{2}.$$

By combining Theorem 1 above and Theorem 14, p. 35 of my former paper cited above, we get the following important theorem:

THEOREM 2. *In order that all limited functions over* $\mathfrak{A}$ *simultaneously satisfy relation* (A), *it is necessary and sufficient that* $\mathfrak{A}$ *be iterable relative to* x.

3. It is to be noticed that iterability of the field is not necessary in order that a single particular limited function satisfy (A), as the following example shows.

Example 2. Let $\mathfrak{A}$ be defined as in Example 1 above. Let $f(x, y) = 1$ when x is rational and $f(x, y) = 2$ when x is irrational.

Here again $\mathfrak{A}$ is not iterable, but relation (A) is satisfied, for

$$\underline{\int}_{\mathfrak{A}} f = 1, \qquad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = 1, \qquad \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = 1, \qquad \overline{\int}_{\mathfrak{A}} f = \tfrac{3}{2}.$$

4. *Correction.* I wish to call attention to an error in Theorem 8, page 31 of my former paper cited above. This theorem as there stated will not hold for certain fields involving a Pringsheim aggregate* which has a two-dimensional content greater than zero while each linear section parallel to the x-axis (or y) has a linear content equal to zero.

The proof as there given will hold if we restate the theorem as follows:

THEOREM 3. *Let* $f(x, y)$ *be limited over the limited field* $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$ *where* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are so defined that on each linear section parallel to the* x-*axis* (*or* y) *the points of* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are each everywhere dense with respect to the other. Let* $f(x, y) \geqq 0$ *over* $\mathfrak{A}_1$, *and* $f(x, y) \leqq 0$ *over* $\mathfrak{A}_2$. *Then relation* (A) *is true.*

Theorem 8 was used in no other place, so this error affects no other part of that paper.

The author is indebted to Prof. R. G. D. Richardson for calling his attention to the above error; also for suggesting the truth and method of attack of Theorem 1 of this paper.

THE UNIVERSITY OF ILLINOIS,
November 7, 1913.

* Pierpont's Lectures on the Theory of Functions of Real Variables, vol. 1, p. 546.

Volume 14

Page 65. Oswald Veblen. *Decomposition of an n-Space by a Polyhedron.*

Page 66, line 4 of §4, for "consisting" read "whose interior consists."

Page 66, line 6 of §4, for "The" read "A."

Page 66, line 9 of §4, before the word "According" insert the sentence "It is easily seen that a region has at most one boundary."

Page 66, line 4 of §5, between the words "two-dimensional" and "*polygonal*" insert "*polyhedral or a.*"

A polyhedron is defined in §5 as a set of n-dimensional polyhedral regions, together with their boundaries, subject to certain restrictions. These restrictions are not properly stated. Instead of modifying the statements as they stand, I propose to make a change which simplifies the whole matter without any loss of generality.

It is easy to prove, by an argument like that in §7, that the set of points on any polyhedral region together with its boundary can be regarded as composed of a finite number of convex polyhedral regions together with their boundaries. Hence any set of points consisting of a finite number of polyhedral regions and their boundaries may be regarded as consisting of a finite set of convex polyhedral regions and their boundaries. In view of this observation, we replace the definition at the bottom of page 66 and the top of page 67 by the following:

An *n-dimensional polyhedron* is a set of points $[P]$ which satisfies the following conditions: (1) $[P]$ consists of the points in and on the boundaries of a finite number of n-dimensional convex polyhedral regions, $F_1, F_2, \cdots, F_k$, no two of which have a point in common, and which are such that whenever there is an $(n-1)$-dimensional convex region contained in the boundary of each of q F's and containing no point in or on the boundary of any other F, q is an even number; (2) there is no subset of $[P]$ which has the property (1).

This change in the definition renders superfluous the corollary (1) of §16 on page 72, but makes no change in the rest of the paper.

It is perhaps desirable to call attention to the fact that, according to the definition employed in this paper, the boundary of a general region need not be a closed set of points. It does, however, require the boundary of a convex or polyhedral region in a number-plane to be closed.

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